## Comments and Addenda

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# Quantization and Density of States of Electromagnetic Waves in a Half-Space Dielectric

D. M. Fradkin* and William B. Rolnick
*Department of Physics, Wayne State University, Detroit, Michigan 48202*
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We quantize the electromagnetic field in a half-space dielectric bounded by conducting planes parallel to the interface. The density of states is obtained for both the propagating and evanescent modes. We show that this density is appropriate for any boundary conditions and hence is applicable to the modes given by Carniglia and Mandel.

Recently,[1,2] there has been considerable interest in the properties of evanescent waves. Carniglia and Mandel[2] have quantized the evanescent modes of the electromagnetic field that can exist throughout a space consisting of a left half-space filled with a homogeneous passive dielectric separated by a plane interface from a right half-space vacuum. They chose the dielectric constant to be real and frequency-independent[3] and the permeability to be that of the vacuum.

In many problems, e.g., calculations of cross sections, the density of states is required. The latter is not given in Ref. 2, and furthermore the modes obtained there do not directly lend themselves to a counting procedure because they are not discrete. One can obtain discrete modes by considering the waves in a finite enclosure subject to appropriate boundary conditions. Because of the asymmetry between left (dielectric) and right (vacuum) half-spaces, periodic boundary conditions are not convenient. In order to obtain an expression for the density of states, we quantize the set of modes which arise when the entire space is bounded by conducting planes parallel to the interface. We then show that this density of states will be the appropriate one for the modes of Ref. 2 as well. We also prove that doubly evanescent waves (i.e., those that damp exponentially on both sides of the interface) cannot exist.

Let $x_3 = 0$ be the interface, $x_3 = \pm L$ be the conducting walls, and $n = \sqrt{\epsilon}$ be the real, constant index of refraction of the nonmagnetic dielectric in the left half-space $(-L \leqslant x_3 \leqslant 0)$. The right half-space is a vacuum. Periodic boundary conditions at $\pm L$ in the $\hat{x}_1$ and $\hat{x}_2$ directions are employed. As usual, the Hermitian electric and magnetic field operators $\vec{E}$ and $\vec{B}$ are related to the vector operator $\vec{A}$ by

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \operatorname{curl} \vec{A}, \tag{1}$$

where $\vec{A}$ is given by

$$\vec{A} = \sum_{\vec{k}, s} [\vec{\alpha}_s(\vec{k}; \vec{r}, t) a_s(\vec{k}) + \vec{\alpha}_s^*(\vec{k}; \vec{r}, t) a_s^\dagger(\vec{k})]. \tag{2}$$

Here, $s$ indicates the polarization and $\vec{k}$ represents the allowed wave numbers. The Fock-space operators $a_s$ satisfy

$$[a_s(\vec{k}), a_{s'}^\dagger(\vec{q})] = \delta_{ss'}\delta_{\vec{k},\vec{q}}. \tag{3}$$

The mode functions $\vec{\alpha}$ are obtained by applying the appropriate boundary conditions at the conducting walls and the dielectric-vacuum interface. The component $\vec{k}_\perp$ of the wave-number vector perpendicular to the $\hat{x}_3$ direction is the same on both sides of the interface. The components in the $\hat{x}_3$ direction are related by

$$k_3' = [n^2 k_3^2 + (n^2 - 1)k_\perp^2]^{1/2}, \tag{4}$$

where the primed and unprimed quantities refer to the dielectric and vacuum, respectively. With this notation, the frequency is $\omega = c(k_3^2 + k_\perp^2)^{1/2}$.

The mode functions for the transverse electric (TE) modes are[4]

$$\vec{\mathcal{A}}_{\mathrm{TE}}(\vec{k};\vec{r},t)=N_{\mathrm{TE}}(\vec{k},L)\frac{c}{2}\left(\frac{\hbar}{\omega L^3}\right)^{1/2}(\vec{k}_\perp\times\hat{x}_3)e^{i(\vec{k}_\perp\cdot\vec{r}-\omega t)}\left(\Theta(x_3)\frac{k_3'\cos k_3'L}{k_3\cos k_3L}\sin k_3(x_3-L)+\Theta(-x_3)\sin k_3'(x_3+L)\right), \quad (5)$$

where to order $L^{-1}$,

$$N_{\mathrm{TE}}(\vec{k},L)\cong n^2+\left(\frac{k_3'^2\cos^2 k_3'L}{k_3^2\cos^2 k_3L}\right)^{-1/2}. \quad (6)$$

