tor corresponds to the Neveu-Schwarz model.

There are several obvious flaws in our picture. First, the ground-state "pion" is a tachyon. Second, baryon states which do not decouple from the spurious ground-state scalar have not yet been identified. On the plus side there exists in this model a well-defined procedure of identifying the vector and axial-vector current operators and thus calculating electromagnetic and weak form factors and amplitudes. This of course would involve the solution of the gauge-invariance problem, which we have not dealt with in this paper. There is the further problem of extending this type of model to include isospin and chirality. A model which includes parton isospin but no chirality constraints will be presented elsewhere. We wish to end with a conjecture that chiral and isospin constraints together may be the ingredient needed to generate a more physical spectrum.

Note added in proof. The problem of coupling photons has been solved and it has been proved that a deep-inelastic photon couples only through transverse components, in contrast to the usual model where it couples only through longitudinal components.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 4 15 FEBRUARY 1972

Impact-Parameter Representations in Potential Scattering*

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(Received 9 August 1971)

Starting from the Lippmann-Schwinger equation the off-energy-shell generalizations of the Glauber amplitude and the Blankenbecler-Goldberger amplitude are derived for potential scattering. The Lippmann-Schwinger equation for partial waves treated as a Fredholm equation is solved and the high-energy limit studied.

I. INTRODUCTION

The eikonal approximation of Glauber[1] has been extensively used in analyzing data involving high-energy scattering. An alternative form for the scattering amplitude at high energies was proposed by Blankenbecler and Goldberger.[2] A great deal of theoretical work has recently been done on the formal aspects of these impact-parameter representations of scattering amplitudes in potential scattering.[3,4] In a recent paper Lévy and Sucher[5] derived the Glauber amplitude in potential scattering (and relativistic scattering) for the on-energy-shell case using the "eikonal approximation" for the propagators. In the present work we have studied the high-energy scattering and derived both the Blankenbecler-Goldberger amplitude and the Glauber amplitude in the potential scattering for the off-energy-shell scattering. The transition to the on-energy-shell case is trivial. Our study makes clear the set of approximations one has to make to get one or the other amplitude.

II. DERIVATION OF BLANKENBECLER-GOLDBERGER AND GLAUBER AMPLITUDES FOR OFF-ENERGY-SHELL SCATTERING

In this section we derive the Blankenbecler-Goldberger (B-G) amplitude and the Glauber (G) amplitude for off-energy-shell scattering. Our starting point is the Lippmann-Schwinger[6] equation,

$$T(\vec{p},\vec{p}\,')=V(\vec{p},\vec{p}\,')+\frac{1}{2\pi^2}\int d\vec{q}\,\frac{V(\vec{p},\vec{q})}{q^2-p^2-i\epsilon}\,T(\vec{q},\vec{p}\,')\,, \tag{2.1}$$

where $\vec{p}$ and $\vec{p}\,'$ are the initial and final center-of-mass momenta and in general $|\vec{p}|\neq|\vec{p}\,'|$. We choose the coordinate system such that $\vec{p}$ is along the z axis and

$$\vec{p}\,'=(p',\theta,\phi)\,,$$
$$\vec{q}=(q,\theta',\phi')\,.$$

We define two new functions H and B as follows:

$$T(p,y,p')=\int_0^\infty b\,db\,J_0(2pby)H(p,b,p') \tag{2.2}$$

and

$$V(p,y,p')=\int_0^\infty b\,db\,J_0(2pby)B(p,b,p')\,, \tag{2.3}$$

where $y=\sin\frac{1}{2}\theta$. Substituting Eqs. (2.2) and (2.3) into Eq. (2.1) we get

$$\int_0^\infty b\,db\,J_0(2pby)[H(p,b,p')-B(p,b,p')]$$
$$=\frac{1}{2\pi^2}\int_0^\infty\frac{q^2dq}{q^2-p^2-i\epsilon}\int_0^\infty b'db'\int_0^\infty b''db''B(p,b',q)H(q,b'',p')\int d\Omega_q J_0(2pb'\sin\tfrac{1}{2}\theta)J_0(2qb''\sin\tfrac{1}{2}\gamma)\,, \tag{2.4}$$

where

$$\hat{p}'\cdot\hat{q}\equiv\cos\gamma=\cos\theta\,\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\,.$$

To carry out the angular integration let us define

$$I=\int d\Omega_q J_0(2pb'\sin\tfrac{1}{2}\theta')J_0(2qb''\sin\tfrac{1}{2}\gamma)\,. \tag{2.5}$$

