the generalization to an arbitrary symmetry group.

[12]J. Grodnik (private communication).

[13]For the purpose of arriving at an expression for the Hamiltonian as a function of the local currents, one may ignore such terms as the $\delta(0)$, which technically ought to appear in the last equality for $\mu=0$; this was already assumed in the defining equation (3.2). The justification for the Hamiltonian expression obtained in this manner lies in the fact that it can be shown to satisfy the desired continuity equation and Schwinger condition using the current commutators.

[14]The Sugawara constant c in Eqs. (4.36) and (4.38) for the case $n=1$ is one-fourth the constant c in Eq. (2.14), since we have defined $J^0_\mu=\frac{1}{2}j_\mu$.

[15]This result differs from an analogous statement in a slightly different context by Y. Freundlich and D. Lurié, Phys. Rev. D 1, 1660 (1970).

[16]See Freundlich and Lurié (Ref. 15).

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Dilatation and Dimensional Transformations*

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The effect of a field redefinition on the dilatation transformation is considered. It is proposed that the generator of the dilatation transformation is modified if the divergence of the dilatation current can be expressed in the form of the divergence of a local current as a consequence of field equations. The notion of the dimensional transformation is introduced to generalize the case when the dilatation current does not satisfy the above condition. The dimensional transformation is worked out in detail in the case of a massive neutral scalar field.

I. INTRODUCTION

In this paper an attempt is made to clarify some fundamental points with regard to the definition of the dilatation transformation, and also to show explicitly a close relation between the dilatation transformation and dimensional analysis.

The first problem arises in the following situation. We consider some dynamical system described by field variables $\phi^{(\kappa)}(x)$ $(\kappa=1, 2, \ldots)$ and some constants such as the mass and the coupling constant with some dimension. In the dilatation transformation, the field variables $\phi^{(\kappa)}$ are transformed according to their dimension whereas the dimensional constants are held fixed.[1] Hence, the appearance of the dimensional constants results in the violation of invariance under the dilatation transformation in the dimensionally consistent theory. Let us suppose, however, that the dimensional constants can be eliminated by a field redefinition. In terms of new field variables thus introduced, we now have a dilatation-invariant theory. Which of the two dilatation transformations, in terms of the old or new variables, is the legitimate one? Or, can we define the dilatation transformation irrespective of the choice of field variables? We shall discuss this point in the following two sections. The argument presented is further generalized in Sec. IV to clarify the relation between the dilatation transformation and the dimensional analysis. For this purpose, we introduce the notion of the *dimensional transformation,* associated with *dimensional invariance*. Dimensional invariance can never be violated in the sense that any dimensionally consistent theory must be invariant under the dimensional transformation.

It may be admitted that the above generalization looks trivial at a glance. Indeed, the dimensional transformation does not provide us with anything new except to show that our theory is dimensionally consistent. We point out, however, that the dimensional transformation plays a vital role in the discussion of the spontaneous breakdown of the dilatation transformation. This will be the subject of a subsequent paper, in which we shall show that the dilatation transformation of Heisenberg operators turns into the dimensional transformation at the level of physical (or asymptotic) fields.

Our argument makes use of the following five relations.[2]

Relation (1). For a spatially closed function F, it holds that

$$\int d\sigma_\mu\, \partial_\nu F(x) = \int d\sigma_\nu\, \partial_\mu F(x)\,. \tag{1.1}$$

The proof of this formula can be found in Ref. 2.

Relation (2). For a spatially closed vector χ_μ,

$$\tfrac{1}{3}\int d\sigma_\mu\, x_\nu(\delta_{\mu\nu}\partial_\lambda\chi_\lambda - \partial_\nu\chi_\mu) = \int d\sigma_\mu\, \chi_\mu\,. \tag{1.2}$$

Proof:

$$\begin{aligned}\tfrac{1}{3}\int & d\sigma_\mu\, x_\nu(\delta_{\mu\nu}\partial_\lambda\chi_\lambda - \partial_\nu\chi_\mu)\\ &= \tfrac{1}{3}\int d\sigma_\mu x_\mu \partial_\lambda\chi_\lambda - \tfrac{1}{3}\int d\sigma_\mu\partial_\nu(x_\nu\chi_\mu) + \tfrac{4}{3}\int d\sigma_\mu\chi_\mu\\ &= \tfrac{1}{3}\int d\sigma_\mu x_\mu \partial_\lambda\chi_\lambda - \tfrac{1}{3}\int d\sigma_\nu\partial_\mu(x_\nu\chi_\mu) + \tfrac{4}{3}\int d\sigma_\mu\chi_\mu\\ &= \int d\sigma_\mu\chi_\mu\,,\end{aligned}$$

where (1.1) has been used.

Relation (3).

$$\tfrac{1}{3}\int d\sigma_\mu x_\nu\left(\delta_{\mu\nu} - \partial_\mu\partial_\nu\frac{1}{\Box}\right)\chi = \int d\sigma_\mu\partial_\mu\frac{1}{\Box}\chi\,. \tag{1.3}$$

The proof is similar to that of (1.2).