The allowed $\vec{k}$ quantities are given by the conditions

$$\exp(2ik_1L)=1=\exp(2ik_2L), \quad (7)$$

$$(k_3+k_3')\sin[(k_3+k_3')L]=(k_3-k_3')\sin[(k_3-k_3')L]. \quad (8)$$

The mode functions for the transverse magnetic (TM) modes are[4]

$$\vec{\mathcal{A}}_{\mathrm{TM}}(\vec{k};\vec{r},t)=N_{\mathrm{TM}}(\vec{k},L)\frac{c}{2}\left(\frac{\hbar}{\omega L^3}\right)^{1/2}e^{i(\vec{k}_\perp\cdot\vec{r}-\omega t)}\frac{c}{\omega}\left(\Theta(x_3)\frac{\cos k_3'L}{\cos k_3L}[\hat{x}_3k_\perp\cos k_3(x_3-L)-\hat{k}_\perp ik_3\sin k_3(x_3-L)]\right.$$
$$\left.+\Theta(-x_3)n^{-2}[\hat{x}_3k_\perp\cos k_3'(x_3+L)-\hat{k}_\perp ik_3'\sin k_3'(x_3+L)]\right), \quad (9)$$

where to order $L^{-1}$,

$$N_{\mathrm{TM}}(\vec{k},L)\cong\left(1+\frac{\cos^2 k_3'L}{\cos^2 k_3L}\right)^{-1/2}. \quad (10)$$

The allowed values of $k_3$ are given by the condition

$$(k_3'+n^2k_3)\sin[(k_3+k_3')L]=(k_3'-n^2k_3)\sin[(k_3-k_3')L], \quad (11)$$

where $k_1$ and $k_2$ again satisfy Eq. (7).

In the preceding, the quantities $k_1$ and $k_2$ range over both positive and negative values. However, to avoid double counting, $k_3', k_3\geq 0$ when $k_3', k_3$ are real, and $\mathrm{Im}k_3', \mathrm{Im}k_3\geq 0$ when they are pure imaginary[5] (the case of evanescent waves). The normalization of the mode functions has been chosen so that the energy is

$$\tfrac{1}{2}\int d^3x(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})=\sum_{\vec{k},s}\tfrac{1}{2}\hbar\omega[a_s^\dagger(\vec{k})a_s(\vec{k})+a_s(\vec{k})a_s^\dagger(\vec{k})]. \quad (12)$$

In order to obtain the density of states we must count the number of solutions of Eqs. (7), (8), and (11) in a momentum-space volume element $\Delta k_1'\Delta k_2'\times\Delta k_3'$. (Here, the dielectric transverse wave numbers $k_1'$, $k_2'$ are equal to the vacuum $k_1$, $k_2$, respectively.) We may express the density as

$$\rho(\vec{k}')=\rho_3(\vec{k}')(L/\pi)^2, \quad (13)$$

where $\rho_3(\vec{k}')$ is the number of allowed values of $k_3'$ in a range $\Delta k_3'$. For the situation where both $k_3$ and $k_3'$ are real, Eqs. (8) and (11) are of the form

$$M_a\sin(1+\Lambda)k_3'L=M_b\sin(1-\Lambda)k_3'L, \quad (14)$$

where $\Lambda\equiv k_3/k_3'$ and $M_a>|M_b|$. In Fig. 1 we plot the left- and right-hand sides of Eq. (14) as a function of $k_3'$. Treating $\Lambda$ as a constant over the range $\Delta k_3'$, it is evident that there are two intersections in each interval of $k_3'L$ of length $2\pi/(1+\Lambda)$. Consequently, it follows that

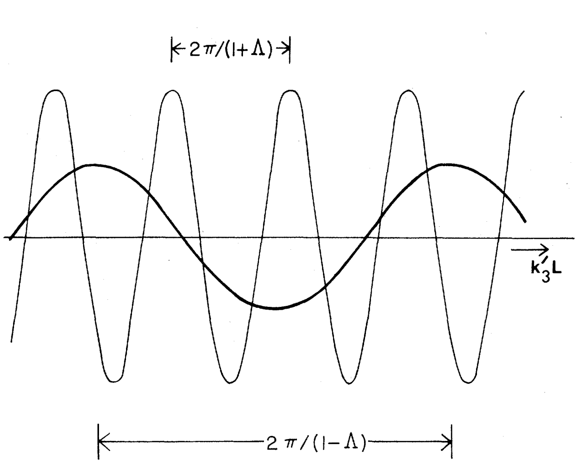


FIG. 1. Plot of the left- and right-hand sides of Eq. (14) as a function of $k_3'L$. The intersections give the allowed values of $k_3'$.

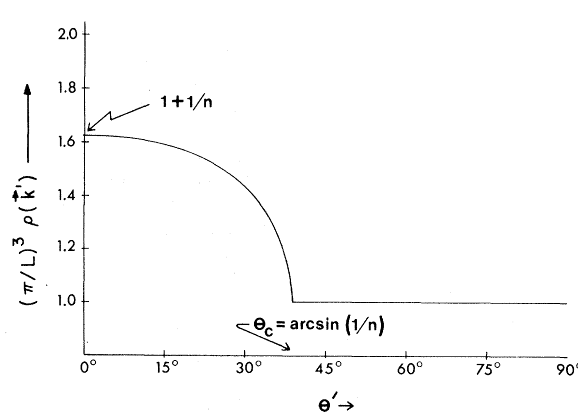


FIG. 2. Plot of the density of states (for either polarization) as a function of $\theta'$. The curve is drawn for $n=1.6$.

$$\rho_3(\vec{k}') = \pi^{-1}L(1+\Lambda) \quad \text{for } \theta' \leqslant \theta_c , \tag{15}$$

with

$$\Lambda = n^{-1}[1-(n^2-1)\tan^2\theta']^{1/2} , \tag{16}$$

where $\theta'$ is the angle between $\vec{k}'$ and the normal (3-direction), and $\theta_c$ is the critical angle $[\arcsin(1/n)]$.

