A clearer picture is provided by the energy-dependent absorptive modification method[14] of the Born amplitude. This method can in principle be applied to any Born amplitude, also to the complicated amplitudes of Secs. II and III. The question of whether this method has also some predictive power depends on whether a *universal* energy-dependent cutoff radius can also be found for peripheral photoproduction processes.

As already noted, the reactions dealt with in Secs. II and III possess no free coupling constants. Modifying the corresponding Born amplitudes by the absorption model and comparing with experiment will therefore show whether the same radius function applies to both processes. The situation will be even more interesting if this radius will turn out to equal the one required in simple hadronic processes.[14]

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Minimal Coupling and Nonlinearization of Chiral Lagrangian Models*

Min-Shih Chen
Brookhaven National Laboratory, Upton, New York 11973

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We investigate the relationship between the linear models and nonlinear models of chiral symmetry in the presence of electromagnetic interaction. It is shown that the minimally coupled linear models and the nonlinear models are related by the same nonlinearization procedure as that in the case without electromagnetic interaction. Therefore, any results involving low-energy photons and pions obtained from these models should be uniquely related.

Recently, calculations have been made on reactions involving low-energy photons and pions by using the linear σ model and nonlinear chiral Lagrangian models.[1] It has been shown that,[2] in the absence of the electromagnetic interaction, the nonlinear models are related to the linear σ model in the limit of infinite σ mass. In the presence of the electromagnetic interaction, one may ask whether these models are still similarly related. If so, then calculations from these different models should give unique results which can serve as an effective-Lagrangian realization of the current algebra involving both low-energy photons and pions. Since the answer has not readily existed in the literature, we investigate in this note the relationship among these models with minimal coupling. Following the procedure of Ref. 2, it can be straightforwardly shown that the minimally coupled nonlinear model is indeed the infinite-σ-mass limit of the minimally coupled linear σ model. In other words, the minimal coupling and nonlinearization procedures are commutative. A more general proof of this relationship is also given.

To proceed, we at first follow Weinberg's nonlinearization procedure[2] of the chiral SU(2)⊗SU(2) σ model,[3,4]

$$\mathcal{L} = -\overline{N}[\gamma^\mu\partial_\mu + m_{N0} - G_0(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)]N - \tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + m_{\pi0}{}^2\vec{\pi}^2) - \tfrac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + m_{\sigma0}{}^2\sigma^2) - (m_{\sigma0}{}^2 - m_{\pi0}{}^2)\left(\frac{G_0{}^2}{8m_{N0}{}^2}(\vec{\pi}^2+\sigma^2)^2 - \frac{G_0}{2m_{N0}}(\vec{\pi}^2+\sigma^2)\sigma\right). \tag{1}$$

A chiral-symmetric nonlinearization can be achieved by defining a new nucleon field N' via a chiral transformation

$$N = (1+\vec{\xi}^2)^{-1/2}(1+i\gamma_5\vec{\tau}\cdot\vec{\xi})N', \tag{2}$$

with $\vec{\xi}$ chosen as

$$\vec{\xi} = G_0\vec{\pi}\{m_{N0} - G_0\sigma + [(m_{N0}-G_0\sigma)^2 + G_0{}^2\vec{\pi}^2]^{1/2}\}^{-1} \tag{3}$$

so that the nonderivative $\overline{N}\gamma_5\vec{\tau}\cdot\vec{\pi}N$ and $\sigma\vec{\pi}^2$ interactions are transformed away. This transformation replaces the $\vec{\pi}$ dependence of $\mathcal{L}$ with a $\vec{\xi}$ dependence. If one defines a new pion field $\vec{\pi}'$ and a chiral scalar field σ' by

$$\vec{\pi}' \equiv 2m_{N0}\vec{\xi}/G_0, \tag{4}$$

$$m_{N0} - G_0\sigma' \equiv [(m_{N0}-G_0\sigma)^2 + G_0{}^2\vec{\pi}^2]^{1/2}, \tag{5}$$

then the Lagrangian (1) can be written in a form such that the chiral scalar field σ' may be dropped everywhere in the limit $m_{\sigma0}\to\infty$. The resulting nonlinear model is

$$\mathcal{L}' = -\overline{N}'\left[\gamma^\mu\partial_\mu + m_{N0} + i\gamma^\mu\left(1+\frac{G_0{}^2\vec{\pi}'^2}{4m_{N0}{}^2}\right)^{-1}\left(\frac{G_0}{2m_{N0}}\gamma_5\vec{\tau}\cdot\partial_\mu\vec{\pi}' + \frac{G_0{}^2}{4m_{N0}{}^2}\vec{\tau}\cdot\vec{\pi}'\times\partial_\mu\vec{\pi}'\right)\right]N' - \frac{1}{2}\left(1+\frac{G_0{}^2\vec{\pi}'^2}{4m_{N0}{}^2}\right)^{-2}\partial_\mu\vec{\pi}'\cdot\partial^\mu\vec{\pi}' - \frac{1}{2}\left(1+\frac{G_0{}^2\vec{\pi}'^2}{4m_{N0}{}^2}\right)^{-1}m_{\pi0}{}^2\vec{\pi}'^2. \tag{6}$$

