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How to Express Production Amplitudes in Exceptional Regions

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A study is made for representations of a production amplitude in exceptional regions inside the physical region. We find that a complete set of independent invariant variables is not sufficient to describe production processes involving more than five external particles, in the exceptional regions. Our result not only influences kinematic singularities as well as the meaning of analyticity of production amplitudes, but also provides a guideline for proposing models and for fitting data. An approach, related to some of these problems, is suggested so that particle distributions for inclusive reactions can be studied with the multi-Regge expansions for all coupling schemes (configuration diagrams).

Production amplitudes for a multiparticle process are in general functions of the four-momenta of the external particles, incoming and outgoing. However, one usually considers scalar amplitudes or helicity amplitudes in the proper Lorentz frames.[1] These scalar amplitudes and helicity amplitudes are functions of the invariant variables only. It is well known that in a process involving four or five external particles, any invariant variable can be written, through the conservation of four-momentum, as a linear combination of a complete set[2] of independent invariant variables. Therefore, the choice of working invariant variables is immaterial. One usually chooses one of the complete sets to express the amplitudes.

Because of dimensionality constraints,[3,4] this situation no longer exists for a process involving six external particles or more. In this note, we will show that production amplitudes for a N-particle process should be functions of at least $\frac{1}{2}N(N-3)$ *linearly* independent invariant variables instead of $3N-10$ independent invariant variables if $N\geq 6$. This not only influences kinematic singularities as well as the meaning[5] of analyticity of production amplitudes, but also provides a guideline for proposing those models with invariant variables as variables. We will, however, show that a complete set of $3N-10$ independent group parameters, such as those in multi-Regge expansions[6] (MRE) and in the multiparticle partial-wave expansion[7] (MPE), can describe the amplitudes faithfully. A complete set of the group parameters can be expressed in general in terms of a complete set of $3N-10$ independent invariant variables except in the exceptional regions, where the transformations between these two sets are undetermined. We will also discuss relationship between the group parameters and the invariant variables.

Recently, Mueller[8] has suggested the generalized optical theorem, which relates the invariant cross sections for inclusive reactions to the absorptive part of the appropriate forward multiparticle scattering amplitudes. The forward scattering region is one of the exceptional regions. Therefore, the invariant cross sections miss the dependences on certain independent variables (as will be shown later) if one chooses some complete sets of independent invariant variables as variables. In addition, the relations between the complete sets of the invariant variables and those of the group parameters in MRE and MPE cannot be continued to the forward scattering region for some coupling schemes (configuration diagrams). We will suggest one method to avoid this difficulty. Based on this method, the studies on inclusive reactions have been done elsewhere, with MRE for all coupling schemes.

We illustrate those problems with a process involving six external particles shown in Fig. 1(a), since this process is the simplest in which dimensionality constraints enter. Let us introduce our notation and conventions. The invariant variables and the determinants are defined as follows:

$$s_{ij}\equiv(p_i+p_j)^2, \quad s_{ijh}\equiv(p_i+p_j+p_h)^2, \tag{1a}$$

$$G(x,y,\ldots,w)\equiv G\begin{pmatrix} x\ y\ \cdots\ w \\ x\ y\ \cdots\ w\end{pmatrix}, \quad k_{xy}\equiv 2p_x\cdot p_y, \tag{1b}$$

$$G\begin{pmatrix} x & y & \cdots & w \\ x' & y' & \cdots & w'\end{pmatrix}\equiv \det(k_{mm'})$$

$$\text{for}\quad m=x,y,\ldots,w \quad\text{and}\quad m'=x',y',\ldots,w', \tag{1c}$$

where p_x, p_y, etc. are linear combinations of the momenta p_i of the external particles. For simplicity, we assume that all the external particles are spinless and have the same mass μ. One thus has only one amplitude[9] for each related reaction. In general, there are eight independent invariant variables ($3N-10=8$ for $N=6$). One of the complete sets of independent invariant variables is

$$S \equiv (s_{12}, s_{123}, s_{56}; s_{23}, s_{34}, s_{45}; s_{24}, s_{35}) . \tag{2}$$

We will show later that set S is closely related to the group parameters in the MRE and in the MPE with a coupling scheme shown in Fig. 1(b).

