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Bias-Free Method for the Detection of Bound, Antibound, and Resonant States from Error-Affected Experimental Data

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It has already been recognized that if the scattering amplitude is known only along a limited part of the cuts and with a limited accuracy, its analytic continuation to other points of its domain of holomorphy is not only nonunique but also unstable. The latter is a crucial fact both for the phenomenological analyses and for the theory unless we are in the fortunate position to have a theory which predicts the amplitude at all energies at once. Indeed, as a consequence of the finiteness of the energy range, it is always possible to set up an infinite set of holomorphic functions inside the error corridor, no matter how narrow it is, which are such that they differ by an arbitrary amount at any other point of the holomorphy domain. However, it is a remarkable mathematical fact that the maxima of the moduli of all these functions have to be greater than a certain lower bound which is uniquely determined and which can be computed numerically from the data function and its errors. This number can be used as a sensitive device for the location of the singularities as well as the zeros of the scattering amplitudes in a way which avoids the false particles that might be produced by specific parametrizations (e.g., Breit-Wigner form). The paper exemplifies the extreme sensitivity of this bias-free method to two model amplitudes, one of which exhibits a pole and the other a zero in the complex energy plane.

I. INTRODUCTION

The object of this paper is to describe a mathematically correct method of determining the location of zeros or poles of a function which is known to be analytic in a certain domain, using its error-affected values on some part of the boundary. The method can be used for determining, in a most unbiased way, bound, antibound, or resonant states of the S matrix, zeros of the form factors, etc.

As is well known, one usually looks for the detection of resonances by fitting the *parameters of a certain formula,* e.g., of the Breit-Wigner type, to the experimental data. However, it has been lately emphasized that this approach contains an unjustified bias, and it can in principle predict properties of particles which might not even exist. Indeed, one can find a set of values of the "pole" parameters, even if the pole does not exist.[1]

The method to be described below is based instead upon a global study of the whole set of possible functions, analytic in the cut plane and consistent with the experimental data within the known experimental errors.

Suppose the data are given on some limited range Γ_1 of the cut in the complex energy or momentum plane, under the form of a complex function ("the histogram") $H(z)$, with nonvanishing experimental errors $\epsilon(z)$. For simplicity, we treat first the case of a constant-error channel and we relegate the variable-error case to the end of this paper.

The correctly posed problem would be to find *the set of all functions* $F(z)$ analytic in the cut plane and satisfying

$$\sup_{z\in\Gamma_1}|F(z)-H(z)|<\epsilon\,. \tag{1}$$

For what follows it is essential to remark[2] that in this set of possible amplitudes there exists a function $F_0(z)$ which is minimal in the sense that it has the smallest maximum of modulus on the remaining part Γ_2 of the cuts. More exactly,

$$M_0=\sup_{z\in\Gamma_2}|F_0(z)|\,, \tag{2}$$

which depends on ϵ, is least among all F's consistent with (1), i.e., it is also below the maximum

$$M_{\text{true}}=\sup_{z\in\Gamma_2}|F_{\text{true}}(z)|$$

corresponding to the true physical amplitude.

The clue to the method is the remark that if the complex function H given on Γ_1 is not the boundary value of a holomorphic function, then, at least for small ϵ, it would be very hard to slip a holomorphic function inside the error corridor.[3] This means that even the minimal admissible amplitude F_0 will have a very high modulus maximum M_0. This should be transparent from the fact that for vanishing error ϵ, H completely defines an analytic function, together with its singularities, and no finite M_0 can be found satisfying (2), with holomorphic function F_0.

To locate the zeros (poles) from the experimental data, one reasons as follows: Assume that the true amplitude has a zero at some point z_0 inside the cut plane and denote the histogram corresponding to it by $H(z)$. Then F can be written as $F(z)=B_{z_0}(z)f(z)$, where $f(z)$ is holomorphic and $B_{z_0}(z)$ is the function (uniquely determined, see below), which is unimodular on the cuts, holomorphic inside the cut plane, and has a simple zero at the point z_0. The problem then comes to looking for f's having no other constraints but