Relation (4).

$$\tfrac{1}{3}\int d\sigma_\mu\,(\delta_{\mu\nu}\partial_\lambda\chi_\lambda - \partial_\nu\chi_\mu) = 0\,. \tag{1.4}$$

Relation (5).

$$\tfrac{1}{3}\int d\sigma_\mu\left(\delta_{\mu\nu} - \partial_\mu\partial_\nu\frac{1}{\Box}\right)\chi = 0\,. \tag{1.5}$$

Relations (4) and (5) follow directly from (1.1).

II. INFINITESIMAL DILATATION TRANSFORMATION

Let us consider a system consisting of several fields $\phi^{(\kappa)}(x)$ ($\kappa = 1, 2, \dots$). Let all the masses and coupling constants involved in this system be $m^{(\kappa)}$ and f_i ($i = 1, 2, \dots$), respectively. We shall write the Lagrangian of this system as

$$\mathcal{L} = \mathcal{L}(\phi^{(\kappa)}(x), \partial_\mu\phi^{(\kappa)}(x), m^{(\kappa)}, f_i)\,. \tag{2.1}$$

The canonical energy-momentum tensor is defined by

$$T_{\mu\nu}(x) = \sum_\kappa \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^{(\kappa)}(x)}\,\partial_\nu\phi^{(\kappa)}(x) - \delta_{\mu\nu}\mathcal{L}\,. \tag{2.2}$$

Its trace can be written as

$$\begin{aligned}T_{\mu\mu}(x) &= \sum_\kappa \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^{(\kappa)}(x)}\,\partial_\mu\phi^{(\kappa)}(x) - 4\mathcal{L}\\ &= -\sum_\kappa l^{(\kappa)}\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^{(\kappa)}(x)}\,\phi^{(\kappa)}(x)\right)\\ &\quad -\sum_\kappa m^{(\kappa)}\left(\frac{\partial\mathcal{L}}{\partial m^{(\kappa)}}\right)_{\text{explicit}} + \sum_i \eta_i f_i\left(\frac{\partial\mathcal{L}}{\partial f_i}\right)_{\text{explicit}}\end{aligned} \tag{2.3}$$

by the aid of Euler's relation.[3,4] Here, the dimension of $\phi^{(\kappa)}(x)$, $m^{(\kappa)}$, and f_i is assumed to be

$$\begin{aligned}&[\phi^{(\kappa)}(x)] = L^{-l^{(\kappa)}}\,,\\ &[m^{(\kappa)}] = L^{-1}\,,\\ &[f_i] = L^{\eta_i}\,.\end{aligned} \tag{2.4}$$

For the purpose of constructing the usual dilatation transformation, it is convenient to introduce[3]

$$\begin{aligned}T'_{\mu\nu}(x) = T_{\mu\nu} &+ \tfrac{1}{3}\delta_{\mu\nu}\partial_\lambda\sum_\kappa l^{(\kappa)}\frac{\partial\mathcal{L}}{\partial\partial_\lambda\phi^{(\kappa)}(x)}\,\phi^{(\kappa)}(x)\\ &- \tfrac{1}{3}\partial_\nu\sum_\kappa l^{(\kappa)}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^{(\kappa)}(x)}\,\phi^{(\kappa)}(x)\,,\end{aligned} \tag{2.5}$$

which satisfies

$$\partial_\mu T'_{\mu\nu}(x) = 0\,, \tag{2.6}$$

$$T'_{\mu\mu}(x) = -\sum_\kappa m^{(\kappa)}\left(\frac{\partial\mathcal{L}}{\partial m^{(\kappa)}}\right)_{\text{explicit}} + \sum_i \eta_i f_i\left(\frac{\partial\mathcal{L}}{\partial f_i}\right)_{\text{explicit}}\,. \tag{2.7}$$

It can be shown[3] that (2.5) may be interpreted as the canonical energy-momentum tensor corresponding to the new Lagrangian

$$\mathcal{L}' = \mathcal{L} - \tfrac{1}{3}\sum_\kappa l^{(\kappa)}\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^{(\kappa)}(x)}\,\phi^{(\kappa)}(x)\right).$$

Let us note here that the canonical energy-momentum tensor is invariant under the field redefinition

$$\psi^{(\kappa)}(x) = \psi^{(\kappa)}[\phi^{(\kappa)}(x), m^{(\kappa)}, f_i]\,. \tag{2.8}$$

To prove this statement, we make use of

$$\partial_\mu\psi^{(\kappa)}(x) = \sum_{\kappa'}\frac{\partial\psi^{(\kappa)}}{\partial\phi^{(\kappa')}}\,\partial_\mu\phi^{(\kappa')}(x)\,. \tag{2.9}$$

Hence

$$\frac{\partial\partial_\mu\psi^{(\kappa)}(x)}{\partial\partial_\nu\phi^{(\kappa')}(x)} = \delta_{\mu\nu}\frac{\partial\psi^{(\kappa)}}{\partial\phi^{(\kappa')}}\,. \tag{2.10}$$

