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Current-Algebra Sum Rules in Neutrino Scattering and Scale Dimension of the Chiral-Symmetry-Breaking Hamiltonian

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Current-algebra sum rules for the structure functions W_4 and W_5 which occur in neutrino scattering because of the nonconservation of the axial-vector current are derived and used to get information about the scaling behavior of W_4 and W_5. This information together with the light-cone analysis is used to determine the scale dimension of the chiral-symmetry-breaking Hamiltonian.

I. INTRODUCTION

The purpose of this paper is to discuss some sum rules based on current-algebra techniques for the structure functions W_4 and W_5 which appear in high-energy neutrino scattering due to nonconservation of the axial-vector current. The sum rules provide information about the scaling behavior of these structure functions. This information is then correlated through light-cone analysis to the scale dimension of the chiral-symmetry-breaking term in the Hamiltonian. In this way one is able to get information[1] about the scale dimension of the chiral-symmetry-breaking Hamiltonian.

If broken scale invariance is a useful concept and SU(3)×SU(3) is assumed to be an approximate symmetry of strong interactions, then it is possible to write the Hamiltonian density as[2]

$$H=\Theta_{00}=\overline{\Theta}_{00}+\lambda\delta+\epsilon u\,, \tag{1.1}$$

where $\overline{\Theta}_{00}$ is both chiral- and scale-invariant and has the scale dimension -4, $\lambda\delta$ is chiral-invariant (but violates scale invariance) with scale dimension $l_\delta > -4$, and ϵu violates both chiral and scale invariance and has the scale dimension $l_u > -4$. Thus

$$\begin{aligned}&[D(0),u(0)]=il_u u(0)\,,\\&[D(0),\delta(0)]=il_\delta\delta(0)\,,\end{aligned} \tag{1.2}$$

where $D(0)$ is the dilation operator. Now the divergence of the axial-vector current $A_{i\lambda}$ is given by (we shall confine ourselves to the hypercharge-conserving currents so that $\partial_\lambda V_{i\lambda}=0$)

$$\partial_\lambda A_{i\lambda}=-i[F_i^5,\Theta_{00}]=-i\epsilon[F_i^5,u]\,. \tag{1.3}$$

Further one can easily see by using Eqs. (1.1), (1.2), and (1.3) and the Jacobi identity that

$$[D,\partial_\lambda A_{j\lambda}]=il_u\partial_\lambda A_{j\lambda}\,, \tag{1.4}$$

so that $\partial_\lambda A_{j\lambda}$ has the same dimension l_u as the ϵu term in Eq. (1.1). It is very desirable to measure l_u since we will then get information about the chiral-symmetry-breaking Hamiltonian. One way to get information about l_u is to study such processes in which $\partial_\lambda A_{j\lambda}$ is directly involved. Thus one can get information about l_u by studying in the deep-inelastic region of neutrino scattering the structure functions W_4 and W_5, which are the ones which appear in the neutrino scattering because of the nonconservation of the axial-vector current and are defined more precisely in the next section. The idea is that if W_4 and W_5 scale ($\nu\to\infty$, $\chi=q^2/m\nu$ fixed) as

$$\begin{aligned}&\nu^p W_4(\nu,q^2)\underset{\nu\to\infty}{\longrightarrow}F_4(\chi)\,,\\&\nu^p W_5(\nu,q^2)\underset{\nu\to\infty}{\longrightarrow}F_5(\chi),\end{aligned} \tag{1.5}$$

then p can be related to the scale dimension of the chiral-symmetry-breaking term in Hamiltonian. This is due to the fact that if the scaling limit exists, it can be related to the light-cone ($z^2\to 0$) analysis in configuration space.[3] The light-cone expansion can be described in terms of the scale dimension l_u. This relationship between p and l_u was studied by Fritzsch and Gell-Mann[4] and by Mandula *et al.*[5] However, to get information about p directly from experiments is not possible at present and is unlikely in the near future. It is, therefore, very desirable to find information about p from other theoretical considerations. We show that one gets information about p and hence about l_u by studying the current-algebra sum rules for the structure functions W_4 and W_5, which are derived in the appendix. Our analysis shows that:

(i) If leading behavior [given in Eq. (1.5)] with respect to ν for W_4 and W_5 cancels in the linear combination

$$D(\nu, q^2) = (q^2)^2 W_4(\nu, q^2) - 2m\nu q^2 W_5(\nu, q^2), \quad (1.6)$$

so that $D(\nu, q^2)$ has a sharper falloff than that implied by noncancellation, then there are two possible values of l_u, $l_u = -2$ and -3.

(ii) If the leading behavior for W_4 and W_5 in $D(\nu, q^2)$ does not cancel then $l_u = -3$.

II. SUM RULES

We define (with $\nu = -p\cdot q/m$)

$$\begin{aligned} A_{\mu\nu} &= \tfrac{1}{2}(2\pi)^3 \frac{p_0}{m}\int d^4z\, e^{-iq\cdot z}\langle p|[J_\mu(z), \bar{J}_\nu(0)]|p\rangle \\ &= 2\pi W_2^{\bar{\nu}p}\frac{1}{m^2}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) + 2\pi W_3^{\bar{\nu}p}\frac{1}{2m^2}\epsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta \\ &\quad + 2\pi W_1^{\bar{\nu}p}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + 2\pi W_4^{\bar{\nu}p}\frac{q_\mu q_\nu}{m^2} + 2\pi W_5^{\bar{\nu}p}(p_\mu q_\nu + p_\nu q_\mu)\frac{1}{m^2}, \end{aligned} \quad (2.1)$$