$$\sup_{z\in\Gamma_1} |f(z)-H(z)/B_{z_0}(z)|<\epsilon\,. \tag{3}$$

Suppose we now forget everything about where the zero of the true amplitude is situated, and regard z_0 in (3) as a parameter at our disposal. If z_0 actually coincides with the zero of the true amplitude, we may state that H/B_{z_0} is the limiting value (error-affected) of a holomorphic function, since the artificial pole produced by B_{z_0} is exactly canceled. Then, according to the previous discussion, $M_0(z_0)$ is small, as it is certainly below $M_{\rm true}$ of the true amplitude. But, as soon as z_0 is moved away from the position of the zero, the histogram $H(z)/B_{z_0}(z)$ no longer comes from a holomorphic function, and we expect the corresponding $M_0(z_0)$ to be very high, at least for small enough ϵ. Therefore, if one draws a graph of M_0 as a function of z_0, one should find a sharp drop of M_0 around the zero of the true amplitude. If the true amplitude has a pole, the above reasoning proceeds similarly; of course, one has now to multiply the histogram by the function $B_{z_0}(z)$ so that the pole will survive until the parameter z_0 coincides with its position.

Actual calculations on test functions, done on a computer, show a spectacular narrow dip around the zero (pole) of the chosen amplitude. In our examples, $F_1(z)=[1-(4-z)^{1/2}]/[1+(4-z)^{1/2}]$ (for Fig. 1) and $F_2(z)=1/F_1(z)$ (for Fig. 2) have only one zero (pole) on the real axis, at $z=3$. Although $F_1(z)$ and $F_2(z)$ have only a right-hand cut, the do-

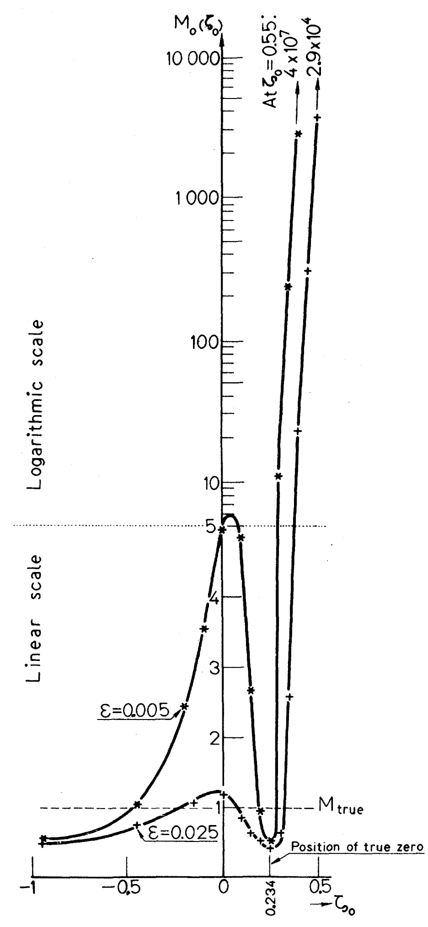

FIG. 1. M_0 vs ζ_0 for model amplitude $F_1(z)$, with a zero at $z_0=3$. Here $\zeta_0=\{30-z_0+i[195z_0(z_0-4)]^{1/2}\}/(14z_0-30)$.

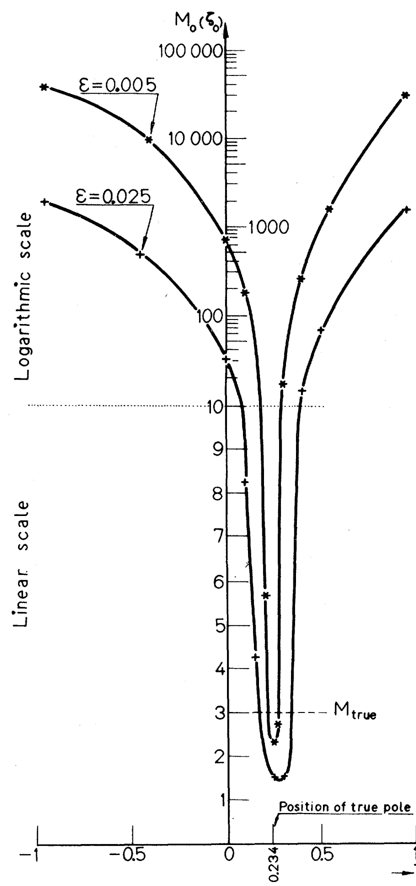

FIG. 2. M_0 vs ζ_0 for model amplitude $F_2(z)$, with a pole at $z_0=3$.

main of holomorphy of the admissible amplitudes was bounded by two "cuts" along the real axis, from 4 to ∞ as well as from $-\infty$ to 0. The "data" were given from $z=4$ to 30; 5% and 1% random errors were added. The maxima of the moduli *on the boundaries* are $M_{\text{true}}=1$ for $F_1(z)$ and $M_{\text{true}}=3$ for $F_2(z)$, the latter being attained on the left-hand boundary.

