*Work supported in part through funds provided by the Atomic Energy Commission, under Contract No. AT(30-1)-2098.

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[9]Using the results of the Appendix and Eq. (2.7) of Ref. 2, it is easy to see that the ϕ^3 ladder model satisfies Eq. (12) and thus our Steinmann assumption.

[10]This is no surprise, of course. The cuts in the asymptotic variables are asymptotic representations of poles and we know the dual model has no simultaneous poles in overlapping variables.

[11]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 256.

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Eikonal Approximation and the Light Cone*

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This comment considers the high-energy behavior of a particle of arbitrary spin interacting with an external potential $A_{\mu_1 \cdots \mu_n}(x)$ by means of a $J^{\mu_1 \cdots \mu_n} A_{\mu_1 \cdots \mu_n}(x)$ interaction, where $J^{\mu_1 \cdots \mu_n}$ is a function of the fields and their derivatives. It is shown that in the eikonal approximation, "exponentiation" occurs when the light-cone commutation relations satisfy

$$\int_{-\infty}^{\infty} dx^- \int_{-\infty}^{\infty} dy^- [J^{+\cdots+}(x),\ J^{+\cdots+}(y)]_{x^+=y^+=0} = 0.$$

It has become apparent that the knowledge of current commutators on the light cone provides useful theoretical and experimental information about the high-energy behavior. The structure of light-cone commutators of currents in various model field theories has been exposed by Cornwall and Jackiw,[1] Fritzsch and Gell-Mann,[2] and Gross and Treiman,[3] and many applications of the theory have already been given.[4-6]

In this comment we wish to indicate a new area of applicability of light-cone techniques – the study of the eikonal approximation. This approximation has become a useful method in calculating high-energy scattering amplitudes in various model theories.[7,8]

That there should be a connection between the light-cone approach and the eikonal approximation can be seen in various ways. For example, the natural variables in the eikonal methods are the + and – components familiar from the light cone. Also, the work of Gross and Treiman[3] abounds in eikonal-type exponentials in their light-cone formulas. More specifically, Lee[9] and Chang[10] have related "exponentiation" of the eikonal approximation in quantum electrodynamics to the properties of current commutators on the light cone.

Recently, Weinberg[11] has shown for the scattering of a fast particle of arbitrary spin in an external electromagnetic field that exponentiation occurs provided that the anomalous magnetic moment obeys a generalized Drell-Hearn sum rule.[12] The Drell-Hearn sum rule can be related to the light-cone commutation relations.[5] Thus there is a general relation between the exponentiation of the eikonal approximation and light-cone commutators. In this comment that relationship is directly obtained. It is found that for a particle of arbitrary spin [field $\phi(x)$] interacting with an arbitrary external potential $A^{\alpha_1 \cdots \alpha_m}_{\mu_1 \cdots \mu_n}(x)$ by the interaction $J^{\mu_1 \cdots \mu_n}_{\alpha_1 \cdots \alpha_m} A^{\alpha_1 \cdots \alpha_m}_{\mu_1 \cdots \mu_n}(x)$, exponentiation of the eikonal approximation to the wave function $\Phi(x; p, \lambda, \alpha; A) \equiv \langle 0\,\text{out} | \phi(x) | p, \lambda, \alpha\,\text{in}\rangle$ is valid provided that

$$\int_{-\infty}^{\infty} dx^- \int_{-\infty}^{\infty} dy^- [J^{+\cdots+}_{\alpha_1 \cdots \alpha_m}(0, \vec{x}_\perp, x^-),\ J^{+\cdots+}_{\beta_1 \cdots \beta_m}(0, \vec{y}_\perp, y^-)] = 0\,.$$

In order to obtain this relation one considers the Lagrangian for a spin-j quantum field ϕ with an external potential $A^{\alpha_1\cdots\alpha_m}_{\mu_1\cdots\mu_n}(x)$, where $\alpha_1, \ldots, \alpha_m$ are indices associated with internal quantum numbers of the field [e.g., isospin, SU(3)] and $\mu_1, \ldots, \mu_n$ are Lorentz indices. We assume that A is a scalar in spinor space; thus under a pure Lorentz transformation Λ,

$$A'^{\alpha_1\cdots\alpha_m}_{\mu_1'\cdots\mu_n'}(x)=\Lambda_{\mu_1'}{}^{\mu_1}\cdots\Lambda_{\mu_n'}{}^{\mu_n}A^{\alpha_1\cdots\alpha_m}_{\mu_1\cdots\mu_n}(\Lambda^{-1}x)\,.$$