Now using the integral

$$\int_0^{2\pi}d\phi'J_0([x^2-2xy\cos(\phi-\phi')+y^2]^{1/2})=2\pi J_0(x)J_0(y)\,, \tag{2.6}$$

where

$$x=2b''q\sin\tfrac{1}{2}\theta\,\cos\tfrac{1}{2}\theta'$$

and

$$y=2b''q\cos\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\theta'\,,$$

and a well-known integral of Sonine,[7] we get

$$I=4\int_0^\pi d\phi\,\frac{J_1(2\beta)}{\beta}\,, \tag{2.7}$$

where

$$\beta^2=p^2b'^2+q^2b''^2-2pqb'b''\cos\tfrac{1}{2}\theta\,\cos\phi\,.$$

Defining

$$G(p,q;b',b'';b)=\frac{p^2}{\pi}\int_0^\pi d\theta'\sin\theta'J_0(2pb\sin\tfrac{1}{2}\theta')\times\int_0^\pi d\phi\,\frac{J_1(2\beta)}{\beta} \tag{2.8}$$

and using the relation

$$\int_0^\infty b\,db\,J_0(2pb\sin\tfrac{1}{2}\theta)J_0(2pb\sin\tfrac{1}{2}\theta')=\frac{\delta(\sin\frac{1}{2}\theta-\sin\frac{1}{2}\theta')}{4p^2\sin\frac{1}{2}\theta}\,, \tag{2.9}$$

we get

$$I=4\pi\int_0^\infty b\,db\,J_0(2pby)G(p,q;b',b'';b)\,. \tag{2.10}$$

Substituting Eq. (2.10) into Eq. (2.4) we get

$$H(p,b,p')=B(p,b,p')+\frac{2}{\pi}\int_0^\infty\frac{q^2dq}{q^2-p^2-i\epsilon}\int_0^\infty\int_0^\infty b'db'b''db''B(p,b',q)H(q,b'',p')G(p,q;b',b'';b)\,. \tag{2.11}$$

A. Blankenbecler-Goldberger Amplitude

In the high-energy limit it can be shown that[2]

$$I\underset{p\text{ or }q\to\infty}{\sim}\frac{2\pi}{pq}J_0(2qb''\sin\tfrac{1}{2}\theta)\frac{\delta(b''-(p/q)b')}{b}\,, \tag{2.12}$$

and then it follows from Eq. (2.10) that

$$G(p,q;b',b'';b)\underset{p\text{ or }q\to\infty}{\sim}\frac{\delta(b-b')\delta(b''-(p/q)b')}{2pqbb'}\,. \tag{2.13}$$

Substituting for G from Eq. (2.13) into Eq. (2.11) we get the integral equation for H:

$$H(p,b,p')=B(p,b,p')+\frac{1}{\pi}\int_0^\infty dq\,\frac{B(p,b,q)}{q^2-p^2-i\epsilon}H(q,(p/q)b,p')\,. \tag{2.14}$$

This integral equation can be represented diagrammatically as a ladder summation. Notice that the impact parameter in the intermediate state, by angular momentum conservation, is $b'=(p/q)b$.

Let us define the momentum transfer at the first rung of the ladder diagram as

$$\vec{k}=\vec{p}-\vec{q}\,.$$

Then $q^2-p^2=-2\vec{p}\cdot\vec{k}+k^2$. If we now assume that the momentum transfer is small ("eikonal approximation") compared to the $\vec{p}\cdot\vec{k}$ term, then the denominator of Eq. (2.14) is approximated by $-2pk_z-i\epsilon$. The physical picture we then have is that at the first rung of the ladder we have an almost on-energy-shell scattering and the off-energy-shell scattering occurs through $H(q,(p/q)b,p')$. With this on-energy-shell scattering at the first rung we have $dq\approx -dk_z$ and also $B(p,b,q)\to B(p,b,p)$ and $H(p,(p/q)b,p')\to H(p,b,p')$. Then Eq. (2.14) takes a particularly simple form,

$$H(p,b,p')=B(p,b,p')+\frac{1}{\pi}B(p,b,p)H(p,b,p')\int_{-\infty}^{\infty}\frac{dk_z}{-2pk_z-i\epsilon}\,. \tag{2.15}$$

In the last integral the upper limit is actually $\frac{1}{2}p$, which we have replaced by $+\infty$ in the high-energy limit. The same replacement will be made in all the following derivations whenever a similar situation arises. On carrying out the integration in Eq. (2.15) we get

$$H(p,b,p')=\frac{B(p,b,p')}{1-(i/2p)B(p,b,p)}\,. \tag{2.16}$$

When substituted into Eq. (2.2) this yields us the B-G amplitude for off-energy-shell scattering. The on-energy-shell case is obtained by writing p for p'.