II. COMPUTATION OF M_0

The computation of M_0 is most easily performed if one first maps the cut z plane onto the unit disk in the ζ plane,[4] in such a way that the region of the data comes onto the right semicircle. In this variable the B function assumes the simple form (Blaschke factor)

$$B_{\zeta_0}(\zeta)=\frac{\zeta-\zeta_0}{1-\zeta\zeta_0^*}\,,\quad \text{where } \zeta_0=\zeta(z_0).$$

Now suppose there exist functions $f(z)$ satisfying apart from (3) a boundedness condition

$$\sup_{\zeta\in\Gamma_2}|f(\zeta)|<M. \tag{4}$$

If we multiply these functions by the so-called[5] Carleman function $C_0(\zeta)$, holomorphic inside the unit disk and of modulus 1 on Γ_1 and ϵ/M on Γ_2, it is easy to see that conditions (3) and (4) merge into

$$\sup_{\zeta\in\Gamma_1\cup\Gamma_2}|\tilde f(\zeta)-\tilde h(\zeta)|<\epsilon\,. \tag{5}$$

Here $\tilde f=fC_0$ and $\tilde h$ is equal to $(H/B_{\zeta_0})C_0$ on Γ_1 and 0 on Γ_2. Denote by ϵ_0 the minimum of the left-hand side of (5) over the class of all holomorphic functions $\tilde f(\zeta)$. It is essential that an explicit formula has been found for the computation of ϵ_0. This is described in Ref. 2, where it is shown that ϵ_0 is the square root of the largest eigenvalue of the infinite matrix $AA^\dagger$, with

$$A=\begin{pmatrix} c_{-1} & c_{-2} & c_{-3} & \cdots \\ c_{-2} & c_{-3} & & \cdots \\ \vdots & \vdots & & \end{pmatrix} \tag{6}$$

being constructed from the negative-frequency Fourier coefficients[6] of the Carleman weighted histogram $(H/B_{\zeta_0})C_0$:

$$c_{-n}=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{H(\theta)}{B_{\zeta_0}(\theta)}\,e^{in\theta}\times\exp\left[-\frac{i}{\pi}\ln\left(\frac{M}{\epsilon}\right)\ln\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]d\theta\,. \tag{7}$$

One can easily see that ϵ_0 depends in a monotonically decreasing way on the ratio M/ϵ, so that if one increases the parameter M (ϵ is fixed and given by the experimental data) there will be a value of M at which ϵ_0 gets smaller than the error ϵ. That value of M for which $\epsilon_0(M/\epsilon)=\epsilon$ is then the required minimal M_0.

When performing this on a computer, one has to treat properly[7] the peculiar behavior of the Carleman function near $\theta=\pm\pi/2$, which is apparent in (7). In practice, one has also to replace (6) by a finite-order matrix, obtained by setting $c_{-k}=0$ for k bigger than a certain N.[8]

III. CONCLUSIONS

(1) If one decreases ϵ (this corresponds to more accurate data) the graphs have to move upwards. At the position of the zero (pole), M_0 goes up very slowly as compared to the rest of the graph, since it is bounded here by the maximum M_{true} of the true amplitude. This has the effect of a dramatic narrowing of the dip towards small ϵ. On the other hand, the graphs are completely stable against the noise compatible with a given ϵ (i.e., random choice of histograms inside the error corridor).

(2) When we detect a zero (i.e., we produce a moving pole in the histogram through the Blaschke factor B_{z_0}) the graph (Fig. 1) exhibits a pronounced asymmetry. The fall of the graph towards the left can be explained by the diminishing influence of the artificial pole as it moves away from Γ_1. On the other hand, the very steep wall on the right (see Fig. 1) shows how sensitive M_0 is to the presence of a pole close to the physical region, despite the fact that it was created so that it does not modify the modulus of the histogram.

(3) In the converse procedure of detecting a pole in the true amplitude, the graph (Fig. 2) has a symmetrical appearance. In this case, the only thing that is changing when z_0 is moved along the real axis is the residue of the pole, rather than its position. So the nonanalyticity of the histogram, determined by the strength of the residue, is expected to be roughly symmetrical around the fixed position of the pole.