The Lagrangian is given by

$$\mathcal{L}(\phi(x),\phi^\dagger(x),x)=\mathcal{L}_{\text{free}}(\phi(x),\phi^\dagger(x)) +J^{\mu_1\cdots\mu_n}_{\alpha_1\cdots\alpha_m}A^{\alpha_1\cdots\alpha_m}_{\mu_1\cdots\mu_n}(x)\,.$$

Consider the wave function Φ for a particle that at $t\to-\infty$ was a free particle of momentum $p^\mu=(p^+,\vec{0}_\perp,m^2/2p^+)$, helicity λ, and other internal quantum numbers denoted by α. One may write

$$\Phi(x;p,\lambda,\alpha;A)\equiv\langle 0\,\text{out}|\phi(x)|p,\lambda,\alpha\;\text{in}\rangle=\left\langle 0\left|T\left(\phi_I(x)\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]\right)\right|p,\lambda,\alpha\right\rangle$$
$$=\left\langle 0\left|T\left(\phi_I(x)\exp\left[i\int d^4z\,J^{\mu_1\cdots\mu_n}_{I\alpha_1\cdots\alpha_m}(z)A^{\alpha_1\cdots\alpha_m}_{\mu_1\cdots\mu_n}(z)\right]\right)\right|p,\lambda,\alpha\right\rangle.$$

For simplicity, the indices $\alpha_1,\ldots,\alpha_m$ are suppressed and n is set equal to 1. The argument is exactly the same in the general case. Thus, one considers

$$\Phi(x;p,\lambda,\alpha;A)=\left\langle 0\left|T\left(\phi_I(x)\exp\left[i\int d^4z\,J_I^{\mu_1}(z)A_{\mu_1}(z)\right]\right)\right|p,\lambda,\alpha\right\rangle.$$

Making use of the translation operator, one obtains

$$\Phi(x;p,\lambda,\alpha;A)=\left\langle 0\left|T\left(\phi_I(0)e^{-ip\cdot x}\exp\left[i\int d^4z\,J_I^{\mu_1}(z-x)A_{\mu_1}(z)\right]\right)\right|p,\lambda,\alpha\right\rangle$$
$$=\left\langle 0\left|T\left(\phi_I(0)e^{-ip\cdot x}\exp\left[i\int d^4z\,J_I^{\mu_1}(z)A_{\mu_1}(x+z)\right]\right)\right|p,\lambda,\alpha\right\rangle.$$

In the limit $p^+\to\infty$, $p^2=m^2$ one would expect Φ to take the eikonal form. One can choose $|p,\lambda,\alpha\rangle=e^{-i\theta\lambda}e^{-i\delta K_3}|p_0,\lambda,\alpha\rangle$, where $p_0^\mu=(m/\sqrt{2},\vec{0}_\perp,m/\sqrt{2})$, and $e^{-i\delta K_3}$ is a boost along the $+z$ direction with rapidity δ. So the relationship between p and p_0 is $p^+=e^\delta p_0^+$, $\vec{p}_\perp=\vec{p}_{0\perp}$, and $p^-=e^{-\delta}p_0^-$. The phase which removes the effect of the boost on the helicity λ is $e^{-i\theta\lambda}$. Thus

$$\lim_{p^+\to\infty;\,p^2=m^2}\Phi(x;p,\lambda,\alpha;A)$$
$$=\lim_{\delta\to\infty}\left\langle 0\left|e^{+i\delta K_3}T\left(\phi_I(0)\exp\left[i\int d^4z\,J_I^{\mu_1}(z)A_{\mu_1}(x+z)\right]\right)e^{-i\delta K_3}\right|p_0,\lambda,\alpha\right\rangle e^{-ip\cdot x}e^{-i\theta\lambda}$$
$$=\lim_{\delta\to\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{n!}\left\langle 0\left|T'_\delta\left(\left[i\int d^4z\,\Theta\left(\frac{z^++z^-}{\sqrt{2}}\right)e^{i\delta K_3}J_I^{\mu_1}(z)e^{-i\delta K_3}A_{\mu_1}(x+z)\right]^m\right)e^{i\delta K_3}\phi_I(0)e^{-i\delta K_3}\right.\right.$$
$$\left.\left.\times T'_\delta\left(\left[i\int d^4z\,\Theta\left(\frac{-z^+-z^-}{\sqrt{2}}\right)e^{i\delta K_3}J_I^{\mu_1}(z)e^{-i\delta K_3}A_{\mu_1}(x+z)\right]^{n-m}\right)\right|p_0,\lambda,\alpha\right\rangle e^{-ip\cdot x}e^{-i\theta\lambda},\tag{1}$$

where

$$T'_\delta(A(x)B(y))\equiv\Theta\left(\frac{(x^+-y^+)e^\delta+(x^--y^-)e^{-\delta}}{\sqrt{2}}\right)A(x)B(y)+\Theta\left(\frac{(y^+-x^+)e^\delta+(y^--x^-)e^{-\delta}}{\sqrt{2}}\right)B(y)A(x)\,.$$

