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Application of Regge Calculus to the Axially Symmetric Initial-Value Problem in General Relativity

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Using Regge calculus some axially symmetric solutions of the initial-value equations of general relativity are constructed at the moment of time symmetry. Examples of infinite and closed spaces are given.

I. INTRODUCTION

In general relativity one is concerned with curved 4-dimensional spaces. Such spaces may be approximated by a collection of 4-dimensional blocks (4-simplexes) such that the geometry inside each block is Euclidean. This collection of blocks will not fit together in a 4-dimensional Euclidean space – there will be "rattle" between the blocks. It has been shown[1] by Regge that Einstein's equations for a curved empty space may be translated into constraints on the blocks. This approach to general relativity using simplectic nets is known as Regge calculus.

The calculational problems involved in constructing 4-dimensional solutions of Einstein's equations using Regge calculus are somewhat daunting, especially since one cannot visualize a set of 4-dimensional blocks hinging together on triangles. Therefore we shall restrict ourselves to the comparative simplicity of the initial-value problem. For an empty space possessing a time-symmetric metric the initial-value equations at the moment of time symmetry reduce to[2]

$$^{(3)}R=0\,. \tag{1}$$

Therefore the solution of the initial-value equations in this particular case reduces to finding a 3-geometry having vanishing curvature scalar. When translated into Regge-calculus terms this condition becomes[2]

$$\sum_i l_i\delta_i=0\,, \tag{2}$$

where l_i is the length of the edge i and the sum is over all the block edges meeting at a given vertex. δ_i is the deficit angle between the sum of the dihedral angles between block faces meeting at the edge i and the normal value (i.e., when there is no "rattle") of 2π.

In an application to the time-symmetric initial-value problem for geometries possessing spherical symmetry, Wong has shown[3] that Regge calculus can reproduce the Schwarzschild and Reissner-Nordström geometries with an accuracy better than 1% by dividing the space into spherical shells. By dividing the space into icosahedral shells errors of about 10% arise.

In this paper we shall consider the initial-value problem for spaces with axial symmetry.[4] Our aim will be to demonstrate the existence of Regge-calculus representations for solutions of Eq. (1) possessing axial symmetry rather than trying to produce good approximations to known analytic solutions. With this purpose in mind we shall use fairly "coarse" subdivisions of spaces into blocks. We shall assume that, by taking a finer subdivision, more accurate representations of the solutions being considered may be obtained.

II. GEOMETRIES POSSESSING A NONSPHERICAL THROAT

Our first application of Regge calculus will be to solutions of Eq. (1) having a geometry possessing two asymptotically Euclidean regions connected by a nonspherical throat. The existence of such solutions may be seen by a simple transformation of the coordinates in the Schwarzschild metric.

The basic problem encountered in setting up a Regge-calculus model for this, or any other, geometry is finding a suitable choice of blocks with which to approximate it. The choice must be such that the satisfaction of the vertex conditions completely determines each block. However this choice must not be so restrictive as to preclude any solutions other than a Euclidean one. In addition we have the requirement of axial symmetry and, in this particular case, asymptotic flatness of the solution.

For the present problem we divide the space into shells constructed from twelve blocks: four blocks around the axis of symmetry and four blocks forming each end. The blocks around the axis of symmetry have rectangular side, top, and bottom faces, while those at the ends have parallel equilateral triangles as their top and bottom faces (see Figs. 1 and 2). We have chosen the distance

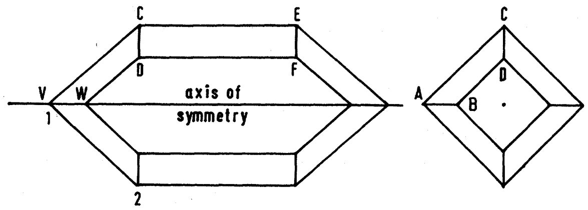

FIG. 1. The basic shell used for the single-throat geometry.