All the invariant variables except s_{25}, s_{15}, s_{16}, and s_{26} are linear combinations of the invariant variables in set S through the conservation of four-momentum ($p_1+p_2+\cdots+p_6=0$). The variable s_{25} which is equivalent to k_{25} is related to set S through the dimensionality constraint,

$$G(1,2,3,4,5)=0 . \tag{3a}$$

From (3a) and the identity[10]

$$G(3,4,(12))\,G(2,5,3,4,(12))=G(2,3,4,(12))\,G(5,3,4,(12))-\left[G\begin{pmatrix}2 & 3 & 4 & (12)\\ 5 & 3 & 4 & (12)\end{pmatrix}\right]^2 \tag{3b}$$

one can easily show

$$\left[k_{25}G(3,4,(12))-k_{23}G\begin{pmatrix}3 & 4 & (12)\\ 5 & 4 & (12)\end{pmatrix}+k_{24}G\begin{pmatrix}3 & 4 & (12)\\ 5 & 3 & (12)\end{pmatrix}-k_{2(12)}G\begin{pmatrix}3 & 4 & (12)\\ 5 & 3 & 4\end{pmatrix}\right]^2=G(2,3,4,(12))\,G(5,3,4,(12)) . \tag{3c}$$

By direct calculation, all the determinants in (3c) can be expressed in terms of set S. From (3c), the variable s_{25} can be expressed[11] in terms of set S, except in the exceptional regions where $G(3,4,(12))=0$; i.e., where the four-momenta p_3, p_4, and p_1+p_2 are linearly related. Similar arguments can be applied to the variables s_{15}, s_{16}, and s_{26}. We shall illustrate the effect of the breakdown of the relation between s_{25} and set S.

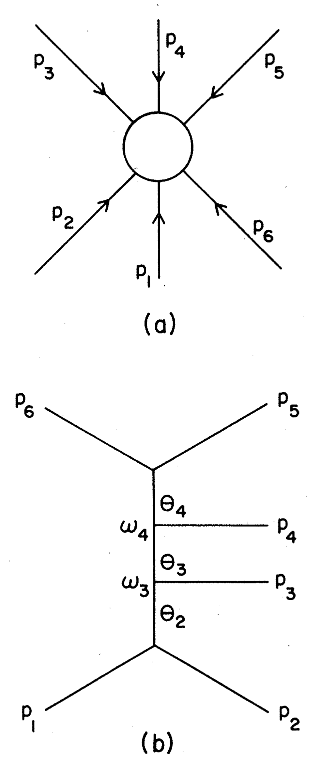

FIG. 1. (a) A system of six external particles. (b) One of the coupling schemes.

One[12] of the simple linear relations among the four-momenta p_i, which lead to $G(3,4,(12))=0$, is $p_3=-p_4$ or $p_3=p_4$. For example, the first case ($p_3=-p_4$) occurs in the physical region of process A, $1+2+3\to\bar{4}+\bar{5}+\bar{6}$, while the second ($p_3=p_4$) in that of process B, $1+6\to\bar{2}+\bar{3}+\bar{4}+\bar{5}$. One can easily show that for the first case one has

$$s_{12}=s_{56}, \quad s_{23}=4\mu^2-s_{24}, \tag{4a}$$

$$s_{34}=0, \quad s_{45}=4\mu^2-s_{35} . \tag{4b}$$

Similarly, for the second case, one has

$$s_{23}=s_{24}, \quad s_{34}=4\mu^2 , \quad s_{45}=s_{35} , \tag{5a}$$

$$s_{12}=2s_{123}-s_{56}+2\mu^2 . \tag{5b}$$

The first relation in (4a) and the relation in (5b) come from the conservation of four-momentum. From (4a), (4b), (5a), and (5b), it follows that set S for both cases contains four independent invariant variables, s_{12}, s_{123}, s_{45}, and s_{23}, in the aforementioned exceptional regions. However, by direct counting, there are five independent invariant variables. They are s_{12}, s_{123}, s_{45}, s_{23}, and s_{25} (or the equivalent). The variable s_{25} is no longer related to set S in the exceptional regions. If one chooses, for example, the complete set $(s_{13}, s_{123}, s_{56}; s_{23}, s_{24}, s_{45}; s_{34}, s_{25})$ at the beginning, the amplitude depends on those five invariant variables in the exceptional regions. However, the latter set encounters similar problems if $p_2=\pm p_4$. In fact, we have shown (a) that no complete sets can serve to express full dependences of the amplitude on the independent variables in the *whole* physical region and (b) that different complete sets have different exceptional regions. These are due to the fact that the physical region is an eight-dimensional hypersurface in the nine-dimensional invariant-variable space (S plus s_{25}). In the general case of N external particles ($N\geq 6$), the corresponding physical region is a $(3N-10)$-dimensional hypersurface in the $\frac{1}{2}N(N-3)$-dimensional invariant-variable space.

In general, the amplitude is a function of all the invariant variables. Since these invariant variables are linear combinations of set S plus s_{25}, the amplitude can also be considered as a function of set S and s_{25}, without introducing additional singularities; i.e., the amplitude can be represented by $T(S, s_{25})$. Therefore, we call the set (S, s_{25}) the complete set of *linearly* independent invariant variables. In the general case of N external particles $(N \geq 6)$, a complete set of *linearly* independent invariant variables has $\frac{1}{2}N(N-3)$ invariant variables.