$$\begin{aligned} A_\nu &= \tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\int d^4z\, e^{-iq\cdot z}\langle p|[\partial_\mu J_\mu(z), \bar{J}_\nu(0)]|p\rangle \\ &= 2\pi W_4^{\bar{\nu}p}\frac{1}{m^2}iq^2 q_\nu + 2\pi W_5^{\bar{\nu}p}\frac{1}{m^2}iq^2\left(p_\nu + \frac{p\cdot q}{q^2}q_\nu\right), \end{aligned} \quad (2.2)$$

$$\begin{aligned} A &= \tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\int d^4z\, e^{-iq\cdot z}\langle p|[\partial_\mu J_\mu(z), \partial_\nu\bar{J}_\nu(0)]|p\rangle \\ &= 2\pi W_4^{\bar{\nu}p}\frac{1}{m^2}(q^2)^2 + 2\pi W_5^{\bar{\nu}p}\frac{1}{m^2}q^2 2p\cdot q \\ &= 2\pi\frac{1}{m^2}D^{\bar{\nu}p}(q^2, \nu), \end{aligned} \quad (2.3)$$

where $J_\mu = V_{1+i2\mu} + A_{1+i2\mu}$ is the hypercharge-conserving weak hadronic current; $\bar{J}_\lambda = J_\lambda^\dagger(1-2\delta_{\lambda 4})$ and $\partial_\mu J_\mu = \partial_\mu A_{1+i2\mu}$. The structure functions W_i satisfy the following crossing relations:

$$\begin{aligned} W_i^{\bar{\nu}p}(\nu, q^2) &= -W_i^{\nu p}(-\nu, q^2), \quad i = 1,2,3,4 \\ W_5^{\bar{\nu}p}(\nu, q^2) &= W_5^{\nu p}(-\nu, q^2). \end{aligned} \quad (2.4)$$

For fixed q^2, we have the following sum rule which is derived in the Appendix.

$$\frac{2}{m}\int_0^\infty W_5'^{(+)}(\nu', q^2)d\nu' + 2\int_0^\infty \frac{\nu'}{q^2}W_2'^{(+)}(\nu', q^2)d\nu' = \mathcal{S}, \quad (2.5)$$

where $\mathcal{S} = (p_0/m)(2\pi)^3\langle p|S|p\rangle$ is the operator part of the Schwinger term defined as

$$\langle p|[J_0(\vec{z},0), \bar{J}_i(0)]|p\rangle = -i\langle p|S|p\rangle \times \frac{\partial}{\partial z_i}\delta^3(\vec{z}) + 4\langle p|J_i^3|p\rangle\delta^3(\vec{z}) \quad (2.6)$$

and

$$\begin{aligned} W_{2,5}^{(+)}(\nu, q^2) &= W_{2,5}^{\bar{\nu}p}(\nu, q^2) + W_{2,5}^{\nu p}(\nu, q^2), \\ \nu W_2'^{(+)}(\nu, q^2) &= \nu W_2^{(+)} - m\sum_i \beta_2^i(q^2)\left(\frac{\nu}{m}\right)^{\alpha_i(0)-1}, \\ W_5'^{(+)}(\nu, q^2) &= W_5^{(+)} - m\sum_i \beta_5^i(q^2)\left(\frac{\nu}{m}\right)^{\alpha_i(0)-1}, \end{aligned} \quad (2.7)$$

where α_i is the trajectory with natural parity and even charge conjugation evaluated at $t=0$ and with $\alpha_i(0) > 0$.

Using Eq. (1.5) and the fact that

$$\nu W_2(\nu, q^2) \xrightarrow[\nu\to\infty]{} F_2(\chi),$$

we get from Eq. (2.5)

$$\lim_{q^2\to\infty}\frac{2}{m}\left(\frac{1}{q^2}\right)^{p-1}\int_0^2\left(\frac{1}{m\chi}\right)^{-p+1}\frac{1}{\chi}F_5'^{(+)}(\chi)d\chi + \frac{2}{m}\int_0^2\frac{1}{\chi^2}F_2'^{(+)}(\chi)d\chi = \mathcal{S}. \quad (2.8)$$

The corresponding sum rule for the conserved current is

$$\frac{2}{m}\int_0^2\frac{1}{\chi^2}F_2'^{(+)}(\chi)d\chi = \mathcal{S}. \quad (2.9)$$

The comparison of Eqs. (2.8) and (2.9) suggests that in the Bjorken scaling limit, the nonconserved pieces W_4 and W_5 should fall off faster than the conserved piece W_2 which scales as $\nu W_2 \to F_2(\chi)$. This means that the conserved part of the axial-vector current behaves in the same manner as the vector current in the scaling limit. One then obtains

$$p > 1 \quad (2.10)$$

and

$$p = 1 \quad (2.11a)$$

if

$$\int_0^2 \frac{1}{\chi} F_5'^{(+)}(\chi)d\chi = 0 . \qquad (2.11b)$$

The possibility (2.11) is excluded by considering the following sum rule (derived in the Appendix):

$$\frac{2q^2}{m}\int_0^\infty W_5'^{(+)}(\nu', q^2)d\nu' = \mathcal{A} , \qquad (2.12)$$

where $\mathcal{A}$ is defined by $(\partial_\mu J_\mu = \partial_\mu A_{1+i2\mu})$

$$\langle p|[\partial_\mu J_\mu(\vec{z},0), \overline{J}_0(0)]\,|p\rangle = i\langle p|w(0)|p\rangle\delta^3(z) = i\mathcal{A}\frac{m}{p_0}\frac{1}{(2\pi)^3}\delta^3(\vec{z}) . \qquad (2.13)$$

Using Eq. (1.5), we have from Eq. (2.12), in the scaling limit,

$$\lim_{q^2\to\infty} 2(q^2)^{-p+2}\int_0^2\left(\frac{1}{m\chi}\right)^{-p+2} F_5'^{(+)}(\chi)d\chi = \mathcal{A} . \qquad (2.14)$$

It follows from (2.14) that

$$p = 2 , \qquad (2.15)$$

which is consistent with (2.10) and excludes (2.11a), (2.11b).