B. Yukawa Potential

It is worthwhile to demonstrate the explicit form of B both for the on-energy-shell and the off-energy-shell cases. This we do for the scattering of spinless particles of mass M scattering through a Yukawa potential,

$$V(r)=-\frac{g^2}{r}e^{-mr}\,. \tag{2.17}$$

Then

$$V(\vec{p},\vec{p}')\equiv V(p,y,p')=-\frac{M}{4\pi}\int d^3r\,V(r)e^{i(\vec{p}-\vec{p}')\cdot\vec{r}}\,. \tag{2.18}$$

For on-energy-shell scattering the vector $\vec{p}+\vec{p}'$ is orthogonal to $\vec{p}-\vec{p}'$. We choose the z direction along $\vec{p}+\vec{p}'$ and a 2-dimensional vector $\vec{b}$ orthogonal to $\vec{p}+\vec{p}'$. Then with $\vec{\Delta}=\vec{p}-\vec{p}'$,

$$\begin{aligned}V(\vec{p},\vec{p}')&=-\frac{M}{4\pi}\iint V((b^2+z^2)^{1/2})\,e^{i\vec{\Delta}\cdot\vec{b}}d^2b\,dz\\&=-\frac{M}{2}\int_0^\infty b\,db\,J_0(\Delta b)\int_{-\infty}^{\infty}V((b^2+z^2)^{1/2})dz\,.\end{aligned} \tag{2.19}$$

If we define

$$\begin{aligned}V(b)&\equiv-\frac{M}{2}\int_{-\infty}^{\infty}V((b^2+z^2)^{1/2})dz\\&=+Mg^2\int_0^\infty\frac{e^{-m(b^2+z^2)^{1/2}}}{(b^2+z^2)^{1/2}}dz\\&=Mg^2K_0(mb)\,,\end{aligned} \tag{2.20}$$

then

$$B(p,b,p)=Mg^2K_0(mb)=V(b)\,. \tag{2.21}$$

For off-energy-shell scattering $\vec{p}+\vec{p}'$ is not orthogonal to $\vec{p}-\vec{p}'$. To get $B(p,b,p')$ we proceed as follows:

$$\begin{aligned}V(\vec{p},\vec{p}')&=\sum V_l(p,p')(2l+1)P_l(\cos\theta)\\&=\sum V_l(p,p')(2l+1)\int_0^\infty d\beta J_{2l+1}(\beta)J_0(\beta y)\,,\end{aligned} \tag{2.22}$$

where $y=\sin\frac{1}{2}\theta$. Now for the Yukawa potential,

$$V_l(p,p')=\frac{Mg^2}{2pp'}Q_l\left(1+\frac{(p-p')^2+m^2}{2pp'}\right) \tag{2.23}$$

so that[4]

$$\begin{aligned}V(\vec{p},\vec{p}')&=\int_0^\infty J_0(\beta y)d\beta\sum(2l+1)J_{2l+1}(\beta)\frac{Mg^2}{2pp'}Q_l\left(1+\frac{(p-p')^2+m^2}{2pp'}\right)\\&=\frac{Mg^2}{2pp'}\int_0^\infty\beta\,d\beta\,J_0(\beta y)\frac{1}{2}\int_0^1\frac{J_0(\beta\xi)\xi\,d\xi}{\xi^2+[(p-p')^2+m^2]/4pp'}\,.\end{aligned} \tag{2.24}$$

Set $\beta = 2pb$ and $\zeta = 2p\xi$; then

$$\int_0^1 \frac{J_0(\beta\xi)\xi\,d\xi}{\xi^2+[(p-p')^2+m^2]/4pp'} = \int_0^{2p} \frac{J_0(b\zeta)\zeta\,d\zeta}{\zeta^2+(p/p')[(p-p')^2+m^2]}$$

$$\underset{p\to\infty}{\simeq} K_0(b(p/p')^{1/2}[(p-p')^2+m^2]^{1/2}). \tag{2.25}$$

Combining Eq. (2.25) with (2.24) we get

$$V(\vec{p},\vec{p}\,') = Mg^2\frac{p}{p'}\int_0^\infty b\,db\,J_0(2pby)K_0\left(b\left(\frac{p}{p'}\right)^{1/2}[(p-p')^2+m^2]^{1/2}\right), \tag{2.26}$$

which gives us

$$B(p,b,p') = Mg^2\frac{p}{p'}K_0\left(b\left(\frac{p}{p'}\right)^{1/2}[(p-p')^2+m^2]^{1/2}\right). \tag{2.27}$$