Now to leading order in e^δ,

$$e^{i\delta K_3}J_I^{\mu_1}(z)e^{-i\delta K_3}=e^\delta J_I^+(z^+e^{-\delta},\vec{z}_\perp,z^-e^\delta)g^{\mu_1-}\,.\tag{2}$$

Thus Eq. (1) becomes to leading order

$$\lim_{p^+\to\infty;\,p^2=m^2}\Phi(x;p,\lambda,\alpha;A)=\lim_{\delta\to\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{n!}e^{-ip\cdot x}e^{-i\theta\lambda}$$
$$\times\left\langle 0\left|T'_\delta\left(\left[i\int d^4z\,\Theta\left(\frac{z^++z^-}{\sqrt2}\right)e^\delta J_I^+(z^+e^{-\delta},\vec z_\perp,z^-e^\delta)A_+(x+z)\right]^m\right)e^{i\delta K_3}\phi_I(0)e^{-i\delta K_3}\right.\right.$$
$$\left.\left.\times T'_\delta\left(\left[i\int d^4z\,\Theta\left(\frac{-z^+-z^-}{\sqrt2}\right)e^\delta J_I^+(z^+e^{-\delta},\vec z_\perp,z^-e^\delta)A_+(x+z)\right]^{n-m}\right)\right|p_0,\lambda,\alpha\right\rangle.$$

Changing variables $z^-e^\delta\to z^-$ and then dropping terms[13] of order $e^{-\delta}$ relative to terms of order 1, one obtains

$$\lim_{p^+\to\infty;\,p^2=m^2}\Phi(x;p,\lambda,\alpha;A)=\lim_{\delta\to\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{n!}e^{-ip\cdot x}e^{-i\theta\lambda}$$
$$\times\left\langle 0\left|T''_\delta\left(\left[i\int d^4z\,\Theta(z^+)A_+(x^++z^+,\vec z_\perp+\vec x_\perp,x^-)J_I^+(z^+e^{-\delta},\vec z_\perp,z^-)\right]^m\right)e^{i\delta K_3}\phi_I(0)e^{-i\delta K_3}\right.\right.$$
$$\left.\left.\times T''_\delta\left(\left[i\int d^4z\,\Theta(-z^+)A_+(x^++z^+,\vec x_\perp+\vec z_\perp,x^-)J_I^+(z^+e^{-\delta},\vec z_\perp,z^-)\right]^{n-m}\right)\right|p_0,\lambda,\alpha\right\rangle,$$

where

$$T''_\delta(A(x)B(y))\equiv\Theta((x^+-y^+)e^\delta+(x^--y^-))A(x)B(y)+\Theta((y^+-x^+)e^\delta+(y^--x^-))B(y)A(x)$$

and $z^+e^{-\delta}$ has not been set equal to zero in order to make T''_δ well defined.

Considering

$$\left\langle 0\left|\int_{-\infty}^{\infty}dz^-J_I^+(z^+e^{-\delta},\vec z_\perp,z^-)\right|N\right\rangle$$

for any physical state $|N\rangle$ with momentum p, we have

$$\left\langle 0\left|\int_{-\infty}^{\infty}dz^-J_I^+(z^+e^{-\delta},\vec z_\perp,z^-)\right|N\right\rangle=\left\langle 0\left|\int_{-\infty}^{\infty}dz^-\,e^{ip^+_{\rm op}z^-}J_I^+(z^+e^{-\delta},\vec z_\perp,0)e^{-ip^+_{\rm op}z^-}\right|N\right\rangle$$
$$=2\pi\delta(p^+)\langle 0|J_I^+(z^+e^{-\delta},\vec z_\perp,0)|N\rangle.$$

But $p^+=0$ implies $p^2\le 0$, so no physical state contributes. (The vacuum does not contribute, since J_I^+ is assumed to be a normal-ordered operator.) Therefore,

$$\left\langle 0\right|\int_{-\infty}^{\infty}dz^-J_I^+(z^+e^{-\delta},\vec z_\perp,z^-)=0.$$

