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Magnetic Spin Monopole from the Yang-Mills Field

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A generalization of the magnetic monopole has been derived in the context of the Yang-Mills field. This is achieved through the observation that the quantization of the ordinary magnetic monopole derives from closure of the angular momentum and magnetic field components on the Lie algebra of the Euclidean group E(3) and the fact that the Yang-Mills field is the generalization of the Maxwell field obtained by replacing the ordinary derivative by a covariant derivative. This leads to defining equations for the generalized monopole which are explicitly solved for the case when SU(2) is the gauge group of the field. The resulting system, referred to as the spin monopole, is investigated in some detail.

I. INTRODUCTION

A conviction in the existence of a symmetry between the electric and magnetic fields leads to the adoption of the magnetic monopole as the fundamental unit of magnetic matter. As first shown by Dirac,[1] the existence of such an entity requires the quantization of the product of the electric and magnetic pole strengths. It has long been supposed that this quantization is somehow responsible for the existence of the unit of charge, though the search for magnetic monopoles has as yet proved entirely unfruitful.[2] Recently Schiff[3] has revived this belief in an attempt to use the monopole quantization to explain the failure of quarks to appear as free individuals, though his method has not gone without criticism.[4] In a different spirit, but with a similar objective, Schwinger[5] has further suggested that the particle which carries the magnetic monopole is at the same time a manifestation of the elusive quark itself.

It is our present objective to develop the notion of a fundamental unit of matter which is capable of producing a charge quantization in the spirit of Dirac's monopole and at the same time carries the SU(3) internal quantum numbers of the hadrons in the spirit of Schwinger's quark. For this purpose we seek a theory which embraces this internal symmetry and yet retains the structure of the Maxwell equations as expressed through the properties of the exterior derivative and Poincaré invariance.

A particularly pleasing example of such a theory is provided by the Yang-Mills field.[6] From a physical standpoint it may be regarded – and indeed was first developed in this fashion[6] – as the natural generalization of the Maxwell field accommodating a noncommutative gauge group. We may choose this gauge group to be SU(3) and in so doing effect an interplay between the internal hadron symmetry and the field equations. Mathematically, the Yang-Mills field represents the generalization of the Maxwell field obtained by replacing the ordinary derivative by a covariant derivative. This structural resemblance to the Maxwell field proves a useful asset in its analysis.

In the following we shall derive a form of magnetic monopole from the Yang-Mills field by exploiting its relationship with the Maxwell field and then investigate the properties of this new system. More specifically, we first develop the quantization of the magnetic monopole through certain algebraic identities which lead to the Lie algebra of the Euclidean group E(3) and the requirement that the representations of this group be unitary. We then replace the Maxwell field by the Yang-Mills field and require closure under the same Lie algebra, defining the generalized magnetic monopole to be the solution of the resulting equations. It turns out that these equations are nonlinear and cannot readily be solved for an arbitrary gauge group, in particular for the interesting case of SU(3). However for SU(2) some simplification is possible owing to the fact that the structure constants for the algebra are also present in the Lie algebra of the Poincaré group itself. As a consequence we are able to obtain a nontrivial solution which we refer to as the magnetic spin monopole. It is shown that this has a corresponding electric-spin-monopole counterpart and the nonrelativistic Schrödinger equation is solved for the combined system.

For the ordinary monopole the quantization can also be regarded as a requirement necessary to remove an inconsistency arising from the fact that the curl of the vector potential differs from the magnetic field by an infinite dipole string.[7,8] This approach has no obvious relationship with the above use of the Euclidean group. However we argue that the former point of view is intimately connected with the requirement that the momentum operator be self-adjoint in the strict sense. This establishes a connection with the Euclidean-group approach since self-adjointness is implicitly assumed in selecting unitary representations.

In contrast to the ordinary monopole we find that no infinite strings of dipoles are involved in the present system. At the same time the quantization is only an apparent one, as it derives merely from the requirement of closure of the algebra and not through the unitarity of the representations. As a consequence, arbitrary values of the pole strength are admissible, though they lead to a loss of rotational invariance as expressed through the failure of an angular momentum vector to be conserved. Unlike the ordinary monopole this does not lead to any inconsistency because the field itself has a directionality induced through the coupling of space-time with SU(2) space. On the other hand loss of rotational invariance does lead to a symmetry-breaking effect in the energy spectrum.

