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PHYSICAL REVIEW D VOLUME 5, NUMBER 4 15 FEBRUARY 1972

Consistency of Current-Algebra Results in the Presence of Electromagnetism*

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We demonstrate in this paper that despite the appearance of dynamical terms in the current-commutation relations as a result of the electromagnetic interaction, one can still use the algebra of currents to derive low-energy results.

INTRODUCTION

It is generally assumed that the algebra of the currents is preserved even in the presence of symmetry-breaking effects. However, when electromagnetic fields are included, two effects combine to make their appearance felt in the commutation relations: (1) The general principles of quantum theory require that gradient terms be included in the commutator of time and space components, and (2) gauge invariance then effectively modifies these commutation relations through the substitution $\partial_\mu \to \partial_\mu - ieA_\mu$. These terms can be explicitly calculated in several models, e.g., spin-zero electrodynamics, the algebra of fields,[1] etc.

In this paper we show that the inclusion of these new dynamical terms does not affect the general results obtained from current algebra by means of low-energy theorems, and that one can still make model-independent calculations provided that one is careful with relativity and gauge invariance. Furthermore, we show that the appearance of noncanonical terms is in fact necessary if one is to avoid certain paradoxes.[2]

A LOW-ENERGY THEOREM

Consider the one-photon–vacuum matrix element of the vector-vector current correlation function $iT^*(V_+^\mu(x)V_-^\nu(0))$, namely,

$$T_{+-}^{\mu\nu}(k,q)=\int dx\, e^{-iq\cdot x} i\langle\gamma(k)|T^*(V_+^\mu(x)V_-^\nu(0))|0\rangle, \tag{1}$$

where the T^* product is the covariant time-ordered product[3]

$$iT^*(\;)=iT(\;)+\rho_{+-}^{\mu\nu}\delta^4(x)$$

and $\rho_{+-}^{\mu\nu}$ is the coefficient of the gradient term in the commutator

$$[V_+^0(x),\ V_-^\mu(0)]\delta(x^0)=2V_3^\mu(x)\delta^4(x) + i(\partial_\lambda+ieA_\lambda)\rho_{+-}^{\lambda\mu}(x)\delta^4(x), \tag{2}$$

with $\rho_{+-}^{\mu\nu}=\rho_{-+}^{\nu\mu}$ and $\rho_{+-}^{0\mu}=0$. The explicit appearance of A_λ in the commutator is required by gauge invariance.[4-6] Calculating the divergence of (1) in the standard way, one obtains

$$iq_\mu T_{+-}^{\mu\nu}=\int dx\, e^{-iq\cdot x} i\langle\gamma|T(D_+(x)V_-^\nu(0))|0\rangle - ie\langle\gamma|A_\lambda\rho_{+-}^{\lambda\nu}|0\rangle+2i\langle\gamma|V_3^\nu(0)|0\rangle. \tag{3}$$

One can view Eq. (3) as a way of calculating the matrix element $\langle\gamma|V_3^\mu(0)|0\rangle$. On the other hand, one can show[7] on the basis of Lorentz and gauge invariance alone that the matrix element of a gauge-invariant four-vector field between the vacuum and the one-photon state must vanish, i.e.,

$$\langle\gamma(\vec{k})|J^\mu(0)|0\rangle=0 \tag{4}$$

whether J^μ is conserved or not. The inclusion of the last term in Eq. (2) is crucial if one is to obtain agreement with this general result proceeding from Eq. (3), and its neglect leads to a paradox.[2] Note that the right-hand side of (3) has Lorentz- and gauge-transformation properties identical to those of the left-hand side, and this again depends on the gradient terms appearing in (2).

The combination

$$iT(D_+(x)V_-^\nu(0)) - ieA_\lambda\rho_{+-}^{\lambda\nu}(0)\delta^4(x) \tag{5}$$

can be shown to transform like a four-vector,[8] and it is the correct expression for the T^* product of the divergence and the current. Now taking the divergence condition

$$\partial_\mu V_\pm^\mu(x) = \mp ieA_\mu V_\pm^\mu(x) \tag{6}$$

and calculating to first order in e, we obtain

$$\langle\gamma|T(D_+(x)V_-^\nu(0))|0\rangle = -ie\frac{\epsilon_\mu(\vec{k})}{(2\pi)^{3/2}}e^{-ik\cdot x}\langle 0|T(V_+^\mu(x)V_-^\nu(0))|0\rangle + O(e^2), \tag{7}$$

$$\langle\gamma|A_\lambda\rho_{+-}^{\lambda\nu}(0)|0\rangle = \frac{\epsilon_\lambda(\vec{k})}{(2\pi)^{3/2}}\langle 0|\rho_{+-}^{\lambda\nu}(0)|0\rangle + O(e^2). \tag{8}$$

Substituting (7) and (8) into (3) gives

$$iq_\mu T_{+-}^{\mu\nu} = 2i\langle\gamma|V_3^\nu(0)|0\rangle - ie\frac{\epsilon_\mu(\vec{k})}{(2\pi)^{3/2}}[\Delta_{+-}^{\mu\nu}(q+k) + \langle 0|\rho_{+-}^{\mu\nu}(0)|0\rangle] + O(e^2), \tag{9}$$

where

$$\Delta_{+-}^{\mu\nu}(q) = \int dx\, e^{-iq\cdot x} i\langle 0|T(V_+^\mu(x)V_-^\nu(0))|0\rangle.$$