C. Glauber Amplitude

In deriving Glauber's amplitude we iterate Eq. (2.14). The term of order $n+1$ in B is

$$H^{(n+1)}(p,b,p') = \left(\frac{1}{\pi}\right)^n \int (dq_1\cdots dq_n)\frac{B(p,b,q_1)B(q_1,pb/q_1,q_2)\cdots B(q_n,pb/q_n,p')}{(q_1^{\,2}-p^2-i\epsilon)\cdots(q_n^{\,2}-p^2-i\epsilon)}. \tag{2.28}$$

There are $n+1$ equivalent ways of writing this expression:

$$\left(\frac{1}{\pi}\right)^n\int\left(\prod_{k=1}^n dq_k\right)\left[\frac{B(p,b,q_1)}{q_1^{\,2}-p^2-i\epsilon}\,\frac{B(q_1,pb/q_1,q_2)}{q_2^{\,2}-p^2-i\epsilon}\cdots\frac{B(q_{l-1},pb/q_{l-1},q_l)}{q_l^{\,2}-p^2-i\epsilon}\right]$$
$$\times B(q_l,pb/q_l,q_{l+1})\left[\frac{B(q_{l+1},pb/q_{l+1},q_{l+2})}{q_{l+1}^{\,2}-p^2-i\epsilon}\cdots\frac{B(q_n,pb/q_n,p')}{q_n^{\,2}-p^2-i\epsilon}\right]. \tag{2.29}$$

There are l terms in the first bracket and $n-l$ in the second bracket. If no approximations are made, then of course, the value of this expression will not depend upon how this decomposition is done. l can vary from 0 to n. Now we write

$$\vec{q}_i = \vec{p}-\vec{k}_1-\vec{k}_2-\cdots-\vec{k}_i = \vec{p}-\vec{K}_i, \quad i=1,2,\ldots,l$$

and

$$\vec{q}_j = \vec{p}\,'+\vec{k}_j+\vec{k}_{j+1}+\cdots+\vec{k}_n = \vec{p}\,'+\vec{K}_j', \quad j=l+1, \quad l+2,\ldots,n.$$

Next we make the "eikonal approximation" to the propagators by writing

$$q_i^{\,2}-p^2-i\epsilon \approx -2pK_{iz}-i\epsilon, \quad i=1,2,\ldots,l$$

and

$$q_j^{\,2}-p^2-i\epsilon \approx +2p'K_{j\xi}-\Delta p^2-i\epsilon, \quad j=l+1, \quad l+2,\ldots,n$$

where $\Delta p^2 = p^2-p'^2$ and K_ξ is the projection of $\vec{K}$ in the direction $\vec{p}\,'$. The essence of our next step is to assume that in the first l rungs of the ladder the scattering is essentially on the energy shell. There is then an off-energy-shell "kick" and for the remaining $n-l$ rungs the scattering is again on the energy shell, albeit the energy is now different from the initial energy. Thus at high energies ($p,p'\to\infty$, but $p\neq p'$) we have $q_i^{\,2}\approx p^2$ and $q_j^{\,2}\approx p'^2$. We also have $dq_i\approx -dk_{iz}$ and $dq_j\approx dk_{j\xi}$. Also for the first l rungs $B(q_i,pb/q_i,q_{i+1})\to B(p,b,p)$ and for the last $n-l$ rungs $B(q_j,pb/q_j,q_{j+1})\to B(p',pb/p',p')$. Thus we write Eq. (2.29) finally in the form

$$H^{(n+1)}(p,b,p') \approx \left(\frac{1}{\pi}\right)^n\int_{+\infty}^{-\infty}\prod_{i=1}^{l}\frac{dk_{iz}}{2pK_{iz}+i\epsilon}\int_{+\infty}^{-\infty}\prod_{j=l+1}^{n}\frac{dk_{j\xi}}{\Delta p^2-2p'K_{j\xi}+i\epsilon}[B(p,b,p)]^l B(p,b,p')[B(p',pb/p',p')]^{n-l}. \tag{2.30}$$