We have not tackled the delicate question of the covariance of the magnetic spin monopole in the context of a field theory though we anticipate less difficulty[9] than with the ordinary monopole owing to the absence of infinite strings. On the other hand we may replace the nonrelativistic Schödinger equation by the Dirac equation to describe the relativistic dynamics of the monopole as in the case of the ordinary Coulomb potential. The problem of covariance is, even without quantization, a difficulty inherent in any system of point charges.[10]

II. THE YANG-MILLS FIELD

We outline below the salient properties of the Yang-Mills field indicating its proper differentio-geometric setting. This will help us to develop the analogy between the Maxwell and Yang-Mills field which will in turn lead to a construction of the generalized monopole.

Starting from the wave equation for a particle in an electromagnetic field it is easily checked that a phase change in wave function is equivalent to a gauge transformation of the vector potential. More generally for an n-component wave function ψ the unitary transformation

$$\psi \rightarrow \psi' = U\psi \tag{2.1}$$

is equivalent to the (non-Abelian) gauge transformation[6]

$$A_\mu \rightarrow A'_\mu = U^{-1} A_\mu U + \frac{i}{e} U^{-1} \partial_\mu U \tag{2.2}$$

on the matrix-valued function A_μ. These are the quantities that form the components of the vector potential of the Yang-Mills theory.

It can be shown that (2.2) is a generalization of the transformation law of the Christoffel symbols of general relativity.[11] Indeed in the language of differential geometry (2.1) is really a transformation on sections of a vector bundle over a differentiable manifold M, where M is identified with space-time (which need not be assumed flat). The Christoffel symbols arise in the special case when the vector bundle is the tangent bundle, the sections are vector fields, and the matrix U is the Jacobian.[12] This formalism enables one to construct quantities which transform covariantly with respect to the transformation on ψ. Thus from the vector potential A_μ, which does not itself transform covariantly owing to the inhomogeneous term $U^{-1}\partial_\mu U$ in (2.2), we may construct the analog of the Riemann-Christoffel tensor. It turns out that this is just the Yang-Mills field itself whose components $B_{\mu\nu}$ take the form

$$B_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + ie[A_\mu, A_\nu]. \tag{2.3}$$

The stated covariance of the field may be verified directly by substitution of (2.2) into (2.3). This gives

$$B'_{\mu\nu} = U^{-1} B_{\mu\nu} U \tag{2.4}$$

as required.

To preserve covariance under differentiation we must apply the covariant derivative of differential geometry. Translated to the present situation this becomes the operator D_μ whose action on the wave function is given by

$$D_\mu \psi = \partial_\mu \psi - ieA_\mu \psi, \tag{2.5}$$

and on an arbitrary matrix V in the space of the ψ components by

$$D_\mu V = \partial_\mu V - ie[A_\mu, V] . \tag{2.6}$$

This last expression is the one usually given for the Yang-Mills covariant derivative. Applied to the field $B_{\mu\nu}$ we obtain

$$D_\lambda B_{\mu\nu} + D_\nu B_{\lambda\mu} + D_\mu B_{\nu\lambda} = 0 , \tag{2.7}$$

which is just the generalization of the Bianchi identities.

It is useful to think of the Yang-Mills field as the extension of the Maxwell field obtained by a replacement of ∂_μ by D_μ. For example, (2.7) can be derived following this prescription. On the other hand $B_{\mu\nu}$ is *not* given, as such an argument might lead us to expect, by the expression $D_\mu A_\nu - D_\nu A_\mu$, though we do have the useful identity

$$ie[B_{\mu\nu}, V] = [D_\mu, D_\nu] V , \tag{2.8}$$

where V is an arbitrary matrix as above. We can think of this as due to the fact that, as the vector potential is not itself covariant, we do not necessarily obtain covariant quantities by applying D_μ to it. Again the conservation of the vector potential of the Maxwell field which is expressed through the relation $\partial^\mu A_\mu = 0$ cannot be translated into $D^\mu A_\mu = 0$ (or even $\partial^\mu A_\mu = 0$) because with respect to gauge transformations neither of these latter quantities is a scalar.

