(1971).
[126]A. Abashian and H. J. Lipkin, Phys. Letters 14, 151 (1965); J. L. Uretsky, *ibid.* 14, 154 (1965); P. K. Kabir and R. H. Lewis, Phys. Rev. Letters 15, 306 (1965); L. B. Okun and I. Ya. Pomeranchuk, Phys. Letters 16, 338 (1965); H. J. Lipkin, Phys. Rev. Letters 22, 213 (1969); P. K. Kabir, *ibid.* 22, 1018 (1969); P. Darriulat, J. P. Deutsch, K. Kleinknecht, C. Rubbia, and K. Tittel, Phys. Letters 29B, 132 (1969); N. N. Nikolaev and R. M. Ryndin, Yadern. Fiz. 12, 865 (1970) [Soviet J. Nucl. Phys. 12, 468 (1971)].
[127]I. Schur, J. Reine Angew. Math. 130, 66 (1905).
[128]P. A. M. Dirac, Proc. Roy. Soc. (London) A180, 1 (1942); Commun. Dublin Inst. Adv. Studies A, No. 1 (1943). Also W. Pauli, Rev. Mod. Phys. 15, 175 (1943).

PHYSICAL REVIEW D VOLUME 5, NUMBER 2 15 JANUARY 1972

One-Loop Diagrams in Yang-Mills Theory*

Rabindra Nath Mohapatra
Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 28 October 1971)

In this paper we discuss the problem of covariance of one-loop diagrams in a canonical formulation of Yang-Mills theory. We show that one-loop diagrams to all orders can be made covariant provided they satisfy certain Ward identities. Verification of these Ward identities to second order in the coupling constant shows that one must introduce renormalization counterterms into the theory.

I. INTRODUCTION

Using the methods of canonical quantization, the present author has recently obtained[1] the noncovariant Feynman rules for the Yang-Mills field in both radiation ($\partial_i \vec{b}_i = 0$) and axial ($\vec{b}_3 = 0$) gauges. It has also been proved[1] that in both cases the tree diagrams to all orders and one-loop diagrams to order g^2 can be described by a covariant set of rules that follow from the above-mentioned noncovariant rules through use of Ward-type identities on tree graphs. Remarkably enough, the covariant scalar loop discovered by many authors[1] (Feynman, DeWitt, Faddeev and Popov, Mandelstam, Fradkin and Tyutin, and others) using other approaches to quantum field theory, was also found to exist within the canonical formulation (at least to lowest order). The aim of the present paper is to show that the one-loop diagrams to all orders can also be made covariant, provided they satisfy certain generalized Ward identities described in the text. The scalar loop is found to exist to all orders. We further investigate whether the Ward identities are really satisfied and, to lowest order (g^2), we find that they are violated by a quadratically divergent term, and one has to renormalize the theory to satisfy the identities. Just such a program for the one-loop case has been recently carried out by t'Hooft[2] and therefore, using t'Hooft's result, we can prove the covariance of one-loop diagrams to all orders.

II. PROOF

We will work in the axial gauge where the propagator is given by[1]

$$D^{ab}_{\mu\nu}(k) = -\frac{i\delta_{ab}}{k^2 - i\epsilon}\left(\delta_{\mu\nu} - \frac{k\cdot\xi(k_\mu\xi_\nu + k_\nu\xi_\mu) - k_\mu k_\nu}{(k\cdot\xi)^2}\right), \tag{1}$$

where $\xi_\mu = \delta_{\mu 3}$ in the frame in which we are working and vertices are given by the following interaction Lagrangian:

$$H_I = -L_I = \tfrac{1}{2} g\vec{g}_{\mu\nu}\cdot(\vec{b}_\mu \times \vec{b}_\nu) + \tfrac{1}{4} g^2(\vec{b}_\mu\times\vec{b}_\nu)\cdot(\vec{b}_\mu\times\vec{b}_\nu), \tag{2}$$

where $\vec{b}_\mu$ denotes the isovector gauge field and

$$\vec{g}_{\mu\nu} = \partial_\mu \vec{b}_\nu - \partial_\nu \vec{b}_\mu. \tag{3}$$

If we can show that for the one-loop diagram all ξ_μ-dependent terms can be dropped, then we have proved our assertion. We will do this in three steps.