Thus

$$\lim_{p^+\to\infty;\,p^2\to m^2}\Phi(x;p,\alpha,\lambda;A)$$
$$=\lim_{\delta\to\infty}\left\langle 0\left|e^{i\delta K_3}\phi_I(0)e^{-i\delta K_3}\,T''_\delta\left(\exp\left[i\int_{-\infty}^0dz^+\int d^2z_\perp A_+(x^++z^+,\vec x_\perp+\vec z_\perp,x^-)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\times\int_{-\infty}^{\infty}dz^-J_I^+(z^+e^{-\delta},\vec z_\perp,z^-)\right]\right)\right|p_0,\lambda,\alpha\right\rangle e^{-ip\cdot x}e^{-i\theta\lambda}.$$

Now the assumption is made that

$$\int_{-\infty}^{\infty}dx^-\int_{-\infty}^{\infty}dy^-[J_I^+(0,\vec x_\perp,x^-),J_I^+(0,\vec y_\perp,y^-)]=0.$$

This assumption means all T''_δ orderings are the same, so that

$$\lim_{p^+\to\infty;\,p^2=m^2}\Phi(x;p,\lambda,\alpha;A)=\lim_{\delta\to\infty}\left\langle 0\left|\phi_I(0)e^{-i\delta K_3}\exp\left[i\int_{-\infty}^0dz^+\int d^2z_\perp A_+(x^++z^+,\vec x_\perp+\vec z_\perp,x^-)\right.\right.\right.$$
$$\left.\left.\left.\times\int_{-\infty}^{\infty}dz^-J_I^+(0,\vec z_\perp,z^-)\right]\right|p_0,\lambda,\alpha\right\rangle e^{-ip\cdot x}e^{-i\theta\lambda}. \tag{3}$$

Equation (3) is an operator form of the eikonal approximation to the wave function. To obtain the more familiar form one assumes a general form for the current $J_I^{\mu_1}$:

$$J_I^{\mu_1}(z) =: \phi_I^\dagger(z)\Gamma^{\mu_1}\phi_I(z): .$$

Then

$$J_I^{\mu_1}(z)\,|r,\,\delta_1,\,\beta_1\rangle = \sum_{\delta_2}\sum_{\beta_2}\int \frac{d^4s}{(2\pi)^3}\,\delta(s^2-m^2)\Theta(s^0)\,|s,\,\delta_2,\,\beta_2\rangle\langle s,\,\delta_2,\,\beta_2\,|J_I^{\mu_1}(z)|\,r,\,\delta_1,\,\beta_1\rangle$$

and

$$\int_{-\infty}^{\infty} dz^-\langle s,\,\delta_2,\,\beta_2\,|J_I^+(0,\,\vec z_\perp,\,z^-)|\,r,\,\delta_1,\,\beta_1\rangle = 2\pi\delta(r^+-s^+)e^{-i\vec z_\perp\cdot(\vec s_\perp-\vec r_\perp)}\langle s,\,\delta_2,\,\beta_2\,|J_I^+(0)|\,r,\,\delta_1,\,\beta_1\rangle\,;$$

thus

$$\begin{aligned}\lim_{p^+\to\infty;\,p^2=m^2}\Phi(x;\,p,\,\lambda,\,\alpha;\,A) = \lim_{\delta\to\infty}\sum_{\alpha_2}\sum_{\lambda_2}\int\frac{d^2p_\perp'}{(2\pi)^2}\,\frac{1}{\sqrt2\,m}\langle 0\,|\,\phi_I(0)e^{-iK_3\delta}\,|\,p_0^+,\,\vec p_\perp';\,\lambda_2,\,\alpha_2\rangle \\ \times\Big\langle p_0^+,\,\vec p_\perp';\,\lambda_2,\,\alpha_2\Big|\exp\Big[i\int_{-\infty}^0 d\tau\int d^2z_\perp A_+(x^++\tau,\,\vec x_\perp+\vec z_\perp,\,x^-)\,\frac{1}{2m^2}\int\frac{d^2r_\perp}{(2\pi)^2}\int\frac{d^2s_\perp}{(2\pi)^2} \\ \times\sum_{\delta_1,\,\delta_2}\;\sum_{\beta_1,\,\beta_2} e^{-i\vec z_\perp\cdot(\vec s_\perp-\vec r_\perp)}\langle p_0^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\,|J_I^+(0)|\,p_0^+,\,\vec r_\perp;\,\delta_1,\,\beta_1\rangle\,|\,p_0^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\rangle \\ \times\langle p_0^+,\,\vec r_\perp;\,\delta_1,\,\beta_1|\Big]\Big|\,p_0^+,\,\vec 0_\perp;\,\lambda,\,\alpha\Big\rangle e^{-ip\cdot x}e^{-i\theta\lambda},\end{aligned}$$

where the normalization of states is

$$\langle p_0^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\,|\,p_0^+,\,\vec r_\perp;\,\delta_1,\,\beta_1\rangle = 2m^2(2\pi)^2\delta^2(\vec r_\perp-\vec s_\perp)\delta_{\delta_1\delta_2}\delta_{\beta_1\beta_2}\,.$$