Unlike the Maxwell field, the Yang-Mills field is not itself gauge-invariant. However with the help of (2.4) gauge-invariant quantities are easily constructed. Thus the term in the Lagrangian

$$\mathcal{L}_1 = \mathrm{tr} B_{\mu\nu} B^{\mu\nu} ,$$

where the trace is over the space of components of ψ, gives rise to the analog of the electromagnetic energy density.[6] Again the term

$$\mathcal{L}_2 = (D^\mu \psi)(D_\mu \psi)$$

gives rise to the momentum operator

$$P_\mu = -i\partial_\mu - eA_\mu \tag{2.9}$$

acting on the ψ field.

III. THE ROLE OF THE EUCLIDEAN GROUP IN THE QUANTIZATION OF THE MAGNETIC MONOPOLE

In the following we develop the quantization of the magnetic monopole through certain algebraic identities[13] satisfied by the angular momentum and magnetic field vectors. Then the requirement that the Yang-Mills field yields the same identities leads to an equation which will determine the form of the generalized monopole. As the Maxwell field is a special case of the Yang-Mills field we may analyze both simultaneously.

From the momentum operator given in (2.9) we define angular momentum components $M_{\mu\nu}$ through the relation

$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu . \tag{3.1}$$

With the help of (2.8) it is easy to verify that P_μ and $M_{\mu\nu}$ are generators for the Poincaré group if and only if $B_{\mu\nu} = 0$, that is, for a free field. More generally we may study their commutation relations for a less trivial choice of $B_{\mu\nu}$. Here we consider the case when the magnetic component has a source term. To this end we set (using Latin letters for the space indices)

$$J_m = \tfrac{1}{2} \epsilon_{mij} M_{ij} , \quad B_m = \tfrac{1}{2} \epsilon_{mij} B_{ij} , \tag{3.2}$$

where ϵ_{mij} is a totally antisymmetric tensor and the summation convention holds. Then $\vec{J}$ is the three-space angular momentum vector and $\vec{B}$ the magnetic field. Independent of the form of $\vec{B}$ the commutation relations of the J_m have the simple expression

$$[J_m, J_n] = i\epsilon_{mnk}(J_k + T_k) , \tag{3.3}$$

where

$$T_k = e x_k \vec{r} \cdot \vec{B} . \tag{3.4}$$

Since, in addition,

$$[T_m, T_n] = 0 , \tag{3.5}$$

there remains to determine the commutator of the components of $\vec{J}$ with $\vec{T}$. These depend on the explicit form of $\vec{B}$ and in general a simple Lie algebraic structure is not obtained. On the other hand, we may select a commutation relation for these operators and impose it as a condition on the form of $B_{\mu\nu}$. The simplest choice and the one we shall adopt here is

$$[J_m, T_n] = i\epsilon_{mnk} T_k . \tag{3.6}$$

This holds in particular for a magnetic monopole. More generally, for any Yang-Mills field (3.2), (3.4), and (3.6) imply

$$[J_m, \vec{r} \cdot \vec{B}] = 0 , \tag{3.7}$$

for $m = 1, 2, 3$. Conversely, when (3.7) holds the components of $\vec{J}$ and $\vec{T}$ close on the Lie algebra of E(3). In this it is convenient to replace $\vec{J}$ by the vector

$$\vec{K} = \vec{J} + \vec{T} \tag{3.8}$$

to convert these commutation relations to a more familiar form. Moreover it is $\vec{K}$ rather than $\vec{J}$ which behaves like an angular momentum vector and which, in the nonrelativistic dynamics, is conserved.

In terms of $\vec{K}$ and $\vec{T}$ the Casimir invariants of the algebra are given by

$$I_1 = \vec{K}\cdot\vec{T}, \quad I_2 = \vec{T}\cdot\vec{T}. \tag{3.9}$$

Since the components of $\vec{K}$ and $\vec{T}$ are self-adjoint operators, their spectral properties are determined through the unitary representations of E(3). From the catalog of representations given for the group by Miller[14] we may set, in an irreducible representation,

$$I_1 = \omega s, \quad I_2 = \omega^2, \tag{3.10}$$

where ω is an arbitrary real constant and $2s$ is a non-negative integer. Substituting for I_1 and I_2 from (3.9) we obtain

$$e^2 r^2 (\vec{r}\cdot\vec{B})^2 = s^2. \tag{3.11}$$

This is the relationship which determines the quantization. For the magnetic monopole we set

$$B_r = g x_r / r^3,$$

where g is the monopole strength. Then (3.7) holds trivially and from (3.11) we obtain

$$eg = 0, \pm\tfrac{1}{2}, \pm 1, \ldots,$$

which on identifying e with the electric charge yields Dirac's[15] quantization.