We now discuss another set of sum rules which reinforce the previous conclusions about the power p in Eqs. (2.10) and (2.15). In addition one obtains sum rules involving longitudinal cross sections in neutrino scattering. Now $q^2 = \vec{q}^2 - q_0^2 = \overline{q}^2 - \nu^2$, where in the rest frame of the nucleon $q_0 = \nu$ and we have put $\overline{q} = |\vec{q}|$. Assuming an unsubtracted dispersion relation in ν for fixed $\overline{q}$, we can derive sum rules for the structure functions W_i. We get the following sum rules (sum rules are derived in the Appendix; $W_i^{(\pm)} = W_i^{\overline{\nu}p} \pm W_i^{\nu p}$):

$$\frac{2}{m}\int_0^\infty W_5^{(+)}(\nu', \overline{q})d\nu' + \frac{2}{m^2}\int_0^\infty \nu' W_4^{(+)}(\nu', \overline{q})d\nu' + 2\int \frac{1}{\nu'} V_L^{(+)}(\nu', \overline{q})d\nu' = \mathcal{S} , \qquad (2.16)$$

$$\frac{2}{m^2}\int_0^\infty W_4^{(-)}(\nu', \overline{q})d\nu' + 2\int_0^\infty V_2^{(-)}(\nu', \overline{q})d\nu' = 0 , \qquad (2.17)$$

where we have defined

$$\frac{\nu}{q^2}\left(W_2 + \frac{\nu^2}{q^2}W_2 - W_1\right) \equiv \frac{1}{\nu}V_L ,$$
$$\frac{1}{q^2}\left(\frac{\nu^2}{q^2}W_2 - W_1\right) \equiv V_2 . \qquad (2.18)$$

In the scaling limit $\chi = q^2/m\nu$ is fixed, therefore $\overline{q} \to \infty$ such that

$$\nu \sim \overline{q} - \tfrac{1}{2}m\chi ,$$
$$q^2 \sim m\chi\overline{q} . \qquad (2.19)$$

Also we note that in this limit [with $F_L(\chi) = F_2(\chi) - \chi F_1(\chi)$]

$$\frac{1}{\nu}V_L(\nu, \overline{q}) \to \frac{1}{m^2\chi^2}F_L(\chi) = \frac{1}{m\chi}\overline{q}(1 - \tfrac{1}{2}\chi)\frac{1}{\pi}\sigma_L(\chi) ,$$
$$V_2(\nu, \overline{q}) \to \frac{\nu}{q^4}F_2 - \frac{1}{mq^2}F_1 = \frac{1}{\overline{q}m^2\chi^2}F_L(\chi) . \qquad (2.20)$$

Now using the variable χ and Eqs. (2.7) and (2.19), we can express the sum rules (2.16) and (2.17) as follows:

$$\lim_{\overline{q}\to\infty} -2\int_2^{-\infty}(\overline{q})^{-p}F_5^{(+)}(\chi)d\chi - \frac{2}{m}\int_2^{-\infty}(\overline{q})^{1-p}F_4^{(+)}(\chi)d\chi - 2\int_2^{-\infty}\frac{1}{m\chi^2}F_L^{(+)}(\chi)d\chi = 2\mathcal{S} , \qquad (2.21)$$

$$\lim_{\overline{q}\to\infty} -\frac{2}{m^2}\int_2^{-\infty}(\overline{q})^{-p}F_4^{(-)}(\chi)d\chi - \frac{2}{\pi}\int_2^{-\infty}\frac{1}{m\chi}(1 - \tfrac{1}{2}\chi)\sigma_L^{(-)}(\chi)d\chi = 0 . \qquad (2.22)$$

Using causality and a method similar to that used by Stack[6] for the electromagnetic current we can write

$$\begin{aligned}
&F_L^{\overline{\nu}p,\nu p}(\chi) = 0 , && -2 > \chi > -\infty \\
&F_L^{\overline{\nu}p}(\chi) = F_L^{\nu p}(-\chi) , && 2 > \chi > -2 \\
&\sigma_L^{\overline{\nu}p}(\chi) = -\sigma_L^{\nu p}(-\chi) , && 2 > \chi > -2 \\
&F_{4,5}^{\overline{\nu}p,\nu p}(\chi) = 0 , && -2 > \chi > -\infty \\
&F_5^{\overline{\nu}p}(\chi) = (-1)^p F_5^{\nu p}(-\chi) , && \\
&F_4^{\overline{\nu}p}(\chi) = (-1)^{p+1}F_4^{\nu p}(-\chi) , && 2 > \chi > -2 .
\end{aligned} \qquad (2.23)$$

Note that as $\chi \to -\chi$, $\overline{q} \to -\overline{q}$. Using Eqs. (2.23), we can write the sum rules (2.21) and (2.22) as follows:

$$\lim_{\overline{q}\to\infty} 4(\overline{q})^{-p}\int_0^2 F_5^{(+)}(\chi)d\chi + \frac{4}{m}(\overline{q})^{1-p}\int_0^2 F_4^{(+)}(\chi)d\chi + 4\int_0^2\frac{1}{m\chi^2}F_L^{(+)}(\chi)d\chi = 2\mathcal{S} , \qquad (2.24a)$$

$$\lim_{\overline{q}\to\infty}\frac{4}{m^2}(\overline{q})^{-p}\int_0^2 F_4^{(-)}(\chi)d\chi + \frac{4}{\pi}\int_0^2\frac{1}{m\chi}\sigma_L^{(-)}(\chi)d\chi = 0 . \qquad (2.24b)$$

The corresponding sum rules for the conserved current are

$$4\int_0^2\frac{1}{m\chi^2}F_L^{(+)}(\chi)d\chi = 2\mathcal{S} , \qquad (2.25a)$$

$$\frac{4}{\pi}\int_0^2\frac{1}{m\chi}\sigma_L^{(-)}(\chi)d\chi = 0 . \qquad (2.25b)$$

The sum rule (2.25a) for neutrino-nucleon scattering is the analog of the sum rule obtained by several authors[7] for electron-nucleon scattering.