We have therefore

$$\begin{aligned}T_{\mu\nu}(x) &= \sum_\kappa \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^{(\kappa)}(x)}\,\partial_\nu\phi^{(\kappa)}(x) - \delta_{\mu\nu}\mathcal{L}\\ &= \sum_{\kappa,\kappa'}\frac{\partial\mathcal{L}}{\partial\partial_\lambda\psi^{(\kappa')}(x)}\,\frac{\partial\partial_\lambda\psi^{(\kappa')}(x)}{\partial\partial_\mu\phi^{(\kappa)}(x)}\,\partial_\nu\phi^{(\kappa)}(x) - \delta_{\mu\nu}\mathcal{L}\end{aligned}$$

$$= \sum_{\kappa,\kappa'} \frac{\partial \mathcal{L}}{\partial\partial_\mu \psi^{(\kappa')}(x)} \frac{\partial \psi^{(\kappa')}(x)}{\partial \phi^{(\kappa)}(x)} \partial_\nu \phi^{(\kappa)}(x) - \delta_{\mu\nu}\mathcal{L}$$

$$= \sum_{\kappa} \frac{\partial \mathcal{L}}{\partial\partial_\mu \psi^{(\kappa)}(x)} \partial_\nu \psi^{(\kappa)}(x) - \delta_{\mu\nu}\mathcal{L} . \qquad (2.11)$$

This implies that the right-hand side of (2.3) is also invariant under the field redefinition of the type (2.8). However, the right-hand side of (2.7) is obviously not invariant.

If we define the generator of the dilatation transformation in the usual manner by[5,6]

$$D(\sigma) = \int d\sigma_\mu(x) T'_{\mu\nu}(x) x_\nu , \qquad (2.12)$$

the scale deficiency

$$\frac{\delta D(\sigma)}{\delta\sigma(x)} = \partial_\mu [T'_{\mu\nu}(x) x_\nu]$$

$$= T'_{\mu\mu}(x)$$

$$= -\sum_\kappa m^{(\kappa)} \left(\frac{\partial \mathcal{L}}{\partial m^{(\kappa)}}\right)_{\text{explicit}} + \sum_i \eta_i f_i \left(\frac{\partial \mathcal{L}}{\partial f_i}\right)_{\text{explicit}} \qquad (2.13)$$

is not invariant under the field redefinition. Then, the violation of the dilatation invariance has no absolute significance. Indeed, as will be seen in the next section, the right-hand side of (2.13) can be made zero by a field redefinition in some cases.

It is easy to show that the dilatation generator $D(\sigma)$ defined by (2.12) satisfies the commutation relation[2,7]

$$[D(\sigma), P_\mu] = -iP_\mu - i\int d\sigma_\mu \partial_\lambda S_\lambda(x) , \qquad (2.14)$$

$$[D(\sigma), M_{\mu\nu}] = i\int d\sigma_\mu x_\nu \partial_\lambda S_\lambda(x) - i\int d\sigma_\nu x_\mu \partial_\lambda S_\lambda(x) , \qquad (2.15)$$

where P_μ and $M_{\mu\nu}$ are the generators of the Poincaré group, and

$$S_\lambda(x) \equiv T'_{\lambda\nu}(x) x_\nu ; \qquad (2.16)$$

consequently

$$\partial_\lambda S_\lambda(x) = T'_{\lambda\lambda}(x) = -\sum_\kappa m^{(\kappa)} \left(\frac{\partial \mathcal{L}}{\partial m^{(\kappa)}}\right)_{\text{explicit}} + \sum_i \eta_i f_i \left(\frac{\partial \mathcal{L}}{\partial f_i}\right)_{\text{explicit}} . \qquad (2.17)$$

III. MODIFIED DILATATION TRANSFORMATION

It may happen that the trace (2.17) turns out to be a divergence of a local vector quantity $\chi_\mu(x)$, say, as a consequence of field equations, namely,

$$T'_{\mu\mu}(x) = \partial_\mu \chi_\mu(x) . \qquad (3.1)$$

In this case it is natural to define

$$T''_{\mu\nu}(x) = T'_{\mu\nu}(x) - \tfrac{1}{3}\delta_{\mu\nu}\partial_\lambda \chi_\lambda(x) + \tfrac{1}{3}\partial_\nu \chi_\mu(x) , \qquad (3.2)$$

which satisfies

$$\partial_\mu T''_{\mu\nu}(x) = 0 , \qquad (3.3)$$

$$T''_{\mu\mu}(x) = 0 . \qquad (3.4)$$

On account of (1.4), the energy-momentum vector is

$$P_\mu = -\int d\sigma_\lambda T_{\lambda\mu}(x)$$

$$= -\int d\sigma_\lambda T'_{\lambda\mu}(x)$$

$$= -\int d\sigma_\lambda T''_{\lambda\mu}(x) . \qquad (3.5)$$

However, the dilatation generator becomes

$$D = \int d\sigma_\mu T''_{\mu\nu}(x) x_\nu$$

$$= \int d\sigma_\mu [T'_{\mu\nu}(x) x_\nu - \chi_\mu(x)] \qquad (3.6)$$

by the aid of (1.2). This generator is evidently conserved:

$$\frac{\delta D}{\delta \sigma(x)} = \partial_\mu [T'_{\mu\nu}(x) x_\nu] - \partial_\mu \chi_\mu(x)$$

$$= T'_{\mu\mu}(x) - \partial_\mu \chi_\mu(x) = 0 \qquad (3.7)$$

because of (3.1). The quantity (3.6) is certainly more convenient than (2.12), since the commutator with P_μ and $M_{\mu\nu}$ is simply

$$[D, P_\mu] = -iP_\mu , \qquad (3.8)$$

$$[D, M_{\mu\nu}] = 0 . \qquad (3.9)$$

The proof of (3.8) and (3.9) involves only the Heisenberg equation of motion,

$$-i\partial_\mu \chi_\lambda(x) = [\chi_\lambda(x), P_\mu] , \qquad (3.10)$$

which is valid for any local operator, and the formula (1.1).