Once we make the "eikonal approximation" the value of $H^{(n+1)}$ depends on the value of l. To overcome this undesirable feature Lévy and Sucher[5] sum over $n+1$ values of l and average over $n+1$. Following Lévy and Sucher, as the integrands are separately symmetric in the k's, we may sum over all permutations of

k's inside each group and divide by $l!(n-l)!$, the total number of permutations. Thus

$$H^{(n+1)}(p,b,p')=\left(\frac{1}{\pi}\right)^n\frac{1}{n+1}\sum_{l=0}^{n}\frac{1}{l!(n-l)!}B(p,b,p')[B(p,b,p)]^l[B(p',pb/p',p')]^{n-l}$$
$$\times\sum_{\pi_i}\int_{+\infty}^{-\infty}\prod_{i=1}^{l}\frac{dk_{iz}}{2pK_{iz}+i\epsilon}\sum_{\pi_j}\int_{+\infty}^{-\infty}\prod_{j=l+1}^{n}\frac{dk_{j\xi}}{\Delta p^2-2p'K_{j\xi}+i\epsilon}, \tag{2.31}$$

where π_i means a permutation of the i's. The use of an identity proven by Lévy and Sucher,[5]

$$\sum_{\pi}\prod_{r=1}^{m}\frac{1}{A_r+x+i\epsilon}=-ix\int_0^{\infty}d\beta\, e^{i\beta x}\prod_{r=1}^{m}\frac{1-e^{+i\beta a_r}}{a_r},$$

where $A_r=a_1+a_2+\cdots+a_r$, enables us to write Eq. (2.31) in the form

$$H^{(n+1)}(p,b,p')=\left(\frac{1}{\pi}\right)^n\frac{B(p,b,p')}{n+1}\sum_{l=0}^{n}\frac{1}{l!(n-l)!}[B(p,b,p)]^l\int_{+\infty}^{-\infty}\prod_{i=1}^{\infty}\frac{dk_{iz}}{2pk_{iz}+i\epsilon}$$
$$\times[B(p',pb/p',p']^{n-l}(-i\Delta p^2)\int_0^{\infty}d\beta\, e^{i\beta\Delta p^2}\int_{+\infty}^{-\infty}\prod_{j=l+1}^{n}\frac{dk_{j\xi}(1-e^{-2i\beta p'k_{j\xi}})}{-2p'k_{j\xi}+i\epsilon}. \tag{2.32}$$

Now

$$\int_{+\infty}^{-\infty}\frac{dk_{iz}}{2pk_{iz}+i\epsilon}=\frac{i\pi}{2p}$$

and

$$\int_{+\infty}^{-\infty}\frac{dk_{j\xi}(1-e^{-2i\beta p'k_{j\xi}})}{-2p'k_{j\xi}+i\epsilon}=\frac{i\pi\theta(\beta)}{2p'},$$

where $\theta(\beta)$ is the usual step function. On carrying out the β integration and summing over l we get

$$H^{(n+1)}(p,b,p')=\frac{i^n}{(n+1)!}B(p,b,p')\chi^n, \tag{2.33}$$

where

$$\chi=\frac{B(p,b,p)}{2p}+\frac{B(p',pb/p',p')}{2p'}. \tag{2.34}$$

Summing over n we get

$$H(p,b,p')=B(p,b,p')\frac{e^{i\chi}-1}{i\chi}. \tag{2.35}$$

Substituting this in Eq. (2.2) we get the off-energy-shell form of Glauber's amplitude. It is clear that on the energy shell $p'\to p$ and one gets the familiar Glauber form

$$T(p,y)=-ip\int_0^{\infty}b\,db\,J_0(2pby)(e^{i\chi}-1), \tag{2.36}$$

and off the energy shell one gets

$$T(p,y)=-i\int_0^{\infty}b\,db\,J_0(2pby)B(p,b,p')\frac{e^{i\chi}-1}{\chi}. \tag{2.37}$$

Let us now look back on the set of approximations we had to make to get the Glauber or the Blankenbecler-Goldberger form of the amplitude. To keep the discussion simple let us consider only the on-energy-shell scattering. In deriving Glauber's form of the amplitude the ladder graph of order $n+1$ was broken into two parts: the first part with l rungs and the second with $n-l$ rungs with the $(l+1)$th rung sandwiched in between. The "eikonal approximation" was then made. This approximation distorts the value of the order-$(n+1)$ term. As Lévy and Sucher point out, the value of the amplitude depends on which rung of the ladder the over-all momentum conservation is imposed. Thus one averages over the $n+1$ possibilities that one has for the term of order $n+1$. This averaging procedure apart from the "eikonal approximation" of the propagator is an essential ingredient in obtaining the Glauber form of the amplitude. In deriving the B-G form, on the other hand, the over-all momentum conservation is imposed at the first rung of the ladder and there is no averaging involved.