The sum rule (2.25b) is a new sum rule for neutrino-nucleon scattering. The sum rule (2.25b) can also be obtained by combining Adler and Bjorken sum rules[8] in the scaling limit. However, our derivation shows that the sum rule (2.25b) is model-independent in the sense of model dependence of the equal-time commutation relation appearing in the Bjorken sum rule. Comparison of Eqs. (2.24a) and (2.25a), and of Eqs. (2.24b) and (2.25b), does not give any better restriction on p than given by (2.10) and (2.11a). Indeed such a comparison of Eqs. (2.24a) and (2.25a) gives $p>1$ and $p=1$ if

$$\int_0^2 F_4^{(+)}(\chi)d\chi=0, \quad p=1 \tag{2.26a}$$

while that of Eqs. (2.24b) and (2.25b) gives $p\geqslant 1$ and $p=0$ if

$$\int_0^2 F_4^{(-)}(\chi)d\chi=0, \quad p=0. \tag{2.26b}$$

However, the possibilities $0\leqslant p\leqslant 1$ implied by Eqs. (2.26a) and (2.26b) are excluded by the result $p=2$ obtained on the basis of sum rule (2.14).

Finally we discuss sum rules which involve $D(\nu,\bar{q})$. Such sum rules (derived in the Appendix) are

$$\frac{2}{m^2}\int_0^\infty \frac{\nu'}{q'^2}D^{(+)}(\nu',\bar{q})d\nu' +\frac{2}{m}\int_0^\infty \bar{q}^2 W_5^{(+)}(\nu',\bar{q})d\nu'=\mathfrak{A}, \tag{2.27a}$$

$$\frac{2}{m^2}\int_0^\infty \frac{1}{q'^2}D^{(-)}(\nu',\bar{q})d\nu'+\frac{2}{m}\int_0^\infty \nu' W_5^{(-)}(\nu',\bar{q})d\nu'=0. \tag{2.27b}$$

We now assume that in the scaling limit

$$(\nu)^{p_1}D(\nu,\bar{q})\to\phi(\chi), \tag{2.28}$$

so that Eqs. (2.27a) and (2.27b) give

$$\lim_{\bar{q}\to\infty}\frac{4}{m}\int_0^2(\bar{q})^{-p_1}\frac{1}{m\chi}\phi^{(+)}(\chi)d\chi +4\int_0^2(\bar{q})^{-p+2}F_5^{(+)}(\chi)d\chi=2\mathfrak{A}, \tag{2.29a}$$

$$\lim_{\bar{q}\to\infty}\frac{4}{m}\int_0^2(\bar{q})^{-p_1-1}\frac{1}{m\chi}\phi^{(-)}(\chi)d\chi +4\int_0^2(\bar{q})^{-p+1}F_5^{(-)}(\chi)d\chi=0. \tag{2.29b}$$

Let us now consider the two cases mentioned in the Introduction:

(i) If the leading behavior with respect to ν for W_4 and W_5 cancels in the linear combination D[cf. Eqs. (1.6) and (2.28)] so that $D(\nu,q^2)$ has a sharper falloff than that implied by the noncancellation, then $p_1>p-2$. Then Eq. (2.29b) gives $p>1$ and $p=1$ if

$$\int_0^2 F_5^{(-)}(\chi)d\chi=0, \quad p=1 \tag{2.30a}$$

while Eq. (2.29a) gives $p=2$ and the sum rule

$$2\int_0^2 F_5^{(+)}(\chi)d\chi=\mathfrak{A}, \quad p=2 \tag{2.30b}$$

i.e., the sum rule (2.14).

(ii) If the leading behavior for W_4 and W_5 in $D(\nu,q^2)$ does not in general cancel, we have from Eqs. (1.6), (1.5), and (2.28)

$$D(\nu,q^2)\to\nu^{-p_1}\phi(\chi)=m^2\nu^{2-p}[\chi^2F_4(\chi)-2\chi F_5(\chi)], \tag{2.31}$$

so that $p_1=p-2$. Then Eq. (2.29b) gives $p>1$ and $p=1$ if

$$\int_0^2[\chi F_4^{(-)}(\chi)-F_5^{(-)}(\chi)]d\chi=0, \quad p=1 \tag{2.32}$$

while Eq. (2.29a) gives, on using Eq. (2.31), $p=2$ and the sum rule

$$2\int_0^2[\chi F_4^{(+)}(\chi)-F_5^{(+)}(\chi)]d\chi=\mathfrak{A}, \quad p=2 \tag{2.33a}$$

which, on using Eq. (2.14), gives

$$\int_0^2 \chi F_4^{(+)}(\chi)d\chi=\mathfrak{A}, \quad p=2. \tag{2.33b}$$

The sum rules (2.26a) for $p=1$, (2.26b) for $p=0$, (2.30a) for $p=1$, and (2.33b) for $p=2$ are the same as have been obtained by Lee and Mandula[9] by different considerations. However, the possibilities $0\leqslant p\leqslant 1$ implied by Eqs. (2.26a), (2.26b), (2.30a), and (2.32) are excluded by the result $p=2$ obtained on the basis of sum rule (2.14) which is not contained in Ref. 9. We may mention that in Ref. 9 it has not been possible to pin down the value of p which determines the scaling behavior of $W_{4,5}$ while in our case the sum rule (2.14) fixes $p=2$.