Let us now investigate the condition under which Eq. (3.1) holds. We assume that all the masses and the dimensional coupling constants are eliminated by the field redefinition of the type (2.8), or only the dimensionless combination of these constants survives after the field redefinition. Then, in terms of new variables $\psi^{(\kappa)}(x)$, we have

$$T_{\mu\mu}(x) = -\sum_\kappa k^{(\kappa)} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\partial_\mu \psi^{(\kappa)}(x)} \psi^{(\kappa)}(x) \right) , \qquad (3.11)$$

where $k^{(\kappa)}$ is the dimension of $\psi^{(\kappa)}$. Since the left-hand side of (3.11) is invariant under the field redefinition, we obtain, from (3.11) and (2.3),

$$
\begin{aligned}
T'_{\mu\mu}(x) &= -\sum_\kappa m^{(\kappa)}\left(\frac{\partial\mathcal{L}}{\partial m^{(\kappa)}}\right)_{\text{explicit}} \\
&\quad+\sum_i \eta_i f_i\left(\frac{\partial\mathcal{L}}{\partial f_i}\right)_{\text{explicit}} \\
&=\partial_\mu\left(\sum_\kappa l^{(\kappa)}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^{(\kappa)}(x)}\phi^{(\kappa)}(x)\right. \\
&\quad\left.-\sum_\kappa k^{(\kappa)}\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^{(\kappa)}(x)}\psi^{(\kappa)}(x)\right),
\end{aligned} \tag{3.12}
$$

which is of the form (3.1). Summarizing, whenever all the dimensional constants can be eliminated by the field redefinition, the trace of $T'_{\mu\nu}$ can be written as a divergence of a local vector $\chi_\mu(x)$. (However, it is not clear whether the converse is true.) In such a case, it is appropriate to define the dilatation generator by (3.6).

Example 1

Take a Lagrangian

$$
\begin{aligned}
\mathcal{L} &= -\tfrac{1}{2}[\partial_\mu\phi(x)\partial_\mu\phi(x)+\mu^2\phi(x)\phi(x)]-\tfrac{1}{2}\partial_\mu B(x)\partial_\mu B(x) \\
&\quad-\frac{1}{2}\frac{g^2}{\mu^2}B^2(x)\phi^2(x)+gB(x)\phi^2(x).
\end{aligned} \tag{3.13}
$$

The field equation for $B(x)$ is

$$
\Box B(x)=\frac{g^2}{\mu^2}B(x)\phi^2(x)-g\phi^2(x). \tag{3.14}
$$

The scale deficiency is, therefore,

$$
\begin{aligned}
\partial_\mu S_\mu(x) &= -\frac{\mu^2}{g}\left(\frac{g^2}{\mu^2}B(x)\phi^2(x)-g\phi^2(x)\right) \\
&= -\frac{\mu^2}{g}\Box B(x).
\end{aligned} \tag{3.15}
$$

Hence, we take

$$
\chi_\mu(x)=-\frac{\mu^2}{g}\partial_\mu B(x) \tag{3.16}
$$

and

$$
T''_{\mu\nu}(x)=T_{\mu\nu}(x)+\tfrac{1}{3}\delta_{\mu\nu}\frac{\mu^2}{g}\Box B(x)-\frac{1}{3}\frac{\mu^2}{g}\partial_\nu\partial_\mu B(x). \tag{3.17}
$$

Obviously,

$$
\begin{aligned}
T''_{\mu\mu}(x) &= T_{\mu\mu}(x)+\frac{\mu^2}{g}\Box B(x) \\
&=\mu^2\phi^2(x)-gB(x)\phi^2(x)+\frac{\mu^2}{g}\Box B(x) \\
&=0,
\end{aligned} \tag{3.18}
$$

because of (3.14).

We note that the Lagrangian (3.13) is changed by $-4\epsilon\mathcal{L}$ under

$$
\begin{aligned}
\phi(x) &\to \phi'(x')=(1-\epsilon)\phi(x), \\
B(x) &\to B'(x')=(1-\epsilon)B(x)-\frac{\mu^2}{g}\epsilon,
\end{aligned} \tag{3.19}
$$

and also that, after the field redefinition

$$
B(x)\to B(x)-\mu^2/g, \tag{3.20}
$$

the Lagrangian becomes

$$
\begin{aligned}
\mathcal{L} &= -\tfrac{1}{2}\partial_\mu\phi(x)\partial_\mu\phi(x)-\tfrac{1}{2}\partial_\mu B(x)\partial_\mu B(x) \\
&\quad-\frac{1}{2}\frac{g^2}{\mu^2}B^2(x)\phi^2(x).
\end{aligned} \tag{3.21}
$$

Here the combination g^2/μ^2 is dimensionless.