III. ALTERNATE DERIVATION OF B-G AND G AMPLITUDES

In this section we present an alternative derivation of the results established in Sec. II. We start with the Lippmann-Schwinger equation for the partial waves,

$$T_l(p,p')=V_l(p,p')+\frac{2}{\pi}\int_0^{\infty}\frac{q^2dq}{q^2-p^2-i\epsilon}V_l(p,q)T_l(q,p'), \tag{3.1}$$

which is obtained from Eq. (2.1) on using the partial-wave expansions

$$T(\vec p,\vec p')=\sum(2l+1)P_l(\cos\theta)T_l(p,p'),$$
$$V(\vec p,\vec p')=\sum(2l+1)P_l(\cos\theta)V_l(p,p'). \tag{3.2}$$

We make the "eikonal approximation" $q^2-p^2\approx-2pk_z$, where k_z is the component of $\vec k$, the momentum transfer, along the direction of $\vec{p}$. As before, we assume that the first rung of the ladder results in almost an on-energy-shell scattering, the off-energy-shell scattering coming from $T_l(q,p')$. In this approximation we get, as in Sec. II with

$$qdq \approx -pdk_z$$

and

$$q \approx p\,,$$

$$T_l(p,p') = V_l(p,p') - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{pdk_z}{k_z + i\epsilon}\, V_l(p,p)T_l(p,p') \tag{3.3}$$

or

$$T_l(p,p') = \frac{V_l(p,p')}{1 - ipV_l(p,p)}\,. \tag{3.4}$$

This is the off-energy-shell form of the partial-wave amplitude at high energy. The on-energy-shell form obtained by writing p for p' has the structure of the unitarized Born approximation.

To see the relationship of the result of Eq. (3.4) and the result of Sec. II let us look at the structure of the partial-wave amplitude for on-energy-shell scattering via the Yukawa potential of (2.17). For the scattering of particles of mass M,

$$\begin{aligned} V_l(p,p) &= \frac{g^2M}{2p^2}\, Q_l\left(1 + \frac{m^2}{2p^2}\right) \\ &\xrightarrow[\text{large } l\text{'s}]{} \frac{g^2M}{2p^2}\, K_0(mb)\,, \end{aligned} \tag{3.5}$$

where $bp = l$ (for large l's). Using the definition of $V(b)$, Eq. (2.20), we get [note that by Eq. (2.21), $V(b) = B(p,b,p)$]

$$V_l(p,p') \xrightarrow[\text{large } l\text{'s}]{} \frac{V(b)}{2p^2}\,. \tag{3.6}$$

Substituting Eq. (3.6) into Eq. (3.4) one gets for on-energy-shell scattering and large l's

$$T_l(p,p') \xrightarrow[\text{large } l\text{'s}]{} \frac{1}{2p^2}\, \frac{V(b)}{1 - (i/2p)V(b)}\,. \tag{3.7}$$

Finally, putting this in Eq. (3.2) and converting the sum over l into an integration together with the replacements

$$\begin{aligned} &pb = l \quad \text{(for large } l\text{'s)}\,, \\ &P_l(\cos\theta) \to J_0(2pb\sin\tfrac{1}{2}\theta)\,, \end{aligned} \tag{3.8}$$

one gets

$$T(p,\cos\theta) \to \int bdb\, J_0(2kb\sin\tfrac{1}{2}\theta)\, \frac{V(b)}{1 - (i/2p)V(b)}\,. \tag{3.9}$$

For the off-energy-shell scattering we get

$$\begin{aligned} V_l(p,p') &= \frac{Mg^2}{2pp'}\, Q_l\left(1 + \frac{(p-p')^2 + m^2}{2pp'}\right) \\ &\xrightarrow[\text{large } l\text{'s}]{} \frac{Mg^2}{2pp'}\, K_0\left(\left(\frac{(p-p')^2 + m^2}{pp'}\right)^{1/2} l\right) \\ &= \frac{Mg^2}{2pp'}\, K_0\left(\left(\frac{p}{p'}\right)^{1/2} [(p-p')^2 + m^2]^{1/2} b\right). \end{aligned} \tag{3.10}$$

Using the definition of $B(p,b,p')$ given in Eq. (2.27) we find

$$V_l(p,p') \xrightarrow[\text{large } l\text{'s}]{} \frac{1}{2p^2}\, B(p,b,p')\,. \tag{3.11}$$

Substituting Eq. (3.11) and Eq. (3.6) in Eq. (3.4) we get

$$T_l(p,p') \xrightarrow[\text{large } l\text{'s}]{} \frac{1}{2p^2}\, \frac{B(p,b,p')}{1 - (i/2p)B(p,b,p)}\,. \tag{3.12}$$

Using Eq. (3.12) together with Eq. (3.8) in Eq. (3.2) and converting the sum into an integral we get the off-energy-shell form of the B-G amplitude

$$\begin{aligned} T(\vec{p},\vec{p}') = \int_0^\infty & bdb\, J_0(2bp\sin\tfrac{1}{2}\theta) \\ &\times \frac{B(p,b,p')}{1 - (i/2p)B(p,b,p)}\,. \end{aligned} \tag{3.13}$$