We note that the term $\mathfrak{A}$ on the right-hand sides of Eqs. (2.14) and (2.33a), (2.33b) is the nucleon matrix element of the so-called σ term which gives information about chiral symmetry breaking, and therefore the sum rules (2.14) and (2.33a), (2.33b) can in principle give information about this term when one will have experimental information about $W_{4,5}$.

We may also note that Eqs. (2.33a) and (2.33b) imply

$$\int_0^2[\chi F_4^{(+)}(\chi)-2F_5^{(+)}(\chi)]d\chi=0. \tag{2.34}$$

Now from the positivity conditions[10] for structure functions

$$q^2W_4-2m\nu W_5\geqslant 0 \tag{2.35a}$$

so that for $p=2$, we have from Eqs. (1.5), (2.31), and (2.35a)

$$\chi F_4^{(+)}(\chi) - 2F_5^{(+)}(\chi) \geq 0 . \qquad (2.35b)$$

Hence Eq. (2.34) implies

$$\chi F_4^{(+)}(\chi) = 2F_5^{(+)}(\chi) . \qquad (2.36)$$

III. LIGHT-CONE ANALYSIS

We now relate p and p_1 with l_μ by the light-cone analysis. In configuration space, we can write Eq. (2.2) as

$$\tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\langle p|[\partial_\mu J_\mu(z),\ \bar{J}_\nu(0)]\,|p\rangle = (-\partial^2)\left(p_\nu + \frac{p_\lambda\partial_\lambda}{\partial^2}\,\partial_\nu\right)\frac{i}{m^2}\,2\pi W_5(p\cdot z, z^2) + (-\partial^2)(-i\partial_\nu)\frac{i}{m^2}2\pi W_4(p\cdot z, z^2) . \qquad (3.1)$$

Thus

$$[\partial_\mu J_\mu(z), \bar{J}_\nu(0)] = -\partial^2\partial_\nu\,\theta_4(z,0) - (\partial^2\delta_{\nu\lambda} + \partial_\nu\partial_\lambda)\,\theta_{5\lambda}(z,0) . \qquad (3.2)$$

We write[11]

$$\theta_4(z,0) = \sum_n E_n(z, c_n)\,\omega_4^n(z,0) ,$$
$$\theta_{5\lambda}(z,0) = \sum_n E_n(z, c_n')\,\omega_{5\lambda}^n(z,0) , \qquad (3.3)$$

where $E_n(z,c_n)$ and $E_n(z,c_n')$ are singular c-number functions and the ω's are operators. Using the scale transformation $z \to \rho z$, it is easy to see that $E_n(z,c_n)$ and $E_n(z,c_n')$ are homogeneous functions of z of degree c_n and c_n', i.e.,

$$E_n(\rho^{-1}z, c_n) = \rho^{-c_n}E_n(z, c_n) ,$$
$$E_n(\rho^{-1}z, c_n') = \rho^{-c_n'}E_n(z, c_n') , \qquad (3.4)$$

where

$$c_n = l_u + l_J - l_{\omega_4^n} + 3 ,$$
$$c_n' = l_u + l_J - l_{\omega_5^n} + 2 . \qquad (3.5)$$

Here l_u is the scale dimension of $\partial_\mu J_\mu$, l_J the scale dimension of the current J_ν, which we take to be -3, and $l_{\omega_4^n}$ and $l_{\omega_5^n}$ are the scale dimensions of ω_4^n and $\omega_{5\lambda}^n$, respectively. The heart of the light-cone expansions is that we can write

$$\theta_4(z,0) = \sum_n [(z^2 + i\epsilon z_0)^{c_n/2} - (z^2 - i\epsilon z_0)^{c_n/2}]\,\omega_4^n(z,0) ,$$
$$\theta_{5\lambda}(z,0) = \sum_n [(z^2 + i\epsilon z_0)^{c_n'/2} - (z^2 - i\epsilon z_0)^{c_n'/2}]\,\omega_{5\lambda}^n(z,0) . \qquad (3.6)$$

By a Taylor expansion near $z=0$, we have

$$\omega_4^n(z,0) = \sum_k z_{\mu_1}z_{\mu_2}\cdots z_{\mu_k}\omega_4^{(n)\mu_1\mu_2\cdots\mu_k}(0) ,$$
$$\omega_{5\lambda}^n(z,0) = \sum_k z_{\mu_1}z_{\mu_2}\cdots z_{\mu_k}\omega_{5\lambda}^{(n)\mu_1\mu_2\cdots\mu_k}(0) . \qquad (3.7)$$

Following Fritzsch and Gell-Mann[4] we assume that operators ω_4^n and $\omega_{5\lambda}^n$ have dimensions $l \leq -J - 2$, where J corresponds to the Lorentz index of the operator (e.g., $J=0$ and 1, respectively, for ω_4 and $\omega_{5\lambda}$). Thus we exclude operators which do not satisfy this rule (for example, a single operator ϕ, with $l=-1$, $J=0$). Therefore we have in general $l_{\omega_4^n} = -r$ $(r \geq 2)$ and $l_{\omega_5^n} = -1-r$. Hence we have from Eqs. (3.5)

$$c_n = l_u + r \equiv -2d ,$$
$$c_n' = l_u + r \equiv -2d . \qquad (3.8)$$

We note that

$$\langle p|\sum_n z_{\mu_1}z_{\mu_2}\cdots z_{\mu_n}\omega_4^{(n)\mu_1\mu_2\cdots\mu_n}(0)\,|p\rangle\tfrac{1}{2}(2\pi)^3\left(\frac{p_0}{m}\right) = \sum_n f_n(p\cdot z)^n \equiv \frac{2\pi}{m^2}f_4(p\cdot z) ,$$
$$\langle p|\sum_n z_{\mu_1}z_{\mu_2}\cdots z_{\mu_n}\omega_{5\lambda}^{(n)\mu_1\mu_2\cdots\mu_n}(0)\,|p\rangle\tfrac{1}{2}(2\pi)^3\left(\frac{p_0}{m}\right) = \sum_n g_n(p\cdot z)^n p_\lambda \equiv \frac{2\pi}{m^2}f_5(p\cdot z)\,ip_\lambda . \qquad (3.9)$$