Example 2

We next consider the system in which a massless scalar field interacts with several matter fields. The Lagrangian is

$$
\begin{aligned}
\mathcal{L} &= -\bar{\psi}(\gamma\cdot\partial+m)\psi-\bar{\Phi}(\gamma\cdot\partial+\mu)\Phi-\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi \\
&\quad+g(\mu\bar{\Phi}\Phi+m\bar{\psi}\psi)\phi.
\end{aligned} \tag{3.22}
$$

The equation for ϕ is

$$
\Box\phi(x)=-g[\mu\bar{\Phi}(x)\Phi(x)+m\bar{\psi}(x)\psi(x)]. \tag{3.23}
$$

The scale deficiency is, therefore,

$$
\begin{aligned}
\partial_\mu S_\mu(x) &= m\bar{\psi}(x)\psi(x)+\mu\bar{\Phi}(x)\Phi(x) \\
&= -\frac{1}{g}\Box\phi(x).
\end{aligned} \tag{3.24}
$$

The modified current is

$$
\hat{S}_\mu(x)=T'_{\mu\nu}(x)x_\nu+\frac{1}{g}\partial_\mu\phi(x). \tag{3.25}
$$

It is not difficult to see that the transformation

$$
\begin{aligned}
x_\mu &\to x'_\mu=(1+\epsilon)x_\mu, \\
\psi(x) &\to \psi'(x')=(1-\tfrac{3}{2}\epsilon)\psi(x), \\
\Phi(x) &\to \Phi'(x')=(1-\tfrac{3}{2}\epsilon)\Phi(x), \\
\phi(x) &\to \phi'(x')=(1-\epsilon)\phi(x)+\frac{1}{g}\epsilon
\end{aligned} \tag{3.26}
$$

is generated by

$$
D=\int d\sigma_\lambda(x)\hat{S}_\lambda(x) \tag{3.27}
$$

and also that the field redefinition

$$
\phi(x)\to\phi(x)-1/g \tag{3.28}
$$

eliminates μ and m.

As is seen in the Lagrangian (3.22), the massless field $\phi(x)$ interacts universally with the mass density. It is interesting to speculate that the apparent scale noninvariance of massive fields may be due to the negligibility of the gravitational interaction.

Returning to the definition of the dilatation transformation, there are a number of generators lying in between $D(\sigma)$ in (2.12) and D in (3.6). Namely, by the use of the field equations, the scale deficiency (2.13) may be transformed partly in the divergence form, which can always be amalgamated into the generator. Since we find no advantage in doing so, we shall not elaborate on this point any further.

IV. INFINITESIMAL DIMENSIONAL TRANSFORMATION

In the modification of the dilatation current discussed in the preceding section, one of the basic requirements was the locality. Namely, the vector current $\chi_\mu(x)$ must be local. Otherwise the Heisenberg equation (3.10) does not necessarily hold and consequently the commutation relations between D, P_μ, and $M_{\mu\nu}$ would possibly change. Nevertheless, let us relax the restriction and allow the nonlocality.

Noting the identity

$$T'_{\mu\mu}(x)=\partial_\lambda \chi_\lambda(x)\,, \tag{4.1}$$

with

$$\chi_\lambda(x)=\partial_\lambda \int d^4x' D^{(\mathrm{ret})}(x-x')T'_{\mu\mu}(x')\,, \tag{4.2}$$

we proceed as before to define

$$\hat{T}_{\mu\nu}(x)=T'_{\mu\nu}(x)-\tfrac{1}{3}\delta_{\mu\nu}\partial_\lambda\chi_\lambda(x)+\tfrac{1}{3}\partial_\nu\chi_\mu(x)\,. \tag{4.3}$$

Evidently,

$$\partial_\mu \hat{T}_{\mu\nu}(x)=0\,, \tag{4.4}$$

$$\hat{T}_{\mu\mu}(x)=T'_{\mu\mu}(x)-\partial_\lambda\chi_\lambda(x)=0\,. \tag{4.5}$$

It is convenient to introduce the symbol

$$\frac{1}{\Box}T'_{\mu\mu}(x)\equiv\int d^4x' D^{(\mathrm{ret})}(x-x')T'_{\mu\mu}(x')\,, \tag{4.6}$$

which enables us to write (4.3) as

$$\hat{T}_{\mu\nu}(x)=T'_{\mu\nu}(x)-\tfrac{1}{3}\left(\delta_{\mu\nu}-\partial_\mu\partial_\nu\frac{1}{\Box}\right)T'_{\rho\rho}(x)\,. \tag{4.7}$$