Step I. In this part, we will show that the $k_\mu k_\nu / k^2(k\cdot\xi)^2$ term in the propagator can be dropped, thereby making the effective propagator look like (for the one-loop case)

$$D'^{ab}_{\mu\nu}(k) = -\frac{i\delta_{ab}}{k^2 - i\epsilon}\left(\delta_{\mu\nu} - \frac{k_\mu\xi_\nu + k_\nu\xi_\mu}{k\cdot\xi}\right). \tag{4}$$

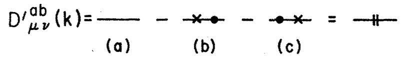

FIG. 1. Decomposition of the noncovariant propagator into its various parts.

Lemma 1: If

$$T^{a_1a_2\cdots a_n}_{\mu_1\mu_2\cdots\mu_n}(k_1,k_2,\cdots,k_n)\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2)\cdots\epsilon_{\mu_n}(k_n)$$

stands for a complete set of covariant tree diagrams [i.e., tree diagrams generated by vertices in Eq. (2) and propagator $-i\delta_{ab}\delta_{\mu\nu}/(k^2-i\epsilon)$] with external legs having isospins $a_1, a_2,\ldots, a_n$ and transverse polarizations $\epsilon_{\mu_1}(k_1),\ldots,\epsilon_{\mu_n}(k_n)$ [satisfying $k_{\mu_i}\epsilon_{\mu_i}(k_i)=0$], then

$$k_{1\mu_1}k_{2\mu_2}\cdots k_{l\mu_l}T^{a_1a_2\cdots a_la_{l+1}\cdots a_n}_{\mu_1\mu_2\cdots\mu_l\mu_{l+1}\cdots\mu_n}\times\epsilon_{\mu_{l+1}}(k_{l+1})\epsilon_{\mu_{l+2}}(k_{l+2})\cdots\epsilon_{\mu_n}(k_n)=0. \quad (5)$$

Proof: This is the statement of the theorem proved in Ref. 1 [I, Eq. (4.1)].

Corollary: Using Lemma 1, it is easy to prove that

$$k_{1\mu_1}k_{2\mu_2}\cdots k_{l\mu_l}T'^{a_1a_2\cdots a_la_{l+1}\cdots a_n}_{\mu_1\mu_2\cdots\mu_l\mu_{l+1}\cdots\mu_n}\times\epsilon_{\mu_l}(k_l)\epsilon_{\mu_{l+1}}(k_{l+1})\cdots\epsilon_{\mu_n}(k_n)=0, \quad (6)$$

where T' stands for the complete set of diagrams with vertices given by Eq. (2) and propagator Eq. (1) or Eq. (4).

It is then obvious that $k_\mu k_\nu/k^2(k\cdot\xi)^2$ can be dropped from the propagator. To see this consider a one-loop diagram with Eqs. (1) and (2) as the rules. Break up the propagator in Eq. (1) into the $k_\mu k_\nu$ term and $D'^{ab}_{\mu\nu}$ and decompose them. Then, the $k_\mu k_\nu$ term can be either within a loop or in a tree attached to the loop. In the latter case, look at the tree attached to the k_μ away from the loop, and there will be a complete set of such trees with a k_μ sitting on one of their external legs, other external legs being on their mass shells. But this is zero by Eq. (6). Therefore, all $k_\mu k_\nu$'s can be dropped from propagators within the trees attached to the loop. However, when the $k_\mu k_\nu$ term appears in a propagator within the loop, break up the loop there; then, what we have is a complete set of trees with k_μ's planted on two of its legs, with other legs on their mass shells, that is, also zero by Eq. (6).