Hence, we get, using Eqs. (3.1), (3.2), (3.3), (3.6), (3.8), and (3.9),

$$2\pi W_{4,5}(p\cdot z, z^2) = 2\pi f_{4,5}(p\cdot z)\left[\frac{1}{(z^2 + i\epsilon z_0)^d} - \frac{1}{(z^2 - i\epsilon z_0)^d}\right] . \qquad (3.10)$$

Now

$$W_{4,5}(q\cdot p, q^2) = \int d^4z\, e^{-iq\cdot z}\,W_{4,5}(p\cdot z, z^2)$$

and we thus obtain

$$W_{4,5}(q\cdot p,\ q^2) \underset{\nu\to\infty}{\sim} \frac{-i\pi^2}{\Gamma(d)}\left(\frac{m\nu}{2}\right)^{d-2}\int_{-\infty}^{\infty} d\lambda\,\lambda^{1-d}\,e^{i\pi d/2}\,e^{i\chi\lambda/2}f_{4,5}(\lambda) . \qquad (3.11)$$

Hence we have finally

$$\nu^{2-d}W_{4,5}(\nu,q^2)\underset{\nu\to\infty}{\longrightarrow}F_{4,5}(\chi)\,, \tag{3.12}$$

so that [cf. Eqs. (1.5) and (3.8)]

$$p=2-d=2+\tfrac{1}{2}(l_u+r)\,. \tag{3.13}$$

Similarly writing the relation (2.3) in configuration space, we have

$$\tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\langle p|[\partial_\mu J_\mu(z),\partial_\nu\bar{J}_\nu(0)]|p\rangle=\frac{2\pi}{m^2}D(p\cdot z,z^2)\,. \tag{3.14}$$

The light-cone expansion then gives

$$\begin{aligned}\tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\langle p|[\partial_\mu J_\mu(z),\partial_\nu\bar{J}_\nu(0)]|p\rangle&=\left[\frac{1}{(z^2+i\epsilon z_0)^{d_1}}-\frac{1}{(z^2-i\epsilon z_0)^{d_1}}\right]\langle p|\sum_n z_{\mu_1}z_{\mu_2}\cdots z_{\mu_n}\omega^{(n)\mu_1\mu_2\cdots\mu_n}(0)|p\rangle\times\tfrac{1}{2}(2\pi)^3\frac{p_0}{m}\\&=\left[\frac{1}{(z^2+i\epsilon z_0)^{d_1}}-\frac{1}{(z^2-i\epsilon z_0)^{d_1}}\right]\frac{2\pi}{m^2}f_1(p\cdot z)\,.\end{aligned} \tag{3.15}$$

Thus

$$D(p\cdot q,q^2)=\int d^4z\,e^{-iq\cdot z}D(p\cdot z,z^2)\underset{\nu\to\infty}{\sim}2\pi\frac{-i\pi^2}{\Gamma(d_1)}\left(\frac{m\nu}{2}\right)^{d_1-2}\int_{-\infty}^{\infty}d\lambda\,\lambda^{1-d_1}e^{i\pi d_1/2}e^{i\chi\lambda/2}f_1(\lambda)\,,$$

giving

$$\nu^{2-d_1}D(\nu,q^2)\underset{\nu\to\infty}{\longrightarrow}\phi(\chi)\,, \tag{3.16}$$

where

$$2l_u+r'=-2d_1\,. \tag{3.17}$$

We have assumed that the scale dimension of ω is $-J-r'=-r'$ ($r'\geq 2$). Comparing with Eq. (2.28), we have

$$p_1=2-d_1=2+l_u+\tfrac{1}{2}r'\,. \tag{3.18}$$

Now using Eq. (3.12), we have

$$\begin{aligned}D(\nu,q^2)&=[(q^2)^2W_4(\nu,q^2)-2m\nu q^2W_5(\nu,q^2)]\\&=m^2\nu^2[\chi^2W_4(\nu,q^2)-2\chi W_5(\nu,q^2)]\\&\underset{\nu\to\infty}{\sim}m^2\nu^d[\chi^2F_4(\chi)-2\chi F_5(\chi)]\,.\end{aligned} \tag{3.19}$$

Equation (3.19) holds if the leading behavior of $W_{4,5}$ as given in Eq. (3.12) does not cancel in $D(\nu,q^2)$. We now consider two cases.

(1) The leading behavior of $W_{4,5}$ given in Eq. (3.12) cancels in $D(\nu,q^2)$, so that $D(\nu,q^2)$ falls off faster than implied in Eq. (3.19). In this case, the simplest possibility[4] is to take $r=2$ so that we have, from Eqs. (2.15) and (3.13), $l_u=-2$. However, if we take $r=3$, then l_u would be -3; $r\geq 4$ is not possible since then l_u would be ≤ -4.

Within this case we now discuss the implication of the sum rule (2.29a). Since $D(\nu,q^2)$ falls off faster than implied in Eq. (3.19), it follows from Eq. (2.29a) that $p_1>p-2$. Then from (3.13) and (3.18) one has

$$l_u>-4+(r-r')\,, \tag{3.20}$$

giving $r\geq r'$.