VW to be b for each shell. This just provides a thickness scale for the shells. The distance d is, by construction, the same for each shell. Given the values of α, β, γ, and a for any one shell the vertex conditions are sufficient to determine their value for all others. It is convenient to start with the shell at the throat for which $\alpha=\beta=\gamma=90°$.

The vertex conditions of Eq. (2) are given by

$$b(\pi-\theta_i-\theta_{i-1})+2a_i(\phi_{i-1}-\phi_i)=0 \tag{3}$$

for vertices of type 1 as indicated in Fig. 1, where

θ_i = angle between faces WVC and WVA

$$=\cos^{-1}\left(\frac{1-2\cos^2\gamma_i}{2\sin^2\gamma_i}\right),$$

ϕ_i = angle between faces WVC and WDB

$$=\cos^{-1}(\cot\gamma_i/\sqrt{3}),$$

and for type-2 vertices by

$$c_i(\pi-2\eta_i)+c_{i-1}(\pi-2\eta_{i-1})+2d(\beta_{i-1}-\beta_i)+2a_i(\phi_{i-1}-\phi_i+\mu_{i-1}-\mu_i)=0, \tag{4}$$

where

η_i = angle between faces WVC and ACD

$$=\cos^{-1}\left(\frac{1-2\cos\alpha_i\cos\beta_i}{2\sin\alpha_i\sin\beta_i}\right),$$

μ_i = angle between faces WDB and ACD

$$=\cos^{-1}\left(\frac{2\cos\alpha_i-\cos\beta_i}{\sqrt{3}\,\sin\beta_i}\right),$$

and

$$c_i=\frac{\sin\gamma_i}{\sin\alpha_i}.$$

In addition to the vertex conditions we have a subsidiary condition arising from the requirement that V, W, D, and C should lie in a plane. This provides a constraint on α, β, and γ:

$$\sin\alpha_i\cot\gamma_i=2\cos\beta_i-\cos\alpha_i\,. \tag{5}$$

Using Eqs. (3)–(5) we may solve for α_i, β_i, and γ_i which allow us to completely specify the ith shell given the $(i-1)$th shell. These equations may be solved numerically on a computer by starting at the throat and first solving Eq. (3) for γ. Equation (4) is then solved subject to the constraint on α and β imposed by Eq. (5). The same procedure is then repeated for subsequent shells.

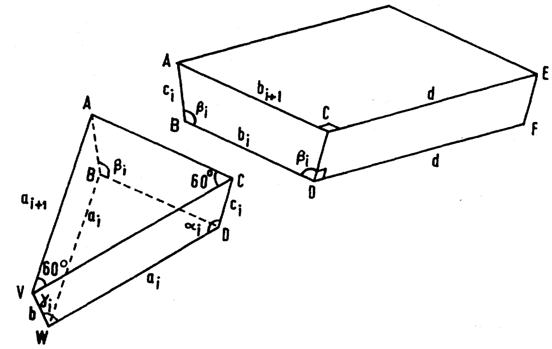

FIG. 2. Details of the blocks used to construct the shell of Fig. 1.

A cautionary remark on solutions to the vertex equations occurring in Regge calculus is appropriate here. In general it is possible that more than one solution may exist. These additional solutions arise because we consider shells of finite thickness. As we decrease this thickness these solutions move on the (α, β) plane. We shall only give solutions for which the values of α and β for all neighboring shells can be made arbitrarily close by making the shell thickness sufficiently small.

In Fig. 3 and Table I we illustrate the results obtained taking $a_0=2$, $a_0/b=5$ and for $d=1$ and 3.

We see that as we move away from the throat α, β, and γ increase, tending asymptotically to a value of 135°. This value corresponds to Euclidean geometry. The sequence of blocks obtained on both sides of the throat is the same. Therefore the solutions we obtain by using shells of the kind shown in Fig. 1 describe the geometry on a space possessing two Euclidean regions connected by a throat. This throat has a circumference of $4a_0$ in a plane perpendicular to the axis of symmetry and $4a_0+2d$ in a plane containing this axis.