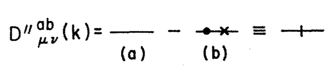

FIG. 2. Decomposition of the noncovariant propagator into its various parts.

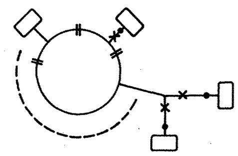

FIG. 3. This figure shows how in the surviving diagrams the crosses are located towards the loop only. The boxes stand for the rest of the diagram.

This then proves that $k_\mu k_\nu/k^2(k\cdot\xi)^2$ can be dropped from the propagator, which now looks like Eq. (4).

Step II. Before going ahead with this step of the proof, let us break up the propagator as in Fig. 1, where Fig. 1(a) stands for the covariant part of $D'^{ab}_{\mu\nu}$: A cross stands for k_μ, a dot stands for $(1/k\cdot\xi)\xi_\mu$, and the Lorentz index on k or ξ is taken to be the same as the Lorentz index of the line on which they are planted. In this step of the proof, we show two things:

(a) The propagator associated with the set of tree diagrams attached to the loop is the one shown in Fig. 2, where the cross in any propagator is towards the loop and the dot is away from the loop. (See Fig. 3 for an example, where we have omitted the covariant parts of the propagator.)

Proof: The proof of this again follows from Eq. (6). Because, whenever we consider the part of the propagator in which the dot is towards the loop, the cross then necessarily sits on one leg of a complete set of trees (T') with other external legs on their mass shells, and, for such a case, Eq. (6) gives zero. The propagator for the trees is, therefore,

$$D''^{ab}_{\mu\nu}(k)=-\frac{i\delta_{ab}}{k^2-i\epsilon}\left(\delta_{\mu\nu}-\frac{\xi_\mu k_\nu}{k\cdot\xi}\right). \quad \text{Q.E.D.}$$

(b) Inside the loop when we break up the propagator as in Fig. 1 and decompose, we will get all kinds of diagrams. What we will prove now is that only those diagrams are nonzero where between any two crosses there is always a dot (and, of course, few of the covariant parts of the propagator). (See Fig. 4 for two examples of such diagrams.)

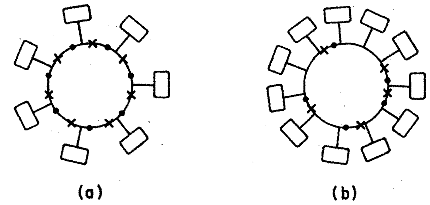

FIG. 4. (a) and (b) represent some typical diagrams which survive after step II(b) of the proof is carried through. Notice the ordering of crosses and dots, and nowhere are two crosses adjacent.

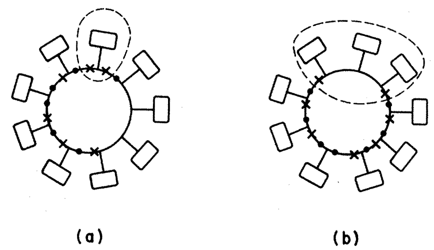

FIG. 5. Shows how certain diagrams vanish when two crosses are adjacent. The dashed line encloses the part of the diagram that vanishes when a complete set then is taken.

Proof: It is easy to convince oneself that the diagrams that do not satisfy the above criterion are of the form where two crosses sit on two legs of a tree diagram (of type T') with all other legs on

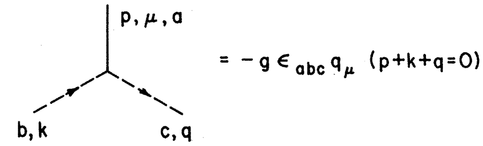

FIG. 6. Denotes the vector-scalar-scalar vertex. The arrow is shown because the momentum does not appear symmetrically. It is this fact which gives rise to one clockwise and one counterclockwise scalar loop in the end.

their mass shells and, in this case, again Eq. (6) tells us that it is zero. Two examples of such diagrams are shown in Fig. 5, where the dashed line encloses the complete set of tree diagrams on whose legs the crosses are planted.