To derive the off-energy-shell Glauber amplitude we iterate Eq. (3.1). The term of order V_l^{n+1} has the form which we can write in $n+1$ equivalent ways, as explained in Sec. II,

$$\begin{aligned} T_l^{(n+1)}(p,p') = \left(\frac{2}{\pi}\right)^n \int \Bigg[\prod_{k=1}^n q_k^2 dq_k & \left(\frac{V_l(p,q_1)}{q_1^2 - p^2 - i\epsilon}\, \frac{V_l(q_1,q_2)}{q_2^2 - p^2 - i\epsilon} \cdots \frac{V_l(q_{l-1},q_l)}{q_l^2 - p^2 - i\epsilon}\right) V_l(q_l,q_{l+1}) \\ & \times \frac{V_l(q_{l+1},q_{l+2})}{q_{l+1}^2 - p^2 - i\epsilon} \cdots \frac{V_l(q_n,p')}{q_n^2 - p^2 - i\epsilon}\Bigg]. \end{aligned} \tag{3.14}$$

We make the "eikonal approximation" as in Sec. II. We also assume that the first l rungs of the ladder give almost on-energy-shell scatterings with an off-energy-shell "kick" at the $(l+1)$th rung followed by $n-l$ on-energy-shell scatterings though at a different energy. We go through exactly the same approximations, permutations, and averaging procedures described in detail in Sec. II and finally get

$$T_l^{(n+1)}(p,p')=\frac{i^n}{(n+1)!}\,V_l(p,p')\times[pV_l(p,p)+p'V_l(p',p')]^n\,. \quad (3.15)$$

This is the analog of Eq. (2.33). To see the correspondence we simply recall that [Eqs. (3.6) and (3.11)]

$$pV_l(p,p)\xrightarrow[\text{large } l\text{'s}]{}\frac{1}{2p}B(p,b,p)$$

and (3.16)

$$pV_l(p,p')\xrightarrow[\text{large } l\text{'s}]{}\frac{1}{2p}B(p,b,p')\,.$$

Substituting Eq. (3.16) in Eq. (3.15) we get

$$T_l^{(n+1)}(p,p')\xrightarrow[\text{large } l\text{'s}]{}\frac{i^n}{(n+1)!}\,\frac{B(p,b,p')}{2p^2}\,\chi^n\,, \quad (3.17)$$

with χ defined in Eq. (2.34). Using Eq. (3.17) in Eq. (3.2), converting the sum over l into integration, and using the small-angle approximation for $P_l(\cos\theta)$ we get the same amplitude as that derived in Eq. (2.37).

IV. FREDHOLM SOLUTION

In this section we present the solution of the Lippmann-Schwinger equation for partial waves treated as a Fredholm equation. We can write the Lippmann-Schwinger equation [Eq. (3.1)] in the form

$$T_l(p,p')=V_l(p,p')+\int q^2\,dq\,K_l(p,q)T_l(q,p')\,, \quad (4.1)$$

where the kernel $K_l(p,q)$ is[8]

$$K_l(p,q)=\frac{2}{\pi}\,\frac{V_l(p,q)}{q^2-p^2-i\epsilon}\,. \quad (4.2)$$

Notice that the propagator involves the square of the second entry in the argument of $K_l(p,q)$ minus p^2 (*not* the square of the first entry). The solution of the Fredholm equation, Eq. (4.1), can be written using a resolvent kernel[9] $H_l(p,q)$,

$$T_l(p,p')=V_l(p,p')-\int q^2dq\,H_l(p,q)V_l(q,p')\,, \quad (4.3)$$

where the resolvent kernel is expressed in a quotient form

$$H_l(p,q)=\frac{N_l(p,q)}{D_l(p)}\,, \quad (4.4)$$

with

$$N_l(p,q)=-\sum_{m=0}^{\infty}\int K\begin{pmatrix}p,\xi_1,\ldots,\xi_m\\ q,\xi_1,\ldots,\xi_m\end{pmatrix}\times\xi_1^2d\xi_1\cdots\xi_m^2d\xi_m \quad (4.5)$$

and

$$D_l(p)=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\int K\begin{pmatrix}\xi_1,\ldots,\xi_m\\ \xi_1,\ldots,\xi_m\end{pmatrix}\times\xi_1^2d\xi_1\cdots\xi_m^2d\xi_m\,, \quad (4.6)$$

where K is defined as the determinant

$$K\begin{pmatrix}p,\xi_1,\ldots,\xi_m\\ q,\xi_1,\ldots,\xi_m\end{pmatrix}=\begin{vmatrix}K(p,q) & K(p,\xi_1)\cdots K(p,\xi_m)\\ K(\xi_1,q) & K(\xi_1,\xi_1)\cdots K(\xi_1,\xi_m)\\ \vdots & \\ K(\xi_m,q) & K(\xi_m,\xi_1)\cdots K(\xi_m,\xi_m)\end{vmatrix}\,. \quad (4.7)$$