(4) For nonelastic resonant states – which cannot be identified as zeros on the first sheet – one can perform similar extrapolations on their appropriate sheets. One takes as regions for extrapolation increasing patches from that sheet, bounded by the histogram on one side. The M_0's which can be found for each of these regions will suddenly increase when the pole is sucked in, provided it is close enough to the real axis.

(5) If there exists some indication about an upper bound M_{true} of the true amplitude, say, $M_{\text{true}}\le M$, the graph of Fig. 1 (or Fig. 2) can receive a probabilistic interpretation. Draw on the graph the line $M_0=M_{\text{true}}$ which intersects the curve $M_0(\zeta_0)$ at ζ_1 and ζ_2. Clearly, there is no function

with a zero (pole) situated outside the range ζ_1, ζ_2. For a point between ζ_1 and ζ_2, the distance between the line $M_0 = M_{\text{true}}$ and the graph can be plausibly regarded as a probability density (normalized to the area below the line) for the zero (pole) to be located at that point.

(6) M_0 is nothing but the L^∞ norm of the minimal possible amplitude on the unknown part of the cuts Γ_2. Another way of locating the singularities is the study of the variation of $\mathfrak{M}_0(z_0)$, the minimum of the L^2 norm on Γ_2, of the analytic functions close to the histogram on Γ_1 (Ref. 9).

(7) The more general case of variable error $\epsilon(z)$ can be easily reduced to the one above (Sec. 4 of Ref. 5) by multiplying the histogram with a new Carleman function C_1, defined as having the modulus equal to $\epsilon/\epsilon(z)$ on Γ_1 and ϵ/M on Γ_2, where ϵ is an arbitrary number.

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[1]G. Callucci, L. Fonda, and G. C. Ghiraldi, Phys. Rev. 166, 1719 (1968).

[2]S. Ciulli and G. Nenciu, J. Math Phys. (to be published).

[3]An analogous phenomenon arose in Cutkosky and Deo's problem of extrapolation of the Chew-Low type [Phys. Rev. Letters 22, 1272 (1968)]. The presence of a singularity required a much larger number of optimal polynomials to fit the histogram to the same desired accuracy.

[4]For explicit formulas, see Sec. 2 of Ref. 5.

[5]S. Ciulli and J. Fischer, Nucl. Phys. B24, 537 (1970).

[6]The negative-frequency part comes mainly from the fact that $\tilde{h}$ is defined as being zero on the boundary Γ_2, *and not* from the irregularity caused by the error, as one might believe.

[7]A possible way of doing this is to isolate the rapidly varying part of the Carleman function and to compute the integral via the moments of $\exp[(i/\pi)\ln(M/\epsilon)\ln\theta]$; we stand by the interested reader with all necessary programs.

[8]This truncation acts as an additional small noise, which lowers the value of M_0.

[9]I. Sabba-Stefanescu, Nucl. Phys. (to be published).

PHYSICAL REVIEW D VOLUME 5, NUMBER 7 1 APRIL 1972

Symmetrization Effects in Spectator Momentum Distributions*

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The traditional identification of the slow nucleon as the spectator in deuteron-breakup collisions is shown to be valid only when there is a significant difference in the speeds of the two nucleons. When both have similar speeds, symmetrization effects cannot be ignored. These effects will be particularly important (a) for low-momentum-transfer events and (b) for "high-momentum spectator" events. A new comparison technique free of these difficulties is suggested. By studying the transverse momentum spectrum for both final-state nucleons, one avoids ambiguous spectator identification. In addition, the effect of the interference in such a distribution reveals the spatial symmetry of the final two-nucleon state. From this information it is possible to determine the relative contributions of spin-flip and -nonflip processes, and thereby to reconstruct the free-nucleon scattering accurately.

In this paper we consider the impulse approximation for the general deuteron disintegration processes $xd \to y'pp$. A traditional element of the experimental data analysis of these reactions has involved comparing the momentum distribution of the slow nucleon, assumed to be the spectator, with the deuteron momentum-space density. We shall show that there is a non-negligible class of events in which that simple comparison is incorrect because of the necessity of symmetrizing the two-nucleon final state.

We begin from the impulse approximation in the form

$$F(xd \to yNN) = \left\langle f \left| \sum_n t_n \right| i \right\rangle, \tag{1}$$

where t_n is the transition matrix for the corresponding free-nucleon process and $|f\rangle$ and $|i\rangle$ are