Step III. This is the final step of the proof. Here, we would like to show the following:

Sum of the remaining diagrams

$$=\sum_l F(k,\xi)k_{1\mu_1}k_{2\mu_2}\cdots k_{l\mu_l}L^{a_1\cdots a_n}_{\mu_1\cdots\mu_l\mu_{l+1}\cdots\mu_n}\epsilon_{\mu_{l+1}}(k_{l+1})\cdots\epsilon_{\mu_n}(k_n)+L^{b_1b_2\cdots b_n}_{\nu_1\nu_2\cdots\nu_n}\epsilon_{\nu_1}(p_1)\epsilon_{\nu_2}(p_2)\cdots\epsilon_{\nu_n}(p_n), \qquad (7)$$

where L stands for a complete set of covariant one-loop diagrams described by the following Feynman rules (F stands for some function of momenta):

(a) 3-vector-particle and 4-vector-particle vertices given in Eq. (2),

(b) vector-particle propagator $-i\delta_{\mu\nu}\delta_{ab}/(k^2-i\epsilon)$,

(c) scalar-particle propagator $-i\delta_{ab}/(k^2-i\epsilon)$,

(d) vector-scalar-scalar vertex (Fig. 6) $-g\epsilon_{abc}q_\mu$ with a (−) sign in front of each scalar loop. It is clear from the above coupling that the scalar particle can occur only inside closed loops.

One can see from Eq. (7) that the condition for the covariance of one-loop diagrams is therefore given by the following Ward identities:

$$k_{1\mu_1}k_{2\mu_2}\cdots k_{l\mu_l}L^{a_1\cdots a_l a_{l+1}\cdots a_n}_{\mu_1\cdots\mu_l\mu_{l+1}\cdots\mu_n}\epsilon_{\mu_{l+1}}(k_{l+1})\cdots\epsilon_{\mu_n}(k_n)=0 \qquad (8)$$

for arbitrary $l\leq n$. To prove Eq. (7), we first note the following (the decomposition is shown in terms of Feynman diagrams by Fig. 7):

$$p_\mu\Gamma^{abc}_{\mu\nu\lambda}(p,q,k)=-g\epsilon_{abc}[(k^2\delta_{\nu\lambda}-k_\nu k_\lambda)-(q^2\delta_{\nu\lambda}-q_\nu q_\lambda)], \qquad (9)$$

where $\Gamma^{abc}_{\mu\nu\lambda}$ stands for the 3-vertex with all momenta outgoing. We also note two algebraic identi-

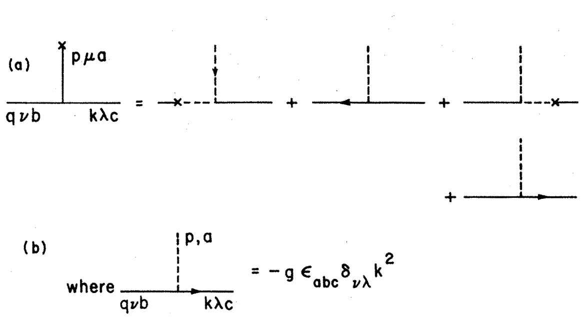

FIG. 7. Effect of a cross on a three-vertex.

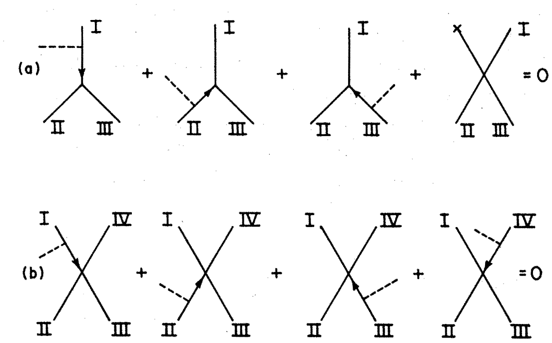

FIG. 8. Algebraic identities obeyed by the VVS vertices.