III. SOLUTIONS WITH TWO THROATS

We now consider solutions to Eq. (1) which possess two throats. In order to do this we shall consider a space built up from blocks as shown in Fig. 4(a). This consists of a series of shells of the type considered in Sec. II enclosing two "spherical" throats each constructed out of eight blocks having identical equilateral-triangular top and bottom faces and rectangular sides (i.e., $a_0=a_1$, $\gamma_0=90°$). The throats are separated by four blocks of the kind shown in Fig. 4(b). In this problem we

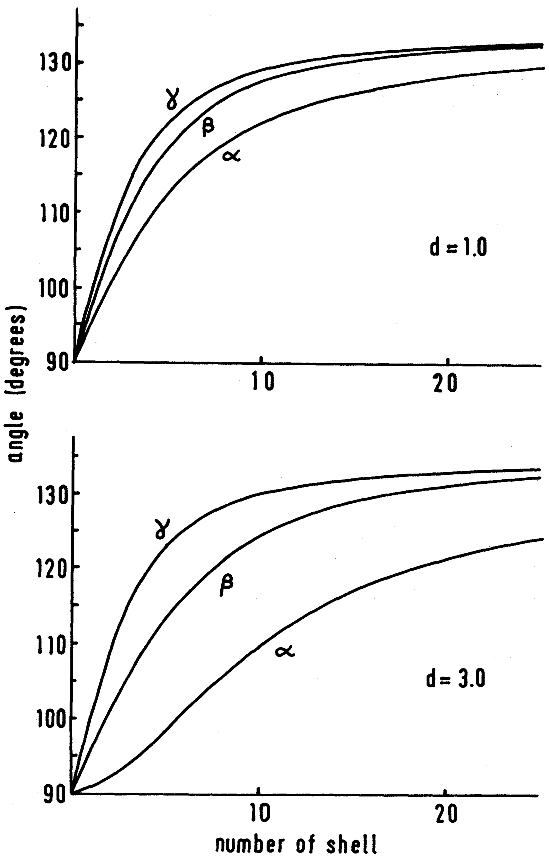

FIG. 3. Values of α, β, and γ for the shells near the throat.

have four different types of vertex at which Eq. (2) must be satisfied. Two of these correspond to vertices 1 and 2 considered in Sec. II. For the vertices 3 and 4 as shown in Fig. 4(a), Eq. (2) becomes

$$\tfrac{1}{3}\pi b + g(\pi - 2r) + 2a(\pi - 2s) = 0\,, \tag{6}$$

where

r = angle between faces LMP and LNQ

$$= \cos^{-1}\left(\frac{1-\cos^2\delta}{2\sin^2\delta}\right),$$

s = angle between faces LNM and LMP

$$= \cos^{-1}\left(\frac{\cot\delta}{\sqrt{3}}\right),$$

and

$$a(3\pi - 2\phi_1 - 2s - 2\mu_1 - 2u) + d(2\pi - 2t - 2\beta_1) + b\left(\frac{2\pi}{3} + (\pi - 2\eta_1)\frac{\sin\gamma_1}{\sin\alpha_1}\right) = 0\,, \tag{7}$$

where

TABLE I. Values of α, β, and γ for $a_0 = 2$, $b = 0.04$.

	$d=1$			$d=3$		
No. of shell	α (deg)	β (deg)	γ (deg)	α (deg)	β (deg)	γ (deg)
0	90.00	90.00	90.00	90.00	90.00	90.00
1	95.67	97.93	100.09	90.82	95.51	100.08
10	122.21	127.70	129.19	109.82	124.40	130.05
100	133.80	134.57	134.58	132.66	134.69	134.74
500	134.77	134.92	134.92	134.57	134.95	134.96
1000	134.89	134.96	134.96	134.79	134.98	134.98

$d = g - 2a\cos\delta$,

t = angle between faces LMP and NMP

$$= \tfrac{1}{2}(\pi - r)\,,$$

u = angle between faces LNM and NMP

$$= \cos^{-1}\left(\frac{-2\cos\delta}{\sqrt{3}}\right).$$

α, β, and γ must also satisfy Eq. (5).