To incorporate an electromagnetic interaction via minimal coupling, replacements on the derivatives of the nucleon and pion fields are made by

$$\partial_\mu N \to (\partial_\mu - ieA_\mu q)N, \tag{7}$$

$$\partial_\mu\vec{\pi} \to (\partial_\mu - ieA_\mu Q)\vec{\pi}, \tag{8}$$

where q and Q are, respectively, the charge matrices for the nucleon and pion fields. In terms of the two- and three-dimensional representations of the isospin matrices τ_3 and T_3, the charge matrices can be written as $q=\frac{1}{2}(1+\tau_3)$ and $Q=T_3$. The nucleon part of the Lagrangian (1) thus should obtain an additional term $ie\overline{N}\gamma^\mu A_\mu qN$. Under the transformation (2), this term becomes

$$\begin{aligned} ie\overline{N}\gamma^\mu A_\mu qN &= ie\overline{N}'(1+i\gamma_5\vec{\tau}\cdot\vec{\xi})\gamma^\mu qA_\mu(1+i\gamma_5\vec{\tau}\cdot\vec{\xi})(1+\vec{\xi}^2)^{-1}N' \\ &= ie\{\overline{N}'\gamma^\mu A_\mu qN' + \tfrac{1}{2}i\overline{N}'\gamma^\mu A_\mu(1+\vec{\xi}^2)^{-1}[\vec{\tau}\cdot\vec{\xi},\tau_3](1+i\gamma_5\vec{\tau}\cdot\vec{\xi})N'\}. \end{aligned} \tag{9}$$

For the commutator we have

$$[\vec{\tau}\cdot\vec{\xi},\tau_3] = 2i\tau_a\xi_{3ab}\xi_b = 2\vec{\tau}\cdot(Q\vec{\xi}), \tag{10}$$

since $i\xi_{3ab}$ is just the charge-matrix element $Q_{ab} = (T_3)_{ab}$. Using the identity

$$(\vec{\tau}\cdot\vec{a})(\vec{\tau}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\tau}\cdot(\vec{a}\times\vec{b}) \tag{11}$$

and the fact that

$$\vec{\xi}\cdot(Q\vec{\xi}) = i\xi_{3ab}\xi_a\xi_b = 0, \tag{12}$$

it is straightforward to show that

$$ie\overline{N}\gamma^\mu A_\mu qN = ie\overline{N}'\{\gamma^\mu A_\mu q + i\gamma^\mu(1+\vec{\xi}^2)^{-1}[\gamma_5\vec{\tau}\cdot(Q\vec{\xi}) + \vec{\tau}\cdot\vec{\xi}\times(Q\vec{\xi})]\}N'. \tag{13}$$

The first term on the right-hand side of Eq. (13) is the term needed for the minimal coupling to the new nucleon field N'. Under the definition of the new pion field (4), the other two terms in Eq. (13) are indeed the terms needed for the minimal coupling to the new pion field $\vec{\pi}'$ in the pion-nucleon interaction terms of the nonlinear Lagrangian (6). For the meson part of the Lagrangian (1), a term

$$e^2A_\mu A^\mu(Q\vec{\pi})\cdot(Q\vec{\pi})$$

should be added to incorporate minimal coupling. Inverting Eqs. (3)–(5), we can express $\vec{\pi}$ in terms of $\vec{\pi}'$ and σ' as

$$\vec{\pi}=\left(1-\frac{G_0\sigma'}{m_{N0}}\right)\left(1+\frac{G_0{}^2\vec{\pi}'^2}{4m_{N0}{}^2}\right)^{-1}\vec{\pi}' . \tag{14}$$

The minimal-coupling term can therefore be written as

$$e^2A_\mu A^\mu(Q\cdot\vec{\pi})\cdot(Q\vec{\pi}) =\left(1-\frac{G_0\sigma'}{m_{N0}}\right)^2\left(1+\frac{G_0{}^2\vec{\pi}'^2}{4m_{N0}{}^2}\right)^{-2}(Q\vec{\pi}')\cdot(Q\vec{\pi}') . \tag{15}$$

In the limit $m_{\sigma0}{}^2\to\infty$, the chiral scalar field σ' can be dropped everywhere and thus the right-hand side of Eq. (4) is also just the term needed for minimal coupling to the pion part of the nonlinear Lagrangian (6).