It should be appreciated from the outset that this remarkable result is of a rather unusual nature in that we have succeeded in quantizing a parameter of the theory, namely eg, which is not itself an eigenvalue of any self-adjoint operator. The origin of this phenomenon might be thought to lie in the simultaneous use of curl $\vec{A}$ and $\operatorname{div}(1/r)$ for the magnetic field $\vec{B}$. Thus strictly speaking $\vec{B}$ has an additional term $\vec{B}'$ which, if we set[8]

$$\vec{A} = \frac{g}{r(r+z)}(-y, x, 0), \tag{3.12}$$

takes the form

$$\vec{B}' = 4\pi g(0, 0, \delta(x)\delta(y)\theta(-z)), \tag{3.13}$$

where θ is the step function. This term, which corresponds to the infinite dipole string, spoils the rotational invariance of $\vec{B}$. Consequently (3.7) and hence (3.6) no longer holds in the whole of three-space. However this does not in fact upset the Lie algebraic argument as we need only require these relations to hold in a simply connected open subset of three-space.[13] Yet there is an additional difficulty introduced by the dipole string which arises from the condition that the operators K_m and T_n are self-adjoint in the strict sense.[16] Indeed what the above analysis essentially shows is that these operators can be self-adjoint only for certain values of eg and it is this which provides the quantization. This may also be demonstrated in a more elementary fashion by explicit solution in polar coordinates of the eigenvalue equation for K^2. A complete set of solutions of this equation is found only for the quantized values of eg and hence K^2 is self-adjoint only when this holds.[17] We may regard the failure of P_μ (and hence K_m and K^2) to be self-adjoint as a consequence of the line singularity in the vector potential A_μ. This singularity effectively restricts the class of functions on which P_μ can act, so that it may fail to have a suitable domain.[16] Indeed if ψ is to be in the domain of P_μ, then it must be a differentiable function and $P_\mu\psi$ must be square-integrable. Now in estimating the contribution from this integral in the region of the line singularity of A_μ, we can perform a gauge transformation on A_μ to shift this singularity to a different region of space. This is effected through a phase transformation in the wave function ψ and we may adopt precisely Dirac's quantization argument to show that this procedure is well defined (i.e., ψ remains single-valued) only if eg is quantized. We remark that whereas the gauge function ϕ is not itself well defined (because it is multivalued) the phase factor $e^{i\phi}$ is well defined for those values of eg. Dirac applied this result to show that whereas the lack of rotational invariance introduced by the dipole string (3.13) was unavoidable in classical mechanics, it could be eliminated in quantum mechanics by quantization of eg. Here we apply this same result in a different way though to the same ultimate end. This question does not arise in the case of the spin monopole, which we discuss in Sec. V, owing to the absence of the infinite line singularity in A_μ, and correspondingly the quantization need not be imposed.

Before turning to the Yang-Mills field itself we determine the most general solution that the Maxwell field admits to (3.7). In this case (3.7) reduces to the statement

$$r(\vec{r}\cdot\vec{B}) = f(r),$$

where f is a function of the radial coordinate r. In addition, since $\vec{B} = \operatorname{curl}\vec{A}$, we have

$$\operatorname{div}\vec{B} = 0.$$

For $f(r)$ nonzero, we may show through Gauss's theorem that these relations cannot both hold in the whole of three-space. On the other hand if we require only that they hold in a simply connected open region of three-space, then simple vector manipulations show that their most general solution takes the form

$$\vec{B} = \alpha\,\operatorname{curl} f\vec{h} + \beta(\vec{r}\times\vec{g}), \tag{3.14}$$

where $\vec{h}$ is a vector satisfying

$$\operatorname{curl}\vec{h} = \vec{r}/r^3$$

in the specified region and $\vec{g}$ is a vector satisfying $\operatorname{div}(\vec{r}\times\vec{g})=0$. For f a constant (and $\beta=0$) we obtain the monopole solution. Otherwise we obtain a generalization which requires a quantization of r. This solution is probably unphysical, though not otherwise inconsistent. It may be excluded by requiring the angular momentum vector $\vec{K}$ to be conserved. This puts an additional constraint on the form of $\vec{B}$ which we determine below and which we shall further impose on the form of the generalized monopole.