Given a_0, b, and g we may use Eqs. (3) and (6) to determine γ and δ and then use Eqs. (5) and (7) to find α and β for the first shell. α, β, and γ for subsequent shells can then be calculated as described in Sec. II.

As a particular example if we choose $a_0 = 2$, $b = g = 1$, we find a solution exists for $\delta = 112.24°$. The values of α, β, and γ are shown in Fig. 5 and Table II.

We see that α, β, and γ again tend to 135°, indicating that the solution is asymptotically flat. To complete the solution we have to consider the geometry "inside" the throats. A certain amount of freedom exists here. For example, we could

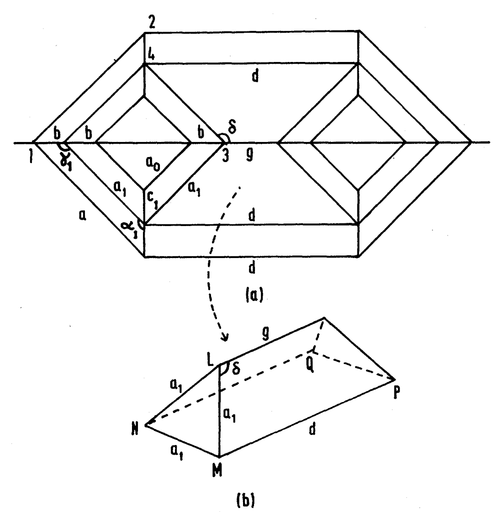

FIG. 4. Block construction used to find solutions with two throats.

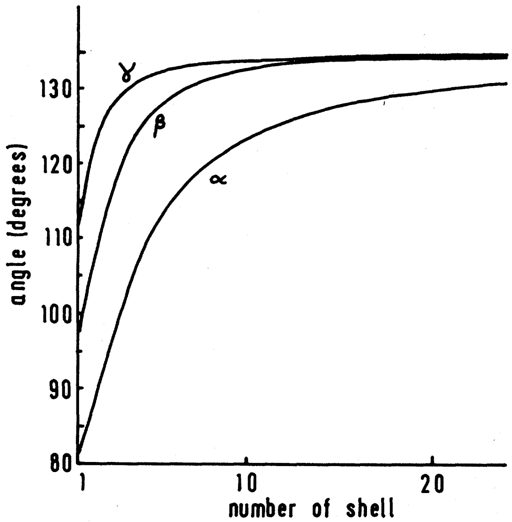

FIG. 5. Values of α, β, and γ for the first few shells for the two-throat solution.

identify the blocks at the two throats which would result in our solution corresponding to a "wormhole".[5] Alternatively we could take the system of blocks on either side of the throats to be identical. This gives rise to a solution corresponding to the geometry on a space having two asymptotically flat regions connected by two throats. This would be an example of an Einstein-Rosen geometry.[6]

Neither of these solutions is static. At the moment of time symmetry they may be interpreted as representing two identical objects of equal mass which are instantaneously at rest. At a later time we expect Einstein's equations to indicate a time dependence which corresponds to a movement of the bodies. As in the case of the Schwarzschild geometry[7] we expect the solution to become singular after a finite time.

IV. CLOSED-SPACE SOLUTIONS

The existence of closed-space topologies satisfying Eq. (2) may easily be seen by considering the following construction:

Take a set of cubes arranged in a rectangular pattern and roll them up into a cylinder by identifying (joining up) the faces of cubes on a pair of opposite edges of the rectangle. Now identify the faces of the blocks at the ends of the cylinder to obtain a toroidal shell. Identifying the inside and outside faces of each cube we obtain a closed space (a 3-dimensional torus, T^3) satisfying Eq. (2) at each vertex.

TABLE II. Values of α, β, and γ for $a_0=2$, $b=g=1$.