We can write the solution given by Eq. (4.3) as

$$T_l(p,p')=\frac{V_l(p,p')D_l(p)-\int q^2dq\,N_l(p,q)N_l(q,p')}{D_l(p)}\,. \quad (4.8)$$

Let us now investigate the high-energy behavior of $T_l(p,p')$ both on the energy shell and off the energy shell.

Denominator of Eq. (4.8). We first look at the term with $m=1$ (first order in the potential) for scattering by a Yukawa potential. This term has a form

$$-\frac{2}{\pi}\int_0^\infty\frac{\xi^2d\xi\,V_l(\xi,\xi)}{\xi^2-p^2-i\epsilon}\,. \quad (4.9)$$

For $l=0$,

$$V_0(\xi,\xi)=\frac{Mg^2}{2\xi^2}\,Q_0\left(1+\frac{m^2}{2\xi^2}\right)=\frac{Mg^2}{4\xi^2}\ln\frac{4\xi^2+m^2}{m^2}\,. \quad (4.10)$$

Putting Eq. (4.10) in (4.9) we find that the principal-value part of the integral yields

$$-\frac{2}{\pi}P\int_0^\infty\frac{\xi^2d\xi\,V_0(\xi,\xi)}{\xi^2-p^2-i\epsilon}=-\frac{Mg^2\pi}{4p} \quad (4.11)$$

while the on-shell part gives

$$-V_0(p,p)=-\frac{Mg^2}{4p}\ln\frac{4p^2+m^2}{m^2}\,. \quad (4.12)$$

Thus for large energies the on-shell part is larger than the principal-part contribution. This is also true of the case with $l=1$ as one can check quite explicitly. Beyond $l=1$ one can use the recurrence relation

$$(l+1)Q_{l+1}(x)=(2l+1)xQ_l(x)-lQ_{l-1}(x)$$

to prove that it is generally true for all angular momenta. Thus at high enough energies one need keep only the on-shell contribution of the term with $m=1$.

One can also show that the higher-order terms ($m\geq 2$) go to zero faster than the terms of order $m=1$. The denominator, $D(p)$, is therefore well approximated at high energies by

$$D_l(p)\underset{p\to\infty}{\sim} 1-\int K(\xi,\xi)\xi^2 d\xi$$
$$=1-\frac{2}{\pi}\int \xi^2 d\xi\, \frac{V_l(\xi,\xi)}{\xi^2-p^2-i\epsilon}\,. \quad (4.13)$$

Numerator of Eq. (4.8). The dominant term of the numerator at high energies is the product of $V_l(p,p')$ (for both on-energy-shell and off-energy-shell scattering) with the unity of $D_l(p)$. Thus at high energies for both on-energy-shell and off-energy-shell scattering one gets

$$T_l(p,p')\underset{p\to\infty}{\sim} V_l(p,p')\left(1-\frac{2}{\pi}\int\frac{\xi^2 d\xi\, V_l(\xi,\xi)}{\xi^2-p^2-i\epsilon}\right)^{-1}. \quad (4.14)$$

If, further, at high energies we retain only the on-shell part of the denominator we get

$$T_l(p,p')\underset{p\to\infty}{\sim}\frac{V_l(p,p')}{1-ipV_l(p,p)}\,, \quad (4.15)$$

which is precisely the result of Eq. (3.4). Thus it appears that the B-G amplitude can also be obtained via the method described in this section. We do not know, however, how the Glauber amplitude could be obtained in an analogous treatment.

Note added in proof. (1) Using Eqs. (2.3) and (2.18) and assuming a spherically symmetric potential it can be shown that

$$B(p,b,p')=-\frac{Mp}{p'}\int_0^\infty dz\cos[z(p-p')]\times V((z^2+pb^2/p')^{1/2})\,.$$

This exact result replaces the procedure adopted in Sec. II B, which is somewhat indirect and approximate. For the Yukawa potential, Eq. (2.27) is readily obtained from the above equation. (2) Recently one of us (ANK) has shown that the averaging procedure, Sec. II, is not necessary for deriving the on-energy-shell version of the Glauber amplitude.[10]

ACKNOWLEDGMENT

We wish to thank Professor M. Razavy for several useful discussions.

*Work supported in part by the National Research Council of Canada.

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