IV. THE DYNAMICS OF THE MONOPOLE

We shall for the moment restrict the dynamical picture to a nonrelativistic setting. The natural choice for the Hamiltonian is then

$$H=P_i P_i + A_0 , \tag{4.1}$$

where the P_i are the momentum operators defined in (2.9).

We shall show that for the ordinary magnetic monopole the angular momentum vector $\vec{K}$ defined in (3.8) and the quantity $r(\vec{r}\cdot\vec{B})$, which is quantized through (3.11), are constants of the motion. In the spirit of the preceding paragraph we then go on to develop equations which determine the most general Yang-Mills field with this property.

For the moment we set $A_0=0$ in (4.1). From the requirement that $\vec{K}$ be a constant of the motion we obtain after a little manipulation

$$i x_j B_m = x_m [P_j, \vec{r}\cdot\vec{B}]. \tag{4.2}$$

Defining

$$C(\vec{r}) = r(\vec{r}\cdot\vec{B}), \tag{4.3}$$

it is easily shown that (4.2) is equivalent to the relations

$$B_m = (x_m/r^3)C , \tag{4.4}$$

$$[P_j, C]=0 . \tag{4.5}$$

These equations imply (3.7) and in addition show that the quantity C is a constant of the motion. Recalling that $-\vec{r}/r^3$ is the gradient of a Green's function for the Laplacian we further obtain

$$D_m B_m = -4\pi\delta(\vec{r})C . \tag{4.6}$$

The left-hand side of (4.6) is precisely the generalized divergence of $\vec{B}$. Hence we may interpret the right-hand side as the source term of the field which, as we see, is concentrated at a single point. This justifies the term monopole to describe the solution of (4.4) and (4.5) above. Should the representation (2.3) for the field hold in all of three-space then by (2.7) this source term must vanish. Clearly this leads to a trivial theory. In order to avoid this difficulty we require only that (2.3) hold in a simply connected open subset of three-space. With this convention the solution to the Maxwell field reduces to the magnetic monopole, the additional terms in (3.14) being excluded. Indeed (4.5) implies that C is a constant which may be identified with the pole strength. At the same time the two expressions (2.3) and (4.4) for $\vec{B}$ differ by an infinite dipole string which may, as in (3.13), be chosen to lie along the negative z axis.

For the Yang-Mills field the solution to (4.5) is less trivial, as the vector potential $\vec{A}$ need not commute with C. It is apparently not easy to obtain the general solution to this problem, partly on account of the essential nonlinearity of the theory. However, in the special case when the gauge group is SU(2), some simplification is possible owing to the fact that the structure constants for the Lie algebra also appear in the expression for $\vec{B}$. This enables us to obtain a nontrivial solution, the details of which we describe below.

V. THE SPIN MONOPOLE

We set out below the equations which must be solved if we are to obtain a generalization of the magnetic monopole according to the prescription outlined in Secs. III and IV. These are

$$B_n = \epsilon_{nij}(\partial_j A_i + ie A_i A_j) , \tag{5.1}$$

$$B_n = (x_n/r^3)C , \tag{5.2}$$

$$D_j C \equiv \partial_j C - ie[A_j, C] = 0 . \tag{5.3}$$

It will be noticed immediately that these equations are nonlinear in the vector potential $\vec{A}$ and so it is hardly surprising that some simplification is needed to obtain a solution. We cannot do this at the expense of losing the nonlinearity since this is essentially the new and interesting feature introduced by the Yang-Mills field. Instead we restrict the gauge group down to SU(2), or what amounts to the same thing, choose the A_j to be 2×2 matrices. As we have already noted this, by coincidence, leads to a simplification. Since our present objective is merely to demonstrate the existence of at least one nontrivial solution, we shall not attempt an exhaustive analysis of the above example, but proceed through an intuitive derivation.