In a region other than the exceptional regions, the variable s_{25} can be replaced through (3c) by a function of set S which is undefined when $G(3, 4, (12)) = 0$. Therefore, the amplitude, when expressed in terms of set S only, is in general undefined in the exceptional regions. Conversely, if in some models the amplitude $T(S)$ is finite in the exceptional regions, either it does not describe the production process faithfully, or it does not depend on the missing variable s_{25}. The latter situation can exist only as a dynamical accident and in general is unlikely to happen.

In the forward scattering of the process $a+b+c \to a+b+c$ one may assign $p_1 = -p_2 = p_a$, $p_5 = -p_6 = p_b$, and $p_3 = -p_4 = p_c$. Set S reduces to two independent invariant variables, $(p_a+p_c)^2$ and $(p_b+p_c)^2$. It is well known that one of the two invariant variables, $(p_a+p_b)^2$ and $(p_a+p_b+p_c)^2$, is also independent. The corresponding amplitude, as a function of set S, misses the dependence on one other independent variable. The generalized optical theorem[8] should be applied to the amplitude which is a function of a complete set of *linearly* independent invariant variables instead of independent invariant variables. For example, a sum of six-point Veneziano terms for the coupling scheme shown in Fig. 1(b) is usually considered as a function of set S. Therefore, the sum alone can not describe the six-particle reactions faithfully, in general, even if the external particles are of different types so that the reaction amplitude need not be symmetrized by adding to it the terms obtained by permutation.

We have shown that at least nine invariant variables are needed to describe the amplitude. However, a set of eight independent group parameters can serve to express the amplitude faithfully ($3N-10$ group parameters for a N-particle system). The typical examples for the group parameters are those in MRE[6] and MPE.[7] We shall illustrate this with the group parameters in MPE. In the physical region of process A one has the MPE in an arbitrary frame.[7]

$$T(s_{12}, s_{123}, s_{56}; \theta_2, \theta_3, \theta_4; \omega_2, \omega_4) = \sum_{j_i \lambda_i} d^{j_2}_{0\lambda_2}(\cos\theta_2) e^{-i\lambda_2\omega_2} d^{j_3}_{\lambda_2\lambda_4}(\cos\theta_3) e^{+i\lambda_4\omega_4} d^{j_4}_{\lambda_4 0}(\cos\theta_4)\, T^{j_2\, j_3\, j_4}_{\lambda_2\, \lambda_4}(s_{12}, s_{123}, s_{56}), \tag{6}$$

where θ_i and ω_i are the scattering angles and the Treiman-Yang angles. The variables s_{12}, s_{123}, and s_{56} can be expressed in terms of the hyperbolic angles if one likes. The ranges of the group parameters for the physical region are

$$0 \leq \omega_i \leq 2\pi, \quad 0 \leq \theta_i \leq \pi, \tag{7a}$$

$$4\mu^2 \leq s_{12}, \quad s_{56} \leq (s_{123}{}^{1/2} - \mu)^2. \tag{7b}$$

This means that the eight independent variables $(s_{12}, s_{123}, s_{56}; \theta_2, \theta_3, \theta_4; \omega_2, \omega_4)$ are sufficient to describe the amplitude.

The group parameters and the invariant variables are related[7,13] by

$$\cos\theta_2 = -G\begin{pmatrix}2 & (12)\\ 3 & (12)\end{pmatrix}[G(2, (12))\, G(3, (12))]^{-1/2}, \tag{8a}$$

$$\cos\theta_3 = -G\begin{pmatrix}3 & (123)\\ 4 & (123)\end{pmatrix}[G(3, (123))\, G(4, (123))]^{-1/2}, \tag{8b}$$

$$\cos\theta_4 = -G\begin{pmatrix}4 & (56)\\ 5 & (56)\end{pmatrix}[G(4, (56))\, G(5, (56))]^{-1/2}, \tag{8c}$$

$$\cos\omega_2 = -G\begin{pmatrix}2 & 3 & (12)\\ 4 & 3 & (12)\end{pmatrix}[G(2, 3, (12))\, G(4, 3, (12))]^{-1/2}, \tag{8d}$$

$$\cos\omega_4 = -G\begin{pmatrix}3 & 4 & (56)\\ 5 & 4 & (56)\end{pmatrix}[G(3, 4, (56))\, G(5, 4, (12))]^{-1/2}. \tag{8e}$$

From (8a)–(8e), it is obvious that the group parameters are functions of the invariant variables in set S, and thus so is the amplitude. One can easily see that both the numerators and the denominators in (8d) and (8e) vanish when $p_3 = \pm p_4$ and thus the relations between set S and the group parameters ω_2 and ω_4 are undetermined. Once again, we have shown that the amplitude, as a function of set S, is undefined in the exceptional regions.