No. of shell	α (deg)	β (deg)	γ (deg)
1	81.26	97.24	112.24
2	89.82	109.46	123.80
10	123.54	132.68	133.94
50	133.12	134.82	134.85
100	134.10	134.92	134.93
200	134.56	134.97	134.97

Now consider the following simpler topology:

Take a row of N rectangular blocks having another block at either end whose upper and lower faces are identical triangles (see Fig. 6). Roll up the strip and join up the two exposed sides of the rectangular blocks. Now join all points marked A in the diagram. Do the same with the points B, H, and G. Having done this we have a shell rather similar to that considered in Sec. I. However, we now identify the internal and external faces of each block to obtain a 3-dimensional doughnut ($S^2\times S^1$).

Unlike the previous example it is not obvious that this construction can satisfy Eq. (2). The constraints obtained by demanding that Eq. (2) be satisfied at a vertex of the type at A and at a vertex of the type at J in Fig. 6 are given by

$$d(2\pi - mN)+Nc(\pi-2n)=0 \tag{8}$$

and

$$a(\pi-2p)+b(\pi-2l)+c(\pi-2n)=0\,, \tag{9}$$

where

$a=2c\cos\beta$,

$d=b-2c\cos\alpha$,

$l=\frac{1}{2}(\pi-m)=$ angle between faces ABC and CDJ

$=\cos^{-1}\left(\dfrac{\cos\beta}{\sin\alpha}\right)$,

$n=$ angle between faces ABC and ACJ

$=\cos^{-1}(-\cot\alpha\cot\beta)$,

$p=$ angle between faces ACJ and CDJ

$=\cos^{-1}\left(\dfrac{\cos\alpha}{\sin\beta}\right)$.

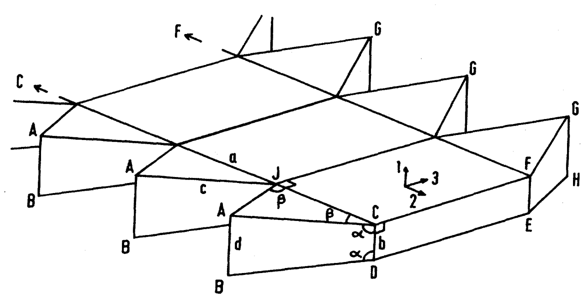

FIG. 6. Blocks used in the construction of doughnut solutions.

We find that, given a/b, these equations are sufficient to determine α and β. In Table III we show the solutions obtained for $a=2$, $b=1$ for various choices of N.

We see that no solutions exist for $N<9$. This may be interpreted as indicating that the ratio of the distance round the doughnut in direction 1 (see Fig. 6) to the distance in direction 2 must exceed some minimum value before solutions corresponding to the construction of Fig. 6 can exist. Numerically we find that this ratio (~18) is approximately the same for all N. We see that the distance round the doughnut in direction 3 can have any value as the distance CF does not enter into the vertex conditions.

TABLE III. Values of α and β for the doughnut solution.

N	α (deg)	β (deg)
7	...	...
8	...	...
9	122.32	63.18
10	118.16	65.08
11	115.25	66.93
12	112.95	68.60
13	111.08	70.08
14	109.51	71.38
15	108.15	72.53

V. CONCLUSIONS

We have seen how axially symmetric solutions to the initial-value problem at the moment of time symmetry may be found using the techniques of Regge calculus. Due to the coarseness of the subdivision of space used, these solutions are only approximate. However, by using a more complicated set of blocks and a finer subdivision better approximations may be obtained. Although it does not provide an exact analytic solution, Regge calculus has several advantages over the usual method of finding solutions to Eq. (1) using coordinates. As we have seen in this paper it provides a means of handling complex topological situations without encountering difficulties from coordinate singularities and at the same time allowing the topology of these solutions to be easily seen. We have only considered one particular application of Regge calculus – to the initial-value problem. The techniques are, of course, applicable to a far wider range of problems. Among these, its use in tracing the time development of the solutions to the initial-value problem seems particularly worthwhile.

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