As we remarked in Sec. IV, the magnetic monopole arises as the solution to (4.5) obtained by setting C equal to a constant. Since, strictly speaking, it is C^2 rather than C which appears in the quantization condition (3.11), we might expect to find a solution as long as C^2 is a constant. In the context of SU(2) this can be achieved (nontrivially) by setting

$$C = \alpha\vec{\sigma}\cdot\hat{r}\,, \tag{5.4}$$

where α is a constant, $\vec{\sigma}$ is the spin vector, and $\hat{r}$ a unit vector in the $\vec{r}$ direction. This does indeed admit a solution as we shall now demonstrate.

Starting from (5.4), we solve (5.3) for the vector potential $\vec{A}$, taking its components to be arbitrary 2×2 matrices. Ignoring terms which are multiples of the identity matrix (these correspond to the ordinary monopole) we obtain

$$\vec{A} = -\frac{1}{2er^2}(\vec{\sigma}\times\vec{r})\,. \tag{5.5}$$

Substituting (5.5) back into (5.1) gives

$$\vec{B} = \frac{1}{2e}\left(\frac{\vec{\sigma}\cdot\vec{r}}{r^4}\right)\vec{r}\,. \tag{5.6}$$

Comparing (5.2), (5.4), and (5.6) we see that (5.4) holds if $\alpha = 1/2e$. This defines the spin monopole. It should be noted that there is no choice of scale here as compared to the ordinary monopole. This is to some extent a consequence of the nonlinearity of the theory. It means that the quantization implied by (3.11) with $2s = \pm 1$ is no more than the statement that $\vec{\sigma}\cdot\hat{r}$ has eigenvalues ± 1. Other choices of the constant α, which correspond to different multiples of $\vec{A}$, still admit a Yang-Mills field of the form given in (5.6) but do not satisfy the condition (5.3) of translation invariance. This in turn means that $\vec{K}$ is no longer a constant of the motion and that it fails to close with $\vec{T}$ under the Euclidean group E(3). Consequently, the quantization obtained here is merely a requirement of closure under E(3). This may be thought a rather more artificial form of quantization compared to that of the ordinary monopole. Yet, as we have already noted, even the latter is somewhat artificial in that it derives from the requirement that the momentum operator, which contains the scaling factor eg in question, be self-adjoint. This difficulty does not arise in the case of the spin monopole owing to the absence of any line singularities in the vector potential. As a consequence, different scaling factors of the field are physically acceptable though they lead to a breakdown of rotational invariance in the sense that $\vec{K}$ fails to be a constant of the motion. By contrast with the ordinary monopole this does not lead to any contradiction, a circumstance which may be regarded as a consequence of the coupling of space with spin space through the term $\vec{\sigma}\cdot\vec{r}$ in the field $\vec{B}$.

Though we have used only the spin operators in (5.4)–(5.6) above, this restriction is not essential within the context of SU(2). Indeed, since we have used only the commutativity of the spin operators in establishing these relations, we may replace $\vec{\sigma}$ by $2\vec{L}$, where $\vec{L}$ has components which close on SU(2). That is

$$[L_i, L_j] = i\epsilon_{ijk}L_k\,. \tag{5.7}$$

With this choice we obtain

$$C = (1/e)(\hat{r}\cdot\vec{L})\,. \tag{5.8}$$

The different quantized values of s given in (3.11) correspond exactly to the eigenvalues of this operator.

Finally, we remark that when the vector potential takes the general form

$$\vec{A} = (2\mu/er^2)(\vec{L}\times\vec{r})\,, \tag{5.9}$$

where μ is an arbitrary real constant, then the field $\vec{B}$ is given by

$$\vec{B} = \frac{-4\mu(\mu+1)}{e}\left(\frac{\vec{L}\cdot\vec{r}}{r^4}\right)\vec{r}\,. \tag{5.10}$$

From (5.10) we see that in general two distinct values of μ give rise to the same field. These may be related through a gauge transformation (2.2) with U given by

$$U = \exp i\pi(\vec{L}\cdot\hat{r})\,. \tag{5.11}$$

This acting on (5.9) sends μ into $-\mu - 1$. The field itself is invariant under the gauge transformation because it commutes with U [see Eq. (2.4)]. From this it follows that for a nonzero value of μ to satisfy the constraint relation (5.3) it must be invariant under the above gauge transformation. The only admissible solution is $\mu = -\frac{1}{2}$ and this agrees with the value previously obtained by explicit computation.