If we now define the dilatation current by

$$\hat{S}_\mu(x)=\hat{T}_{\mu\nu}(x)x_\nu\,, \tag{4.8}$$

it is easy to see that it is conserved:

$$\partial_\mu \hat{S}_\mu(x)=\hat{T}_{\mu\mu}(x)=0 \tag{4.9}$$

because of (4.5). The generator

$$\hat{D}=\int d\sigma_\mu(x)\hat{S}_\mu(x) \tag{4.10}$$

can be cast into the form

$$\begin{aligned}\hat{D}&=\int d\sigma_\mu(x)[T'_{\mu\nu}(x)x_\nu-\chi_\mu(x)]\\&=\int d\sigma_\mu(x)\left(T'_{\mu\nu}(x)x_\nu-\partial_\mu\frac{1}{\Box}T'_{\rho\rho}(x)\right)\end{aligned} \tag{4.11}$$

by the aid of (1.3). For the relation (1.3) to be valid, the integrand must tend to zero sufficiently rapidly. To ensure this, we consider $T'_{\mu\nu}(x)$ multiplied by a spatially closed function to be put equal to unity at the end of the calculation. The integration (4.2) must also be defined, strictly speaking, by

$$\chi_\lambda(x)=\lim_{\epsilon\to+0}\partial_\lambda\int d^4x' D^{(\mathrm{ret})}(x-x')T'_{\rho\rho}(x')e^{-\epsilon|t'|}\,. \tag{4.12}$$

Let us now see how the commutator of $\hat{D}$ and P_μ is modified:

$$\begin{aligned}[\hat{D},P_\mu]&=-i\int d\sigma_\rho\partial_\mu T'_{\rho\nu}(x)x_\nu\\&\quad+i\int d\sigma_\rho\partial_\rho\int d^4x' D^{(\mathrm{ret})}(x-x')\partial'_\mu T'_{\lambda\lambda}(x')e^{-\epsilon|t'|}\\&=i\int d\sigma_\lambda T'_{\lambda\mu}(x)\\&\quad-i\int d\sigma_\lambda\partial_\lambda\int d^4x' D^{(\mathrm{ret})}(x-x')T'_{\rho\rho}(x')\partial'_\mu e^{-\epsilon|t'|}\,.\end{aligned} \tag{4.13}$$

However, the nondiagonal terms in the second term vanish, whereas the diagonal terms survive at the limit $\epsilon\to 0$. We then have

$$[\hat{D},P_\mu]=-iP_\mu-i\left(\int d\sigma_\mu T'_{\lambda\lambda}(x)\right)^{\text{diagonal}}\,. \tag{4.14}$$

If we use the relation

$$\left(\int d\sigma_\mu T'_{\lambda\lambda}(x)\right)^{\text{diagonal}}=\left(-\sum_\kappa m^{(\kappa)}\frac{\partial}{\partial m^{(\kappa)}}+\sum_i \eta_i f_i\frac{\partial}{\partial f_i}\right)P_\mu\,, \tag{4.15}$$

which was established in Ref. 4, we arrive at

$$[\hat{D},P_\mu]=-i\left(1-\sum_\kappa m^{(\kappa)}\frac{\partial}{\partial m^{(\kappa)}}+\sum_i \eta_i f_i\frac{\partial}{\partial f_i}\right)P_\mu\,. \tag{4.16}$$

A similar calculation gives

$$[\hat{D},M_{\mu\nu}]=-i\left(-\sum_\kappa m^{(\kappa)}\frac{\partial}{\partial m^{(\kappa)}}+\sum_i \eta_i f_i\frac{\partial}{\partial f_i}\right)M_{\mu\nu}\,. \tag{4.17}$$

These derivatives will act on explicit as well as implicit variables contained in what follows except on the creation and annihilation operators by which our Fock space is constructed. Hence, when we calculate matrix elements, the state vectors can be freely inserted under these derivatives. This is the reason why the right-hand side of (4.16) or (4.17) does not vanish identically because of the dimensional reason.

From (4.16), we have

$$[\hat{D}, P^2] = -i\left(2 - \sum_{\kappa} m^{(\kappa)} \frac{\partial}{\partial m^{(\kappa)}} + \sum_i \eta_i f_i \frac{\partial}{\partial f_i}\right) P^2 . \tag{4.18}$$

For the state $|M\rangle$ such that

$$P^2|M\rangle = -M^2|M\rangle , \tag{4.19}$$

we obtain

$$P^2 e^{i\hat{D}\epsilon}|M\rangle = -M^2 e^{i\hat{D}\epsilon}|M\rangle \tag{4.20}$$

because of the identity

$$\left(2 - \sum_{\kappa} m^{(\kappa)} \frac{\partial}{\partial m^{(\kappa)}} + \sum_i \eta_i f_i \frac{\partial}{\partial f_i}\right) M^2 = 0 . \tag{4.21}$$

In other words, the state

$$|M'\rangle \equiv e^{i\hat{D}\epsilon}|M\rangle \tag{4.22}$$

is also the eigenstate of P^2 with the same eigenvalue $-M^2$. It is this property which guarantees that our transformation does not contradict the existence of the discrete mass spectrum.

As can be seen above, any theory is invariant under the transformation generated by the $\hat{D}$, as long as it is dimensionally correct. This is due to the fact that $\hat{D}$ induces scale changes of all the dimensional quantities. We shall call this the *dimensional transformation*, and the $\hat{D}$ its generator.

The dimensional invariance does not restrict the physics beyond the dimensional consistency which states, for example, that the S matrix is a function of dimensionless ratios made up of $m^{(\kappa)}$ and f_i, etc.