Thus we have demonstrated that the minimally coupled linear σ model and Weinberg's nonlinear model are related by the limit $m_{\sigma0}{}^2\to\infty$ just as they are in the absence of the minimal coupling. This relationship is also true for more general linear and nonlinear chiral Lagrangian models. In general, the Lagrangian for a linear chiral SU(2) $\otimes$SU(2) model can be written as

$$\mathcal{L}=\mathcal{L}(\phi, \partial_\mu\phi, N, \partial_\mu N) , \tag{16}$$

where ϕ is the $(\frac{1}{2},\frac{1}{2})$ representation of the chiral SU(2)$\otimes$SU(2) group and transforms as a four-vector. This four-vector ϕ can be expressed in terms of Φ and $U(\vec{\xi})$, where Φ is the length of the four-vector and $U(\vec{\xi})$ represents a rotation which leaves the length of a four-vector invariant. The linear Lagrangian (16) can be rewritten as

$$\mathcal{L}=\mathcal{L}(\Phi, \vec{\xi}, \partial_\mu\Phi, \partial_\mu\vec{\xi}, N, \partial_\mu N) . \tag{17}$$

To nonlinearize the model, we make a chiral transformation

$$N=u(\vec{\xi})N' \tag{18}$$

such that the nonderivative meson-nucleon coupling term is rotated into the form $\bar{N}'\Phi N'$ and therefore can be eliminated in a chiral-invariant manner. By letting the mass of the chiral scalar field Φ go to infinity, the Φ field can be dropped everywhere and the Lagrangian (17) becomes

$$\mathcal{L}'=\mathcal{L}(\vec{\xi}, \partial_\mu\vec{\xi}, u(\vec{\xi})N', u(\vec{\xi})\partial_\mu N'+[\partial_\mu u(\xi)]N') , \tag{19}$$

which is a general nonlinear model with a new pion field $\vec{\pi}'$ defined in terms of the $\vec{\xi}$ as

$$\vec{\pi}'=\vec{\xi}f(\vec{\xi}^2) . \tag{20}$$

To incorporate minimal coupling into the linear Lagrangian (17), we make the substitution given by Eqs. (7) and (8). Under the chiral transformation (18), we have

$$\begin{aligned}(\partial_\mu-ieA_\mu q)N&=u(\vec{\xi})\partial_\mu N'+(\partial_\mu u)N'-ieA_\mu q u(\vec{\xi})N'\\ &=u(\vec{\xi})(\partial_\mu-ieA_\mu q)N'\\ &\quad+\{\partial_\mu u(\vec{\xi})-ie[q,u(\vec{\xi})]\}N' .\end{aligned} \tag{21}$$

Since $u(\vec{\xi})$ is a two-dimensional chiral transformation matrix, it can in general be written as

$$u=u(i\gamma_5\vec{\tau}\cdot\vec{\xi}) . \tag{22}$$

Using the commutation relation (10) and Eqs. (21), (22), it is straightforward to show that

$$(\partial_\mu-ieA_\mu q)N=u(\vec{\xi})(\partial_\mu-ieA_\mu q)N' +iu'\gamma_5\vec{\tau}\cdot(\partial_\mu\vec{\xi}-ieA_\mu Q\vec{\xi})N' , \tag{23}$$

where $u'\equiv\delta u/\delta(i\gamma_5\vec{\tau}\cdot\vec{\xi})$. For the $\vec{\xi}$ field, we just make the substitution

$$\partial_\mu\vec{\xi}\to\partial_\mu\vec{\xi}-ieA_\mu Q\vec{\xi} . \tag{24}$$

As a result, we can see that the Lagrangian for the minimally coupled linear model in the infinite-Φ-mass limit takes the form

$$\mathcal{L}=\mathcal{L}(\vec{\xi}, (\partial_\mu-ieA_\mu Q)\vec{\xi}, N, u(\vec{\xi})(\partial_\mu-ieA_\mu q)N' +iu'\gamma_5\vec{\tau}\cdot[(\partial_\mu-ieA_\mu Q)\vec{\xi}]N') . \tag{25}$$