VI. THE ELECTRIC SPIN MONOPOLE

We recall that the original motivation for investigating the magnetic monopole was a belief in the existence of a symmetry between the electric and magnetic fields. Now that we have obtained a generalization of the monopole it is an obvious step to go back and look for its electric field analog. This will enable us to identify e with a generalized electric charge. As before, the calculation is straightforward. We set

$$A_0 = -\frac{e}{2}\left(\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\right), \tag{6.1}$$

choosing $\vec{A}$ to be given by (5.5). Then from (2.3) making the identification $B_{0i} = E_i$, we obtain

$$\vec{E} = \frac{e}{2}\left(\frac{\vec{\sigma}\cdot\vec{r}}{r^4}\right)\vec{r}\,, \tag{6.2}$$

which is the expression corresponding to (5.6). It should be noted that in contrast to the Maxwell field an expression of this form is obtained from (6.1) only if the magnetic-spin-monopole term is present. As we have not used in Sec. V the anticommutativity properties of the spin vector in establishing this result, we may for greater generality replace $\vec{\sigma}$ in (5.4)–(5.6), (6.1), and (6.2) by $2\vec{L}$ where $\vec{L}$ is defined through (5.7).

If we replace $\vec{\sigma}\cdot\hat{r}$ by either of its eigenvalues ± 1 in (6.1), then A_0 becomes the potential for a Coulomb field of strength $\pm\frac{1}{2}e$. Furthermore (6.1) gives rise to a point source in the electric field $\vec{E}$ as may be verified by application of the generalized Laplacian operator $D^\mu D_\mu$. Thus

$$D^\mu D_\mu A_0 = D_i D_i A_0 = -4\pi e\,\delta(\vec{r})\tfrac{1}{2}\vec{\sigma}\cdot\hat{r},$$

as required. Though it does not affect the present discussion, we mention for the sake of further application that the quantity $\vec{\sigma}\cdot\hat{r}\delta(\vec{r})$ cannot be regarded as a distribution. This is because $\vec{\sigma}\cdot\hat{r}$ fails to be smooth in the region where $\delta(\vec{r})$ gives its contribution to the integral.

The difficulties encountered through the requirement of Lorentz covariance with this model can be expressed through the fact that A_μ as given by (5.5) and (6.1) is not a Lorentz vector. In particular $A^\mu A_\mu$, which is easily shown to be given by

$$A^\mu A_\mu = \tfrac{1}{4}(e^2 - e^{-2})r^{-2},$$

is not a Lorentz scalar (except when $e=1$, but then A_μ is still not a Lorentz vector). On the other hand, it is rotationally invariant, which is one degree better than the ordinary magnetic monopole. Indeed for the latter we obtain from (3.12), setting $A_0 = e/r$,

$$A^\mu A_\mu = \left[e^2 - g^2\left(\frac{r-z}{r+z}\right)\right]r^{-2}$$

which is not even rotationally invariant. Though for certain values of eg a gauge transformation can be introduced to restore rotational invariance (this being the basis of Dirac's quantization) the validity of this procedure has been challenged in the context of field theory,[9] where the question of covariance is most deeply felt.

VII. ENERGY SPECTRUM FOR A COMBINED ELECTRIC- AND MAGNETIC-SPIN-MONOPOLE FIELD

It is a simple matter to verify that the Hamiltonian (4.1) with A_μ given by (5.5) and (6.1) admits the angular momentum vector $\vec{K}$ and C as constants of the motion. We may use this fact to determine the energy spectrum of this system.

From the relations

$$\begin{aligned} K^2 &= J^2 + e^2 r^2(\vec{r}\cdot\vec{B})^2, \\ J^2 &= r^2p^2 - (\vec{r}\cdot\vec{p})^2 + i\vec{r}\cdot\vec{p}, \end{aligned} \tag{7.1}$$

we obtain the following Schrödinger equation for the Hamiltonian (4.1):

$$\left[-\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}(K^2 - \tfrac{1}{4})\right) + \frac{e}{2}\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\right]\Psi = E\Psi. \tag{7.2}$$

