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PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

Drell-Hearn Sum Rule from Equal-Time Commutators

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(Received 7 July 1971)

It is shown that the Drell-Hearn sum rule can be obtained from the equal-time commutator

$$[D^+(t),\ D^-(t)]=0, \quad \text{where} \quad D^\pm(t)=\int d^3x\,\frac{x_1\pm ix_2}{\sqrt{2}}\,\rho(x)\,,$$

$\rho(x)$ being the electric charge density.

Some time back, Drell and Hearn[1] derived a sum rule for the proton anomalous magnetic moment by combining the unsubtracted dispersion relation for the spin-flip Compton scattering amplitude with the low-energy theorem for the same amplitude. Since sum rules of the same type have been derived[2] from equal-time commutation relations, it should be possible to obtain the Drell-Hearn sum rule from some equal-time commutation relation. In this note we show that the desired commutation rule is

$$[D^+(t),D^-(t)]=0, \tag{1}$$

where

$$D^\pm(t)=\int d^3x\frac{x_1\pm ix_2}{\sqrt{2}}\,\rho(x), \tag{2}$$

$\rho(x)$ being the electric-charge-density operator. To obtain the sum rule we take the matrix element of Eq. (1) between single-proton states $|P(p')\rangle$ and $\langle P(p)|$:

$$\sum_{\text{spin}}\left[\int\frac{d^3k}{(2\pi)^3}\langle P(p)|D^+(t)|P(k)\rangle\langle P(k)|D^-(t)|P(p')\rangle-(+\longleftrightarrow-)\right]+\sum_{S\neq P}\left[\langle P(p)|D^+(t)|S\rangle\langle S|D^-(t)|P(p')\rangle-(+\longleftrightarrow-)\right]=0. \tag{3}$$

Since

$$\begin{aligned}\langle P(p)|D^i(t)|P(k)\rangle&=\int d^3x\,x_i\langle P(p)|\,\rho(x)|P(k)\rangle\\&=-(2\pi)^3\left(\frac{\partial}{\partial q_i}\,\delta(\vec{q})\right)\frac{eM}{\sqrt{p_0k_0}}\,\bar{u}_p[\gamma_0F_1(q^2)+\sigma_{0\nu}\,q_\nu\,F_2(q^2)]\,u_k\,,\end{aligned} \tag{4}$$

where $q=p-k$, and the single-proton intermediate-state contribution to the commutator is

$$\begin{aligned}&\langle P(p)|[D^+(t),D^-(t)]|P(p')\rangle_{\text{SPC}}\\&=\Bigg\{\int d^3k(2\pi)^3\left(\frac{\partial}{\partial q_+}\,\delta(\vec{q})\right)\left(\frac{\partial}{\partial q'_-}\,\delta(\vec{q}\,')\right)\frac{e^2M^2}{k_0\sqrt{p_0p'_0}}\,\bar{u}_p\bigg[F_1(q^2)F_1(q'^2)\gamma_0\frac{\gamma\cdot k+iM}{2iM}\gamma_0+F_1(q^2)F_2(q'^2)\gamma_0\frac{\gamma\cdot k+iM}{2iM}\sigma_{0\nu}\,q'_\nu\\&\qquad+F_2(q^2)F_1(q'^2)\sigma_{0\mu}\,q_\mu\frac{\gamma\cdot k+iM}{2iM}\gamma_0+F_2(q^2)F_2(q'^2)\sigma_{0\mu}\,q_\mu\frac{\gamma\cdot k+iM}{2iM}\sigma_{0\nu}q'_\nu\bigg]u_{p'}-(+\longleftrightarrow-)\Bigg\}\\&=-2(2\pi)^3\delta(\vec{p}-\vec{p}\,')\left[\frac{e^2x^2}{4M^2}\left(1-\frac{M^2}{p_0{}^2}\right)+\frac{e^2xM}{2p_0{}^2}\right].\end{aligned} \tag{5}$$

Here $q' = k - p'$, and x is the anomalous magnetic moment of the proton. In the limit $p_0 \to \infty$, Eq. (5) reduces to

$$\langle P(p)|[D^+(t), D^-(t)]|P(p')\rangle_{\mathrm{SPC}} = -(2\pi)^3\delta(\vec{p}-\vec{p}')\frac{e^2x^2}{2M^2}\,. \tag{6}$$