The amplitude should be a function of set S and at least one of the invariant variables s_{25}, s_{26}, s_{15}, and s_{16} in the exceptional regions. We shall show how to obtain such a function from the MPE. When $p_3 = -p_4$, one has $\theta_3 = \pi$. It follows from (6) that

$$T = \sum_{j_i \lambda} d^{j_2}_{0\lambda}(\cos\theta_2) e^{-i\lambda\omega} d^{j_4}_{-\lambda 0}(\cos\theta_4)(-1)^{j_3-\lambda} T^{j_2\, j_3\, j_4}_{\lambda\, -\lambda}, \tag{9}$$

where $\omega \equiv \omega_2 + \omega_4$. This is the MPE for the $p_3 = -p_4$ case. The angle ω is the azimuthal angle of the three-momentum $\vec{p}_5$ ($\vec{p}_2$) in the rest frame of $p_1 + p_2 + p_3$, in which $\vec{p}_4$ ($\vec{p}_3$) is along the z axis and $-\vec{p}_2$ ($-\vec{p}_5$) in the xy plane, and is given[7,14] by

$$\cos\omega = +G\begin{pmatrix} 2 & 4 & (123) \\ 5 & 4 & (123) \end{pmatrix} [G(2, 4, (123))\, G(5, 4, (123))]^{-1/2} \tag{10}$$

It is obvious that $\cos\omega$ is related to variable s_{25}. Substituting (10) into (9), one has the amplitude as a function of set S plus s_{25}, in the exceptional regions.

One can also express the amplitude for process B in terms of the group parameters of the MRE for the coupling scheme shown in Fig. 1(b). These group parameters are the same as those for the MPE, but the former are in different ranges. The amplitude may be viewed as the continuation of the MPE in (6), by the Sommerfeld-Watson transform and by crossing. Similarly, the amplitude is undetermined when $p_3 = p_4$, if it is considered as a function of set S. With a similar approach which leads to (10), one can express the amplitude in terms of set S and s_{25}, when $p_3 = p_4$. This approach, which connects a complete set of group parameters to that of *linearly* independent invariant variables, is very useful for investigating particle distributions for inclusive reactions with the framework of the multi-Regge theory. A detailed study[15] of single-particle distributions has been made elsewhere.

We have discussed representations of the amplitude for a system of six equal-mass spinless particles, in the exceptional regions. It is obvious that our discussions can be carried through to the systems of more than six external spinning particles of arbitrary nonzero mass.

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[1]The proper Lorentz frame is the c.m. frame of the system of the incoming (or outgoing) particles, in which two of the external particles are along the z axis and in the xz plane, respectively. In general, all the four-momenta in the above definition can be replaced by linear combinations of the four-momenta of the external particles.

[2]By a complete set of independent invariant variables, we mean a set of $3N-10$ independent invariant variables for a N-particle system ($N \geq 4$).

[3]Since the dimensionality of Lorentz space is four, any five momenta are linearly dependent. We call this the dimensionality constraint.

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[5]K.-H. Wang, Pennsylvania State University report, 1971 (unpublished).

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[7]K.-H. Wang, Phys. Rev. D 4, 489 (1971).

[8]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).

[9]Of course, there is only one amplitude for all the related reactions if one assumes that the amplitude is continuable as in the analytic S-matrix theory.

[10]See Ref. 4, Eq. (4.3.14) or Ref. 7, Eq. (16b).

[11]It is well known that the replacement of s_{25} by set S in the amplitude introduces square-root singularities. We shall not discuss analytic structure of the amplitude here.

[12]One does not have the relations $p_3 = \pm p_4$ if particles 3 and 4 have different masses. However, we may take the relations $p_3 \pm p_4 = \pm(p_1 + p_2)$ and make a similar argument.

[13]Since the period of the Treiman-Yang angles is 2π, the equations in (8d) and (8e) cannot determine them uniquely. In Ref. 7, some methods have been provided for determining them.

[14]Using the dimensionality constraint and then imposing the condition $p_3 = -p_4$, we have obtained the expression (10) by calculating $\cos(\omega_2 + \omega_4)$ directly (the calculation is very tedious). The use of the dimensionality constraint here asserts that the amplitude is a function of set S and s_{25} when the condition $p_3 = -p_4$ is imposed. We have not succeeded in expressing $\cos\omega_2$ and $\cos\omega_4$, respectively, in terms of set S and s_{25} such that at $p_3 = -p_4$, the MPE in (6) automatically becomes the MPE in (9) with $\cos\omega$ given by (10).

[15]K.-H. Wang, Pennsylvania State University report, 1971 (unpublished).