For the situation where $k_3$ is purely imaginary, Eqs. (8) and (11) may be rewritten as

$$(k_3')^{-1}\tan k_3'L = -(\kappa_3)^{-1}\tanh\kappa_3 L , \tag{8$'$}$$

$$k_3'\tan k_3'L = n^2\kappa_3\tanh\kappa_3 L , \tag{11$'$}$$

where $\kappa_3 \equiv \mathrm{Im}\,k_3$. Equations (8′) and (11′) have *no* solution for $k_3'$ imaginary. Hence there are no doubly evanescent waves. For $k_3'$ real and $k_3$ imaginary (i.e., $\theta' \geqslant \theta_c$), a plot of the relevant functions in Eq. (8′) or (11′) shows that there is one solution in each interval of $k_3'L$ of length $\pi$. Thus, we obtain

$$\rho_3(\vec{k}') = \pi^{-1}L \quad \text{for } \theta' \geqslant \theta_c . \tag{17}$$

Combining Eqs. (15) and (17), and noting that $k_3' \geqslant 0$ dictates that $0 \leqslant \theta' \leqslant \pi/2$, we obtain the density

$$\rho(\vec{k}') = (L/\pi)^3[(1+\Lambda)\Theta(\theta_c-\theta') + \Theta(\theta'-\theta_c)\Theta(\pi/2-\theta')] \tag{18}$$

for the modes of *each* polarization (TE or TM). The density is plotted as a function of $\theta'$ in Fig. 2. For $n \to 1$, one obtains $\rho(\vec{k}') \to 2(L/\pi)^3$. (The factor of 2 appears because each of our modes contains waves propagating in both positive and negative 3-directions.) For $n \to \infty$, the dielectric represents a conductor. In this limit, one obtains $\rho(\vec{k}') \to (L/\pi)^3$, as expected, since the region in which the modes are not identically zero occupies half the volume of the total space. Using the relation $\omega = ck'/n$ and Eq. (18), we obtain

$$\begin{aligned}\rho(\omega)d\omega &= \int_{\Omega'} \rho(\vec{k}')k'^2dk'd\Omega' \\ &= (2\pi)^{-2}(n/c)^3(2L)^3 \\ &\quad\times\{1+n^{-1}[1-(n^2-1)^{1/2}\theta_c]\}\omega^2 d\omega \end{aligned} \tag{19}$$

for the density of states in a given range of frequencies $d\omega$ for each polarization.

Von Laue has shown[6] that the density of states for the normal modes of radiation in a cavity is independent of boundary conditions for wavelengths less than the dimensions of the enclosed region. Therefore, when $L \to \infty$, the density of states that we have obtained is independent of our assumed boundary conditions. Consequently, this density of states is appropriate for the modes of Ref. 2 when one takes account of the fact that two of those propagating modes ($\pm k_3$) are equivalent to our single propagating mode ($k_3 \geqslant 0$).

If one employs our modes to calculate transition probabilities, the results will be $L$-dependent even though the $L^3$ factors in $\rho$ and in the $\mathfrak{a}$'s cancel. However, in a physical problem where $L$ effectively becomes infinite, $k_3$ is spread over a range large compared with $L^{-1}$ and the rapidly oscillating functions that appear will average to yield an $L$-independent result.

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[1]R. Asby and E. Wolf, J. Opt. Soc. Am. 61, 52 (1971).

[2]C. K. Carniglia and L. Mandel, Phys. Rev. D 3, 280 (1971). We adopt the Heaviside-Lorentz units that these authors have employed.

[3]Although this cannot represent a physical situation (since causality is violated), the resulting set of orthogonal modes may be useful in a restricted range of frequencies for problems involving dielectrics.

[4]Care must be taken in going to the limit $n \to 1$. For those solutions such that $\cos k_3L \to 0$, one finds that $(\cos k_3'L)/(\cos k_3L) \to -1$. For those solutions such that $\sin k_3L \to 0$, one finds that $(\sin k_3'L)/(\sin k_3L) \to -1$.

[5]Since $k_3 = [(\omega/c)^2 - k_\perp^2]^{1/2}$, and both $\omega$ and $k_\perp$ are real, then $k_3$ can be either real or pure imaginary. Also, since we are taking $n$ to be real, the same conclusion follows for $k_3'$.

[6]M. von Laue, Ann. Physik 44, 1197 (1914). We thank Professor A. M. de Graaf for bringing this paper to our attention.