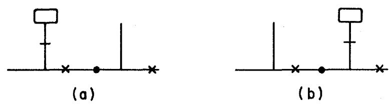

FIG. 9. (a) and (b) denote part of a loop where the horizontal lines are internal to the loop and the vertical ones belong to trees attached to the loop.

ties involving the new vector-vector-scalar (VVS) vertex Fig. 7(b), which actually follow from gauge invariance of the theory, in Fig. 8 (see Ref. 1). From Eq. (9), it follows that

$$q_\nu p_\mu \Gamma^{abc}_{\mu\nu\lambda}(p,q,k) = -g\epsilon_{abc}\, q_\nu (k^2\delta_{\nu\lambda} - k_\nu k_\lambda). \tag{10}$$

Next, we notice that we are left with only diagrams of the type shown in Fig. 4. Of course, some of the lines attached to the loop could be external lines. Therefore, we will have the following two kinds of situations in the loop:

(i) a 3-vertex with one external line, another line with a cross, and another line with a dot [Fig. 9(a)],

(ii) a 3-vertex with one D''-type propagator as one line, another line with a cross, and another with a dot [Fig. 9(b)].

Of course, in each of the cases, one or both of the lines in the loop could be covariant propagators. Note that we have not said anything about 4-vertices for reasons which will become apparent soon. We will complete the rest of the proof diagrammatically. From Fig. 10, we see that a vertex of the type Fig. 9(a) merely slides the cross along to the adjacent vertex, leaving behind a VSS vertex (Fig. 6) at the previous position. From Fig. 11, we observe that 11(c) and 11(d) merely slide the cross along the loop as before; and when we take a complete set of diagrams, 11(a) cancels by virtue of either Fig. 8(a) or 8(b). We did not mention 4-vertices before because a 4-vertex always cancels in the manner just described, with appropriate diagrams with VVS vertices. Figure 11(b) plants a cross on a leg of a complete set of trees T' and therefore is zero by virtue of Eq. (6). Of course, one could ask what would happen if we had a diagram of the type Fig. 12(a). Here again Figs. 12(b) and 12(d) cancel with suitable other diagrams by virtue of Fig. 8(a) or 8(b). Figure 12(e) also plants a cross on a tree diagram and is therefore zero for the same reason as Fig. 11(b). Figure 12(c), of course, merely slides the cross along.

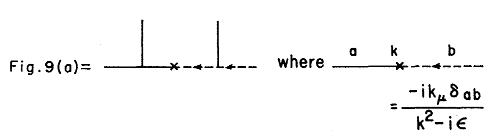

FIG. 10. Shows the action of a cross at a vertex.

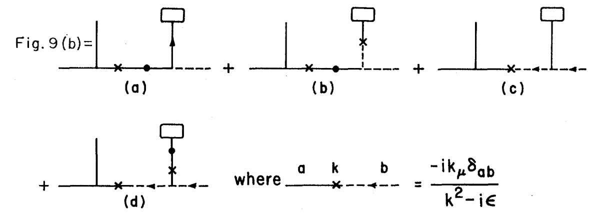

FIG. 11. Shows the action of a cross at a vertex.

The final conclusion then is that a cross always slides along a 3-vertex and, applying the above arguments successively, it leads to a diagram of the type shown in Fig. 13. Notice that here the arrow goes counterclockwise. There will, of course, be a similar set of diagrams where the arrow goes clockwise. Now the question is that one such diagram arises from each configuration of the type shown in Fig. 4 as long as there is one cross present. If there are n internal lines in the loop, then a diagram with l propagators of the type Fig. 1(b) will have a weight of $(-1)^l n!/(n-l)!\,l!$. But l can go from unity to n and we get