It may sound that the dimensional invariance is a trivial concept. We shall see in a separate paper, however, that this very concept plays an important role in the spontaneous breakdown of the dilatation invariance.

V. MASSIVE SCALAR FIELD

As an example of the dimensional transformation, we shall discuss the massive scalar field in detail. From the Lagrangian

$$\mathfrak{L} = -\tfrac{1}{2}[\partial_\lambda \phi(x)\partial_\lambda \phi(x) + m^2\phi(x)\phi(x)] \tag{5.1}$$

we can derive

$$\begin{aligned} T'_{\mu\nu}(x) &= -\partial_\mu\phi(x)\partial_\nu\phi(x) + \tfrac{1}{2}\delta_{\mu\nu}[\partial_\lambda\phi(x)\partial_\lambda\phi(x) + m^2\phi(x)\phi(x)] - \tfrac{1}{3}\delta_{\mu\nu}\partial_\lambda[\partial_\lambda\phi(x)\phi(x)] + \tfrac{1}{3}\partial_\nu[\partial_\mu\phi(x)\phi(x)] \\ &= -\partial_\mu\phi(x)\partial_\nu\phi(x) + \tfrac{1}{2}\delta_{\mu\nu}[\partial_\lambda\phi(x)\partial_\lambda\phi(x) + m^2\phi(x)\phi(x)] - \tfrac{1}{6}(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\phi^2(x) , \end{aligned} \tag{5.2}$$

which agrees with the Callan-Coleman-Jackiw tensor.[6] A short calculation gives

$$T'_{\mu\mu}(x) = m^2\phi^2(x) . \tag{5.3}$$

If we use the expression

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{(2\omega_k)^{1/2}} [a(\vec{k})e^{ikx} + a^\dagger(\vec{k})e^{-ikx}] , \tag{5.4}$$

with

$$k_0 = \omega_k = (\vec{k}^2 + m^2)^{1/2} , \tag{5.5}$$

the generator of the dilatation transformation becomes, after a straightforward but tedious calculation,

$$\begin{aligned} D(t) &= \frac{1}{i}\int d\vec{x}\, T'_{4\nu}(x)x_\nu \\ &= \frac{i}{2}\int d\vec{k}\,\vec{k}\cdot\{[\vec{\nabla}_k a^\dagger(\vec{k})]a(\vec{k}) - a^\dagger(\vec{k})\vec{\nabla}_k a(\vec{k})\} + \frac{i}{4}\int d\vec{k}\frac{m^2}{\omega_k^2}[a(\vec{k})a(-\vec{k})e^{-2i\omega_k t} - a^\dagger(\vec{k})a^\dagger(-\vec{k})e^{2i\omega_k t}] \\ &\quad + t\int d\vec{k}\frac{m^2}{\omega_k}a^\dagger(\vec{k})a(\vec{k}) . \end{aligned} \tag{5.6}$$

From this result, we can compute that

$$i[a(\vec{k}), D(t)] = \tfrac{3}{2}a(\vec{k}) + \vec{k}\cdot\vec{\nabla}_k a(\vec{k}) + \frac{1}{2}\frac{m^2}{\omega_k^2}a^\dagger(-\vec{k})e^{2i\omega_k t} + it\frac{m^2}{\omega_k}a(\vec{k}) \tag{5.7}$$

and

$$i[a^\dagger(\vec{k}), D(t)] = \tfrac{3}{2}a^\dagger(\vec{k}) + \vec{k}\cdot\vec{\nabla}_k a^\dagger(\vec{k}) + \frac{1}{2}\frac{m^2}{\omega_k^2}a(-\vec{k})e^{-2i\omega_k t} - it\frac{m^2}{\omega_k}a^\dagger(\vec{k}). \tag{5.8}$$

The configuration-space version is

$$[\phi(x), D(t)] = i(\vec{x}\cdot\vec{\nabla}+1)\phi(x) + it\pi(x), \tag{5.9}$$

$$[\pi(x), D(t)] = i(\vec{x}\cdot\vec{\nabla}+2)\pi(x) + it(\nabla^2 - m^2)\phi(x) \tag{5.10}$$

at equal time.

To see the generator of the dimensional transformation, we calculate

$$\begin{aligned}
&\int d^3x\,\partial_t \int d^4x'\,D^{(\mathrm{ret})}(x-x')T'_{\lambda\lambda}(x')e^{-\epsilon|t'|}\\
&\quad= \int d^3x\,\partial_t\int d^4x'\,D^{(\mathrm{ret})}(x-x')m^2\phi^2(x')e^{-\epsilon|t'|}\\
&\quad= -i\frac{m^2}{4}\int d\vec{k}\,\frac{1}{\omega_k^2}[a(\vec{k})a(-\vec{k})e^{-2i\omega_k t} - a^\dagger(\vec{k})a^\dagger(-\vec{k})e^{2i\omega_k t}]\\
&\qquad - i\frac{m^2}{2}\int d\vec{k}\left[a(\vec{k})a^\dagger(\vec{k})\int dk_0\,\frac{\delta(k_0-\omega_k)}{k_0-\omega_k}e^{-i(k_0-\omega_k)t} + a^\dagger(-\vec{k})a(-\vec{k})\int dk_0\,\frac{\delta(k_0+\omega_k)}{k_0+\omega_k}e^{-i(k_0+\omega_k)t}\right]\\
&\quad= -\frac{i}{4}\int d\vec{k}\,\frac{m^2}{\omega_k^2}[a(\vec{k})a(-\vec{k})e^{-2i\omega_k t} - a^\dagger(\vec{k})a^\dagger(-\vec{k})e^{2i\omega_k t}] - t\int d\vec{k}\,\frac{m^2}{\omega_k}a^\dagger(\vec{k})a(\vec{k}),
\end{aligned} \tag{5.11}$$