We now introduce the Källén-Lehmann representation for the vacuum expectation value, namely,

$$\begin{aligned}\Delta_{+-}^{\mu\nu}(q) &= \int_0^\infty ds\left(\rho^{(1)}(s)\frac{\delta^{\mu\nu}+q^\mu q^\nu/s}{q^2+s-i\epsilon} + \rho^{(0)}(s)\frac{q^\mu q^\nu}{q^2+s-i\epsilon}\right) + \delta_0^\mu\delta_0^\nu\int_0^\infty ds\left(\frac{\rho^{(1)}(s)}{s}+\rho^{(0)}(s)\right)\\ &= \int_0^\infty ds\,\rho^{(1)}(s)\frac{\delta^{\mu\nu}+q^\mu q^\nu/s}{q^2+s-i\epsilon} + \delta_0^\mu\delta_0^\nu\int_0^\infty ds\frac{\rho^{(1)}(s)}{s} + O(e^2).\end{aligned} \tag{10}$$

Likewise, one can show

$$\langle 0|\rho_{+-}^{\mu\nu}(0)|0\rangle = -(\delta^{\mu\nu}+\delta_0^\mu\delta_0^\nu)\int ds\frac{\rho^{(1)}(s)}{s} + O(e^2). \tag{11}$$

Combining (10) and (11),

$$\Delta_{+-}^{*\mu\nu} \equiv \Delta_{+-}^{\mu\nu} + \langle 0|\rho_{+-}^{\mu\nu}|0\rangle = \int_0^\infty ds\,\rho^{(1)}(s)\frac{\delta^{\mu\nu}+q^\mu q^\nu/s}{q^2+s-i\epsilon} - \delta^{\mu\nu}\int_0^\infty ds\frac{\rho^{(1)}(s)}{s} + O(e^2). \tag{12}$$

Note that the effect of the Schwinger term in (2) is to produce a covariant vacuum expectation value, but that the seagull terms must still be included in the right-hand side of (12). Finally, substituting (12) into (9) and going to the limit $q\to 0$, one obtains

$$0 = 2i\langle\gamma|V_3^\nu(0)|0\rangle - ie\frac{\epsilon_\mu(\vec{k})}{(2\pi)^{3/2}}(k^\mu k^\nu - k^2\delta^{\mu\nu})\int_0^\infty ds\,\rho^{(1)}(s)\frac{1}{s(k^2+s-i\epsilon)} + O(e^2), \tag{13}$$

but since $k^2=0$, $k\cdot\epsilon=0$ there is a cancellation to order e in the right-hand side of (13), and one finally concludes

$$\langle\gamma(\vec{k})|V_3^\nu(0)|0\rangle = O(e^2), \tag{14}$$

and therefore to lowest order there is no conflict with Eq. (4). This cancellation should occur to all orders in e, and although explicit calculations to higher orders are complicated by the necessity of carefully accounting for all terms in the commutators of the electromagnetic field with the currents, which are field-dependent,[9] they are straightforward in principle.

It is perfectly obvious that the argument presented above is completely independent of the parity of the currents (barring the effect of Adler terms of the form $F_{\mu\nu}\tilde{F}^{\mu\nu}$ which we have neglected since there appears to be no evidence that they occur in the charged currents), and the result expressed in Eq. (14) holds for the axial-vector currents as well.

Therefore, we conclude that on the basis of low-energy theorems such as the above, one does not need to introduce any new terms in either the commutation relations or the divergence conditions provided one consistently enforces gauge invariance and relativity. Whether this conclusion is valid for other current-algebra calculations such as those involving hard-pion processes is an open question.

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$$\langle \vec{k}\lambda | J^{\mu}(0) | 0\rangle = \epsilon_{\nu}(\vec{k}\lambda)\, T^{\nu\mu}(k),$$

where $k\cdot\epsilon = 0$, and $T^{\nu\mu}(k)$ is a second-rank tensor satisfying $k_{\nu}T^{\nu\mu}(k) = 0$. [See S. Weinberg, Phys. Rev. 135, B1049 (1964), for a discussion of these gauge conditions.] From these two conditions it follows that

$$T^{\nu\mu}(k) = (\delta^{\nu\mu}k^2 - k^{\nu}k^{\mu})\, T(k^2).$$

Hence on the mass shell

$$\epsilon_{\nu}\, T^{\nu\mu} = 0 = \langle \vec{k}\lambda | J^{\mu} | 0\rangle .$$

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Multiparticle Sum Rules*

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Sum rules are derived by applying soft-pion theorems to the dispersion relations for the reactions $\pi + I \to \pi + F$, $I \to 2\pi + F$, $2\pi + I \to F$, with I and F general multiparticle states. In the special case where I contains one particle and F contains two particles, these sum rules have a simple algebraic interpretation: They require that the transition matrix elements for reactions $\pi + I \to F$ or $I \to F + \pi$ can receive contributions only from the representations (A, B) of chiral $SU(2)\otimes SU(2)$ with $A = B$, but not $A = B \pm 1$. Prescriptions are given for dealing with the complications caused by the nonvanishing matrix elements of chiral generators between states of different mass and spin.

I. INTRODUCTION AND SUMMARY

Sum rules of one sort or another have played a large part in the recent development of elementary-particle theory. Given their great importance, it is perhaps surprising that these sum rules have generally been restricted to a limited class, involving transitions between a *single*-particle state and other states, induced by a current or another particle. For instance, the Adler-Weisberger sum rules[1] are derived either by evaluating matrix elements of axial-vector-current-density commutators between *single*-particle states, or by using soft-pion theorems in conjunc-