Since $\vec{K}$ behaves as an angular momentum vector and is a constant of the motion we may replace it in (7.2) by its eigenvalues $k(k+1)$; $2k$ is a positive integer. [The eigenvalue $k=0$ is excluded through (7.1) and the positivity of J^2.] Likewise we may replace $\vec{\sigma}\cdot\hat{r}$ by its eigenvalues ± 1. Then (7.2) reduces to the pair of uncoupled radial equations

$$\left[-\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}[k(k+1) - \tfrac{1}{4}]\right) \pm \frac{e}{2r}\right]\psi_\pm(r) = E_\pm\psi_\pm(r). \tag{7.3}$$

The plus sign in (7.3) gives rise to a positive continuous energy spectrum. The minus sign also gives a continuous spectrum for positive energies; but in addition there is a discrete spectrum for negative energies given by[18]

$$E_- = -\tfrac{1}{4}e^2\{(2p+1) + [(2k+1)^2 - 1]^{1/2}\}^{-2},$$

where p is a positive integer.

It is quite straightforward to extend the above analysis to include a contribution from ordinary electric- and magnetic-monopole terms. Again there is no special difficulty involved in treating the spin monopole in the context of the relativistic Dirac equation. Identical results are obtained as in the case of the ordinary monopole[19] so that we do not need to repeat the details here.

Finally we remark that when α as defined by (5.4) does not take the value of $1/2e$, then $\vec{K}$ both fails to behave as an angular momentum vector and to be a constant of the motion. This produces a symmetry-breaking effect in the Hamiltonian. We have not attempted to work out the details of this because, apart from being rather difficult, it is the case of SU(3) which is physically more interesting and where experimental comparison is possible. Though our present model is rather simple, we hope that it serves to indicate a program for future development.

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[12]For the mathematics which underlies a global description of the Yang-Mills field see, for example, S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, London, 1963), Vol. I, Chap. III, pp. 113–153.

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[14]W. Miller, Jr., *Lie Theory and Special Functions* (Academic, New York, 1968), Chap. 6, Theorem 6.3, p. 259. These quantum numbers refer to the unitary representations of the simply connected covering group of E(3). For E(3) itself, s must be an integer. It is not obvious from the outset which choice is required physically and indeed this depends on the choice of $\vec{A}$.

[15]Schwinger's quantization in which half-integer values of eg are excluded [J. Schwinger, Phys. Rev. 144, 1087 (1966)] is obtained if the additional requirement that the magnetic field be a pseudovector be imposed. For a monopole field this requires g to be a pseudoscalar and the remaining part of $\vec{A}$ to be antisymmetric under parity. Thus, with the properly symmetrized form of (3.12) exponentiation of the infinitesimal generators yields the Euclidean group rather than its covering group and consequently half-integer values of eg are excluded from the representation.

[16]See, for example, M. A. Naimark, *Normed Rings* (Noordhoff, The Netherlands, 1964), Chap. I, p. 102. In particular, a self-adjoint operator is required to have a dense domain in the underlying Hilbert space.

[17]The explicit calculation can be found in Ref. 2.

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Space-Time Code. II

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Quantum concepts can be applied to space-time processes to make a quantum (q) theory that is free of the possibility of divergencies inherent in classical continuum theories, yet causal, Lorentz-invariant, and asymptotically Poincaré-invariant for large times. A general technique, algebraic quantization, is provided for going from classical (c) paradigms, typically discrete logical structures, to q analogs. Applied to the two-dimensional c checkerboard, algebraic quantization gives a q theory of time and space asymptotic to the four-dimensional Minkowski c theory in the limit of large time. Applied to the simplest dynamics on such a checkerboard, a piece that makes the same move again and again, algebraic quantization gives a q dynamics asymptotic to a massless spin-$\frac{1}{2}$ two-component dynamics in the same limit. The quantum of time, if it exists, must have spin $\frac{1}{2}$. Some features of general relativity such as curvature seem plausible consequences of a quantum theory of space-time processes.

I. INTRODUCTION

We turn now from quantum geometry to quantum dynamics.[1] The development of physics, if we see it right, looks like

$$\cdots c \cdots cq \cdots q.$$

The c period was the epoch of Newton, Faraday, Maxwell, Einstein. In it, time and matter were both classical systems.

The cq period is now, the time of Heisenberg, Dirac, Tomonoga, Schwinger, and Feynman; a brief interregnum during which vigorous hybrids of c (classical) time and space with q (quantum) matter have been created. In the cq period we have learned to specify the kinematics of matter M