(2) If the leading behavior for W_4 and W_5 in $D(\nu,q^2)$ does not cancel so that $D(\nu,q^2)$ falls off as $\nu\to\infty$ in the way given by Eq. (3.19), then from Eqs. (3.16) and (3.19)

$$-2+d_1=d$$

or

$$l_u=-4+(r-r')\,. \tag{3.21}$$

Since we want $l_u>-4$, r must be greater than r'. If we take[4] $r'=2$, then we get from (3.21)

$$r=l_u+6\,. \tag{3.22}$$

Substituting Eq. (3.22) in Eq. (3.13) and using Eq. (2.15), we obtain $l_u=-3$. If we assume that the fractional dimensions are not admissible, then the value $l_u=-3$ is unique in the sense that any other value of r' greater than 2 is not possible. For example $r'=3$ would give $l_u=-\frac{7}{2}$ while $r'>3$ would give $l_u\leq -4$.

Recently Brown[12] has obtained the result $l_u=-3$ using a different analysis for d_1.

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APPENDIX: DERIVATION OF THE SUM RULES

1. Fixed-q^2 sum rules

Define:

$$F_{\mu\nu} \equiv (2\pi)^3 \frac{p_0}{m} \tilde{F}_{\mu\nu} = (2\pi)^3 \frac{p_0}{m} i \int d^4z\, e^{-iq\cdot z}\, \theta(z_0)\langle p|[J_\mu(z), \bar{J}_\nu(0)]|p\rangle$$
$$= [B_1 p_\mu p_\nu + B_2(q_\mu p_\nu + q_\nu p_\mu) + B_3 q_\mu q_\nu + B_4 \delta_{\mu\nu} + C\epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta], \quad (A1)$$

$$F_\nu \equiv (2\pi)^3 \frac{p_0}{m} \tilde{F}_\nu = (2\pi)^3 \frac{p_0}{m} i \int d^4z\, e^{-iq\cdot z} \theta(z_0)\langle p|[\partial_\mu J_\mu(z), \bar{J}_\nu(0)]|p\rangle$$
$$= i(C_1 p_\nu + C_2 q_\nu), \quad (A2)$$

$$F \equiv (2\pi)^3 \frac{p_0}{m} \tilde{F} = (2\pi)^3 \frac{p_0}{m} i \int d^4z\, e^{-iq\cdot z}\, \theta(z_0)\langle p|[\partial_\mu J_\mu(z), \partial_\nu \bar{J}_\nu(0)]|p\rangle . \quad (A3)$$

Correspondingly we can write

$$A_{\mu\nu} = [b_1 p_\mu p_\nu + b_2(q_\mu p_\nu + q_\nu p_\mu) + b_3 q_\mu q_\nu + b_4 \delta_{\mu\nu} + C\epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta], \quad (A4)$$

$$A_\nu = i(c_1 p_\nu + c_2 q_\nu). \quad (A5)$$

From Eqs. (A.1) and (A.2) we have

$$iq_\mu \tilde{F}_{\mu i} = i\int d^4z\, e^{-iq\cdot z}\, \theta(z_0)\langle p|[\partial_\mu J_\mu(z), \bar{J}_i(0)]|p\rangle + i\int d^4z\, e^{-iq\cdot z} \delta(z_0)\langle p|[J_0, \bar{J}_i(0)]|p\rangle$$
$$= \tilde{F}_i + 4i\langle p|J_i^3|p\rangle + i\langle p|S|p\rangle q_i , \quad (A6)$$

$$-iq_\nu \tilde{F}_\nu = i\int d^4z\, e^{-iq\cdot z}\, \theta(z_0)\langle p|[\partial_\mu J_\mu(z), \partial_\nu \bar{J}_\nu(0)]|p\rangle - i\int d^4z\, e^{-iq\cdot z}\, \delta(z_0)\langle p|[\partial_\mu J_\mu(z), \bar{J}_0(0)]|p\rangle$$
$$= \tilde{F} + \mathfrak{A}\, \frac{m}{p_0}\frac{1}{(2\pi)^3} , \quad (A7)$$

where $J_i^3 = V_{3i} + A_{3i}$ and $(2\pi)^3(p_0/m)\langle p|J_i^3|p\rangle = p_i/m$. Also, we have

$$iq_\mu A_{\mu\nu} = A_\nu , \quad (A8)$$

$$-iq_\nu A_\nu = A . \quad (A9)$$

From Eqs. (A6) and (A8), we have

$$-m\nu B_1 + q^2 B_2 = C_1 + 4/m ,$$
$$-m\nu B_2 + q^2 B_3 + B_4 = C_2 + \mathcal{S} , \quad (A10)$$

$$-m\nu b_1 + q^2 b_2 = c_1 ,$$
$$-m\nu b_2 + q^2 b_3 + b_4 = c_2 , \quad (A11)$$

where

$$\frac{m}{p_0}\frac{1}{(2\pi)^3}\, \mathcal{S} = \langle p|S|p\rangle .$$

Equations (A7) and (A9) give

$$-m\nu C_1 + q^2 C_2 = F + \mathfrak{A} , \quad (A12)$$

$$-m\nu c_1 + q^2 c_2 = A = \frac{2\pi}{m^2} D . \quad (A13)$$

Now the comparison of Eqs. (A4), (A5), and (A13) with the corresponding Eqs. (2.1), (2.2), and (2.3) gives

$$m^2 b_1 \equiv 2\pi W_2 ,$$
$$b_4 \equiv 2\pi W_1 ,$$
$$C \equiv \frac{2\pi}{2m^2} W_3 ,$$
$$\frac{m^2 c_1}{q^2} \equiv 2\pi W_5 , \quad (A14)$$
$$b_2 = \frac{2\pi}{m^2} W_5 + \frac{m\nu}{q^2}\frac{2\pi}{m^2} W_2 ,$$
$$\frac{m^2}{q^2}\left(c_2 - \frac{p\cdot q}{q^2} c_1\right) = 2\pi W_4 ,$$

$$b_3 = \frac{2\pi}{m^2} W_4 + \frac{\nu^2}{q^4} 2\pi W_2 - \frac{2\pi W_1}{q^2} .$$