The multiparticle intermediate-state contribution to the commutator can be worked out by using the relation $\partial D^i/\partial t = \int d^3x\, j^i(x)$, which follows from the defining equation (2) and the continuity equation after performing an integration by parts. This is done as follows:

$$\begin{aligned}\langle P(p)|[D^+(t), D^-(t)]|P(p')\rangle_{\mathrm{MPC}} &= \int\frac{d^3p_S}{(2\pi)^3}\sum_{S\neq P}[\langle P(p)|D^+(t)|S\rangle\langle S|D^-(t)|P(p')\rangle - (+\longleftrightarrow -)]\\ &= \int\frac{d^3p_S}{(2\pi)^3}\sum_{S\neq P}\left[\frac{\langle P(p)|\int d^3x j^+(x)|S\rangle\langle S|\int d^3x' j^-(x')|P(p')\rangle}{(p_0-p_{S0})(p_0'-p_{S0})} - (+\longleftrightarrow -)\right]\\ &= (2\pi)^3\delta(\vec{p}-\vec{p}')\int d^3p_S\sum_{S\neq P}\delta(\vec{p}-\vec{p}_S)\left[\frac{|\langle S|j^-(0)|p\rangle|^2 - |\langle S|j^+(0)|p\rangle|^2}{(p_0-p_{S0})^2}\right].\end{aligned} \tag{7}$$

Therefore, in the limit $p_0 \to \infty$,

$$\begin{aligned}\langle P(p)|[D^+(t), D^-(t)]|P(p')\rangle_{\mathrm{MPC}} &= (2\pi)^3\delta(\vec{p}-\vec{p}')\int d^3p_S\sum_{S\neq P}\int_0^\infty dk_0\,\delta^4(p+k-p_S)\left[\frac{|\langle S|j^-(0)|p\rangle|^2 - |\langle S|j^+(0)|p\rangle|^2}{k_0^2}\right]\\ &= \frac{1}{\pi}(2\pi)^3\delta(\vec{p}-\vec{p}')\int_0^\infty\frac{dk_0}{k_0}[\sigma^P(\nu,0)-\sigma^A(\nu,0)]\left(\frac{|\vec{p}|}{p_0}+\frac{|\vec{k}|}{k_0}\right).\end{aligned} \tag{8}$$

Here k is a dummy-variable four-vector such that $k^2 = 0$ and $\nu M = p\cdot k$, and

$$\left(\frac{|\vec{p}|}{p_0}+\frac{|\vec{k}|}{k_0}\right)\sigma^{A,P}(\nu, k^2) = (2\pi)\int d^3p_S\sum_{S\neq P}\frac{|\langle S|j^{+,-}(0)|p\rangle|^2}{2k_0}\delta^4(p+k-p_S), \tag{9}$$

where $\sigma^{A,P}(\nu, k^2)$ is the total cross section for collision of circularly polarized photons with protons. Since

$$\lim_{p_0\to\infty}\left(\frac{|\vec{k}|}{k_0}+\frac{|\vec{p}|}{p_0}\right) = 1,$$

Eq. (8) can be written as

$$\langle P(p)|[D^+(t), D^-(t)]|P(p')\rangle_{\mathrm{MPC}} = (2\pi)^3\delta(\vec{p}-\vec{p}')\frac{1}{\pi}\int_0^\infty\frac{d\nu}{\nu}[\sigma^P(\nu)-\sigma^A(\nu)]\,. \tag{10}$$

Substituting Eqs. (6) and (10) into Eq. (1), we obtain

$$\frac{2\pi^2\alpha x^2}{M^2} = \int_0^\infty\frac{d\nu}{\nu}[\sigma^P(\nu)-\sigma^A(\nu)], \tag{11}$$

which is the sum rule obtained by Drell and Hearn from the unsubtracted dispersion relation and zero-energy theorem for the spin-flip Compton amplitude. In fact, Eq. (11) obtained by us is valid for any spin-$\frac{1}{2}$ target including the proton.

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