For the new pion field $\vec{\pi}'$ defined by Eq. (20), we can invert the definition as

$$\vec{\xi}=\vec{\pi}'g(\vec{\pi}'^2) . \tag{26}$$

Differentiating Eq. (26) gives

$$\partial_\mu\vec{\xi}=(\partial_\mu\vec{\pi}')g(\vec{\pi}'^2)+\vec{\pi}'\frac{dg}{d\vec{\pi}'^2}\vec{\pi}'\cdot(\partial_\mu\vec{\pi}') , \tag{27}$$

while multiplying Eq. (26) by the charge matrix gives

$$\begin{aligned}Q\vec{\xi}&=Q\vec{\pi}'g(\vec{\pi}'^2)\\ &=Q\vec{\pi}'g(\vec{\pi}'^2)+\vec{\pi}'\frac{dg}{d\vec{\pi}'^2}\vec{\pi}'\cdot(Q\vec{\pi}') ,\end{aligned} \tag{28}$$

where we have used the fact that $\vec{\pi}'\cdot(Q\vec{\pi}')=0$. Thus we have

$$(\partial_\mu-ieA_\mu Q)\vec{\xi}=(\partial_\mu-ieA_\mu Q)\vec{\pi}'g(\vec{\pi}'^2) +\vec{\pi}'\frac{dg}{d\vec{\pi}'^2}\vec{\pi}'\cdot(\partial_\mu-ieA_\mu Q)\vec{\pi}' . \tag{29}$$

From Eqs. (25) and (29), it can be seen that the Lagrangian of the linear model with minimal coupling, in the limit of infinite Φ mass, does take the form

$$\mathcal{L}'=\mathcal{L}'(\vec{\pi}', (\partial_\mu-ieA_\mu Q)\vec{\pi}', N', (\partial_\mu-ieA_\mu q)N') , \tag{30}$$

which is the minimally coupled nonlinear model.

To conclude, we have shown that the minimal-coupling and nonlinearization procedures are commutative, that is, the minimally coupled nonlinear model is indeed the infinite-chiral-scalar-mass limit of the linear model. Therefore, just as is the case without the electromagnetic interaction, any results obtained from the linear models, independent of the chiral scalar mass, should be uniquely related to the corresponding results obtained from the nonlinear models. This conclusion can also be straightforwardly generalized to models concerning the chiral SU(3)⊗SU(3) symmetry.

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η and Pion Decays into Lepton Pairs*

Mahendra Pratap and Jack Smith
The Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790
(Received 14 January 1972)

A recent measurement of the branching ratio $\Gamma(\eta\to\mu^+\mu^-)/\Gamma(\eta\to 2\gamma)$ yielded the value $(5.9\pm2.2)\times10^{-5}$. We examine here the implication of this result on theoretical predictions for the real part of the $\eta\to\mu^+\mu^-$ decay amplitude and conclude that the branching ratio is uncomfortably large for most of the models proposed so far. We then evaluate the $\eta\to\mu^+\mu^-$ decay rate in the baryon loop model. Predictions are also made for the associated decays $\eta\to\gamma\mu^+\mu^-$, $\pi^0\to e^+e^-$, and $\pi^0\to\gamma e^+e^-$.

I. INTRODUCTION

Recently, measurements have been reported by Hyams *et al.*[1] on the decay $\eta\to\mu^+\mu^-$ and by Clark *et al.*[2] on the decay $K_L^0\to\mu^+\mu^-$. Since these decays can proceed electromagnetically through a two-photon state it is instructive to give the experimental numbers as branching ratios, i.e., $\Gamma(\eta\to\mu^+\mu^-)/\Gamma(\eta\to2\gamma)=(5.9\pm2.2)\times10^{-5}$ and $\Gamma(K_L^0\to\mu^+\mu^-)/\Gamma(K_L^0\to2\gamma)<0.31\times10^{-5}$ (90% confidence level). To the extent that neutral currents are neglected the only other physical states permissible in these decays are $2\pi\gamma$, 3π, and $3\pi\gamma$. We will assume for the moment that these states can be neglected completely. Then the decay amplitudes for $\eta\to2\gamma$ and $K_L^0\to2\gamma$ are real and can be taken from experiments. If we further assume *CP* invariance and standard quantum electrodynamics for the muon interaction then the above branching ratios should be almost identical because the masses of the η and K mesons are approximately equal. Numerically one finds[3] $\Gamma(\eta\to\mu^+\mu^-)/\Gamma(\eta\to2\gamma)\geq1.07\times10^{-5}$ and[4] $\Gamma(K_L^0\to\mu^+\mu^-)/\Gamma(K_L^0\to2\gamma)\geq1.17\times10^{-5}$. These values, the so-called "unitarity bounds," are expected to be very reliable. There will be of course a small variation due to interferences with other physical states in the imaginary parts of the amplitudes. However, the addition of the real parts of the amplitudes can only increase the branching ratios given above.

It is obvious that the experimental results are significantly different from the unitarity bounds. The result for η decay is compatible with its bound whereas the result for K_L^0 decay is not. As regards the latter, several attempts have been made to explain the discrepancy by allowing interference with other channels, i.e., the $2\pi\gamma$, 3π, and $3\pi\gamma$. Original estimates made by Martin, de Rafael, and Smith[5] have been improved by Farrar and Treiman,[6] Gaillard,[7] Aviv and Sawyer,[8] and Adler,