If we assume unsubtracted dispersion relations in ν for fixed q^2 for the amplitudes B_1, B_2, and C_1 and B_2, $B_4 + q^2 B_3$, and C_2, we get the following sum rules from Eqs. (A10) and (A11):

$$\frac{m}{\pi} \int_{-\infty}^{\infty} b_1^{\overline{\nu}p}(\nu', q^2) d\nu' = \frac{4}{m} \tag{A15}$$

and

$$\frac{m}{\pi} \int_{-\infty}^{\infty} b_2^{\overline{\nu}p}(\nu', q^2) d\nu' = \mathcal{S} . \tag{A16}$$

Similarly from Eqs. (A12) and (A13) we get the sum rule

$$\frac{m}{\pi} \int_{-\infty}^{\infty} c_1^{\overline{\nu}p}(\nu', q^2) d\nu' = \mathcal{a} . \tag{A17}$$

Thus, using Eqs. (A14) and the crossing relations (2.4), we get the sum rules

$$\int_0^{\infty} W_2^{(-)}(\nu', q^2) d\nu' = 2 \tag{A18}$$

and

$$\frac{2}{m} \int_0^{\infty} W_5^{(+)}(\nu', q^2) d\nu' + 2 \int_0^{\infty} \frac{\nu'}{q^2} W_2^{(+)}(\nu', q^2) d\nu' = \mathcal{S} , \tag{A19}$$

$$\frac{2}{m} \int_0^{\infty} q^2 W_5^{(+)}(\nu', q^2) d\nu' = \mathcal{a} , \tag{A20}$$

where

$$W_i^{(\pm)} = W_i^{\overline{\nu}p} \pm W_i^{\nu p} .$$

We recognize that (A18) is the Adler sum rule.[8]

The assumption of unsubtracted dispersion relations may not be applicable to the amplitudes $q^2 B_3 + B_4$ and C_2. However if $B_4 + q^2 B_3$ has Regge asymptotic behavior

$$-\sum_i \beta^i(q^2) \nu^{\alpha_i(0)} \frac{1 + e^{-i\pi\alpha_i}}{\sin\pi\alpha_i} ,$$

then it is possible to construct a modified function $B_4' + q^2 B_3'$ such that

$$B_4' + q^2 B_3' = B_4 + q^2 B_3 + \sum_i \beta^i(q^2) \nu^{\alpha_i(0)} \frac{1 + e^{-i\pi\alpha_i(0)}}{\sin\pi\alpha_i(0)} , \tag{A21}$$

for which we can assume an unsubtracted dispersion relation. Similarly we can construct a modified function C_2'. Hence instead of (A19) and (A20), we get the sum rules (2.5) and (2.13).

In deriving the above sum rules we have taken the point of view that the only fixed poles present are those dictated by the Ward identities (A10) and (A12) where their residues $4/m$, $\mathcal{S}$, and $\mathcal{a}$ are determined by the equal-time commutation relations appearing in (A6) and (A7). As a justification of this point of view we may mention that, as seen above, it has given the Adler sum rule (A18). Had there been fixed poles in addition to what is dictated by the Ward identity (A10) in case of the Adler sum rule, for example, the right-hand side of the Adler sum rule would have contained terms in addition to 2, which is determined by the equal-time commutation relation of currents in Eq. (A6).

2. Fixed-$\overline{q}$ sum rules

We can write Eqs. (A10)–(A13) as

$$-m\nu B_2(\nu, \overline{q}) + (\overline{q}^2 - \nu^2) B_3(\nu, \overline{q}) + B_4(\nu, \overline{q}) = C_2(\nu, \overline{q}) + \mathcal{S} ,$$

$$-m\nu b_2(\nu, \overline{q}) + (\overline{q}^2 - \nu^2) b_3(\nu, \overline{q}) + b_4(\nu, \overline{q}) = c_2(\nu, \overline{q}) , \tag{A22}$$

$$-m\nu C_1(\nu, \overline{q}) + (\overline{q}^2 - \nu^2) C_2(\nu, \overline{q}) = F(\nu, \overline{q}) + \mathcal{a} ,$$

$$-m\nu c_1(\nu, \overline{q}) + (\overline{q}^2 - \nu^2) c_2(\nu, \overline{q}) = A(\nu, \overline{q}) = \frac{2}{m^2} D(\nu, \overline{q}) . \tag{A23}$$

Assuming unsubtracted dispersion relations in ν for B's and C's for fixed $\overline{q}$, we get the sum rule from Eqs. (A22):

$$\frac{m}{\pi} \int_{-\infty}^{\infty} b_2^{\overline{\nu}p}(\nu', \overline{q}) d\nu' + \frac{1}{\pi} \int_{-\infty}^{\infty} (\nu' + \nu) b_3^{\overline{\nu}p}(\nu', \overline{q}) d\nu' = \mathcal{S} .$$

Using the crossing relations (2.4) and Eqs. (A14) and (2.18) we get the sum rules (2.16) and (2.17).

Similarly from Eq. (A23) we get the sum rule

$$\frac{m}{\pi} \int_{-\infty}^{\infty} c_1(\nu', \overline{q}) d\nu' + \frac{1}{\pi} \int_{-\infty}^{\infty} (\nu' + \nu) c_2(\nu', \overline{q}) d\nu' = \mathcal{a} .$$

Using Eqs. (A14), (2.3), and the crossing relations (2.4), we get the sum rules (2.27a) and (2.27b).

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