where the c-number term has been dropped. Substituting (5.6) and (5.11) into (4.11), we obtain

$$\hat{D} = \frac{i}{2}\int d\vec{k}\,\vec{k}\cdot\{[\vec{\nabla}_k a^\dagger(\vec{k})]\,a(\vec{k}) - a^\dagger(\vec{k})\vec{\nabla}_k a(\vec{k})\}, \tag{5.12}$$

whence

$$i[a(\vec{k}), \hat{D}] = \tfrac{3}{2}a(\vec{k}) + \vec{k}\cdot\vec{\nabla}_k a(\vec{k}), \tag{5.13}$$

$$i[a^\dagger(\vec{k}), \hat{D}] = \tfrac{3}{2}a^\dagger(\vec{k}) + \vec{k}\cdot\vec{\nabla}_k a^\dagger(\vec{k}). \tag{5.14}$$

Transforming into configuration space, we obtain

$$i[\phi(x), \hat{D}] = -(x_\mu\partial_\mu + 1)\phi(x) + m\frac{\partial}{\partial m}\phi(x). \tag{5.15}$$

As was mentioned before, the derivative with respect to the mass acts only on the wave functions but not on the creation and annihilation operators.

We can also verify the relation (4.16) directly by calculating

$$\begin{aligned}
[\hat{D}, H] &= -iH + i\int d\vec{k}\,\frac{m^2}{\omega_k}a^\dagger(\vec{k})a(\vec{k})\\
&= -i\left(1 - m\frac{\partial}{\partial m}\right)H
\end{aligned} \tag{5.16}$$

with

$$H = \int d^3k\,\omega_k a^\dagger(\vec{k})a(\vec{k}), \tag{5.17}$$

where again the derivative acts only on ω_k in (5.17).

Finally, we make a few remarks. The expression (5.12) can be generalized to a general free field with spin s as

$$\hat{D} = \frac{i}{2}\sum_{r=1}^{2s+1}\int d\vec{k}\,\vec{k}\cdot\{[\vec{\nabla}_k a^{(r)\dagger}(\vec{k})]\,a^{(r)}(\vec{k}) - a^{(r)\dagger}(\vec{k})\vec{\nabla}_k a^{(r)}(\vec{k}) + [\vec{\nabla}_k b^{(r)\dagger}(\vec{k})]\,b^{(r)}(\vec{k}) - b^{(r)\dagger}(\vec{k})\vec{\nabla}_k b^{(r)}(\vec{k})\}, \tag{5.18}$$

where $b^{(r)}(\vec{k})$ and $b^{(r)\dagger}(\vec{k})$ are annihilation and creation operators, respectively, of antiparticles. Consequently, the relation (5.15) holds true for free fields with any spin, i.e.,

$$i[\phi^{(\kappa)}(x),\hat{D}]=-\left(x_\mu\partial_\mu+l^{(\kappa)}-m\frac{\partial}{\partial m}\right)\phi^{(\kappa)}(x)\,. \tag{5.19}$$

It should also be pointed out that the relation (5.19) is true even for any c-number function of x_μ and m (in this case, both sides of the equation are identically zero). Hence, for any functional $F(x_\mu, m, \phi^{(\kappa)}(x))$,

$$i[F(x_\mu, m, \phi^{(\kappa)}(x)),\hat{D}]=-\left(x_\mu\partial_\mu+l-m\frac{\partial}{\partial m}\right)F(x_\mu, m, \phi^{(\kappa)}(x)), \tag{5.20}$$

where

$$[F]=L^{-l}\,.$$

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Zero-Mass Divergences in Expansions about Chiral and Scale Symmetry

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Singularities of expansions in the symmetry-breaking parameters of chiral- and scale-asymmetric theories are studied, in the tree approximation, in polynomial Lagrangian models. The singularities are related to the appearance of zero-mass scalar fields, not necessarily Goldstone particles, at the radius of convergence of the expansion. Methods of avoiding these singularities are presented.

I. INTRODUCTION

Analyticity of Nambu-Goldstone symmetry realizations[1] and the possibility of expansions[2] in symmetry-breaking parameters have recently been studied in the context of pion loop diagrams[3] and the solution, in the tree approximation, of specific polynomial Lagrangian models[4-7] built on $(3,3^*)\oplus(3^*,3)$ fields.[8] In the latter approach, with which we are concerned here, linear but spontaneously broken realizations of chiral and scale symmetry result in partial-conservation conditions with pseudoscalar masses determined quite simply by the symmetry nature of the ground state and the form