$$\sum_{l=1}^{n} (-1)^l \frac{n!}{(n-l)!\,l!} = -1. \tag{11}$$

Remember that when $l=0$, all internal propagators are covariant $[-i\delta_{ab}\delta_{\mu\nu}/(k^2-i\epsilon)]$, whereas in Fig. 13 all propagators are given by $-i\delta_{ab}/(k^2-i\epsilon)$ (see Figs. 10 and 11). Therefore, $l=0$ gives the required diagram with all 3- and 4-vector-particle vertices and all covariant propagators. In summary, we proved that the propagators and vertices inside the loop can be made covariant, and a scalar loop with weight factor of (-1) appears (there are two such scalar loops, one clockwise and one counterclockwise). Equation (7) then follows if we break up the propagators in the tree diagrams attached to the loop and multiply them and group the diagrams appropriately. This completes our proof of covariance of loop diagrams, provided the Ward identities given by Eq. (8) are satisfied.

III. CONCLUSION

We would like to conclude by adding a few words on the condition Eq. (8). These are the generalized

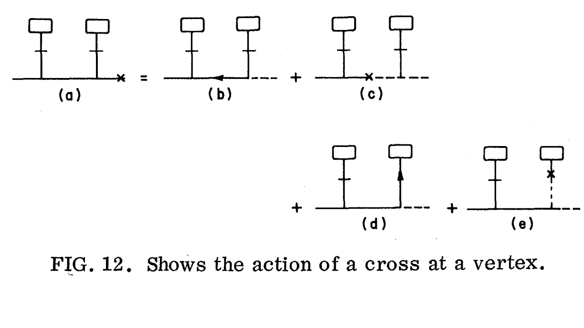

FIG. 12. Shows the action of a cross at a vertex.

Ward identities of the Yang-Mills theory and ought to hold in a suitably renormalized version of the theory.[2] However, we tried to investigate it to order g^2, and we calculated the $\pi_{\mu\nu}(p)$ to order g^2.

$$\pi_{\mu\nu}(p) = -\frac{g^2\delta_{ab}}{(4\pi)^2}\{-4\delta_{\mu\nu}(M^2 - \tfrac{1}{3}p^2) + \tfrac{10}{3}(p^2\delta_{\mu\nu} - p_\mu p_\nu) \times [\ln(M^2/p^2) + \text{finite terms}]\}, \quad (12)$$

where M is the cutoff which goes to infinity. One can see from this that, to order g^3, the diagrams will be covariant regardless of the way we renormalize the theory. But for higher orders, one will have to introduce counterterms into the theory in order to be able to prove the covariance. So, in conclusion, we would like to say that, although we have not strictly proved covariance of one-loop diagrams, we have been able to show that covariance follows provided we accept the proof of the generalized Ward identities given by t'Hooft. It is clear from Eq. (12) that one needs counterterms in the Lagrangian to implement the renormalization program; but such a counterterm would, of course, violate gauge invariance and would lead to trouble. However, one could adopt the viewpoint that renormalization only fixes the ambiguity in the definition of the T product at the apex of the light cone and has nothing to do with the Lagrangian. Then the results of this paper become rigorous (using of course t'Hooft's proof of Ward identities).

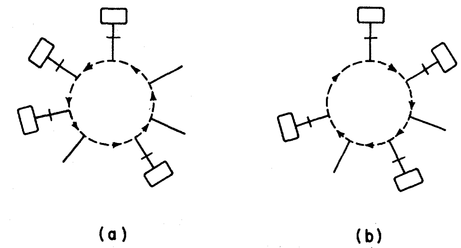

FIG. 13. Shows the scalar loop that appears in the end.

ACKNOWLEDGMENT

The author would like to thank Professor S. Oneda for comments and a critical reading of the manuscript.

*Work supported in part by the National Science Foundation under Grant No. GP-8748.

[1]R. N. Mohapatra, Phys. Rev. D 4, 378 (1971), referred to as I; 4, 1007 (1971), referred to as II; 4, 2215 (1971), referred to as III. An extensive list of references can be found in both I and II.

[2]G. t'Hooft, University of Utrecht report, 1971 (unpublished).