Noting that

$$\lim_{\delta\to\infty}\langle 0\,|\,\phi_I(0)e^{-i\delta K_3}\,|\,p_0^+,\,\vec p_\perp';\,\lambda_2,\,\alpha_2\rangle = \lim_{\delta\to\infty}\langle 0\,|\,\phi_I(0)\,|\,p^+,\,\vec p_\perp';\,\lambda_2,\,\alpha_2\rangle e^{i\theta\lambda_2} \simeq \lim_{\delta\to\infty}\langle 0\,|\,\phi_I(0)\,|\,p^+,\,\vec 0_\perp;\,\lambda_2,\,\alpha_2\rangle\,e^{i\theta\lambda_2}$$

and letting

$$\sqrt2\,mf(p,\,\lambda,\,\alpha)\equiv\langle 0\,|\,\phi_I(0)\,|\,p^+,\,\vec 0_\perp;\,\lambda,\,\alpha\rangle\,,$$

one obtains

$$\begin{aligned}\lim_{p^+\to\infty;\,p^2=m^2}\Phi(x;\,p,\,\lambda,\,\alpha;\,A) = \lim_{\delta\to\infty}\sum_{\alpha_2}\sum_{\lambda_2} f(p,\,\lambda_2,\,\alpha_2)e^{-ip\cdot x}e^{-i\theta(\lambda-\lambda_2)} \\ \times\int\frac{d^2p_\perp'}{(2\pi)^2}\Big\langle p_0^+,\,\vec p_\perp';\,\lambda_2,\,\alpha_2\Big|\exp\Big[i\int_{-\infty}^0 dz^+\int d^2z_\perp A_+(x^++\tau,\,\vec x_\perp+\vec z_\perp,x^-)\frac{1}{2m^2}\int\frac{d^2r_\perp}{(2\pi)^2}\int\frac{d^2s_\perp}{(2\pi)^2} \\ \times\sum_{\delta_1,\,\delta_2}\;\sum_{\beta_1,\,\beta_2} e^{-i\vec z_\perp\cdot(\vec s_\perp-\vec r_\perp)}\langle p_0^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\,|J_I^+(0)|\,p_0^+,\,\vec r_\perp;\,\delta_1,\,\beta_1\rangle\,|\,p_0^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\rangle \\ \times\langle p_0^+,\,\vec r_\perp;\,\delta_1,\,\beta_1|\Big]\Big|\,p_0^+,\,\vec 0;\,\lambda,\,\alpha\Big\rangle\,.\end{aligned}$$

Now by Eq. (2)

$$\begin{aligned}\lim_{\delta\to\infty}\langle p_0^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\,|J_I^+(0)e^\delta g^{\mu_1 -}|\,p_0^+,\,\vec r_\perp;\,\delta_1,\,\beta_1\rangle &= \lim_{\delta\to\infty}\langle p_0^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\,|\,e^{i\delta K_3}J_I^{\mu_1}(0)e^{-i\delta K_3}\,|\,p_0^+,\,\vec r_\perp;\,\delta_1,\,\beta_1\rangle \\ &= \lim_{p^+\to\infty}\langle p^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\,|J_I^{\mu_1}(0)|\,p^+,\,\vec r_\perp;\,\delta_1,\,\beta_1\rangle\,e^{i\theta(\delta_1-\delta_2)}.\end{aligned}$$

By performing a boost to the rest frame of the state $|p^+,\,\vec r_\perp;\,\delta_1,\,\beta_1\rangle$ it is clear that one can write[14] to leading order

$$\lim_{p^+\to\infty}\langle p^+,\,\vec s_\perp;\,\delta_2,\,\beta_2\,|J_I^{\mu_1}(0)|\,p^+,\,\vec r_\perp;\,\delta_1,\,\beta_1\rangle\,e^{i\theta(\delta_1-\delta_2)} = \lim_{p^+\to\infty}\frac{p^+\sqrt2}{m}F(\vec s_\perp-\vec r_\perp,\,\delta_1,\,\delta_2,\,\beta_1,\,\beta_2)g^{\mu_1 -}e^{+i\theta(\delta_1-\delta_2)},$$

so that

$$\lim_{\delta\to\infty}\langle p_0^+, \vec{s}_\perp; \delta_2, \beta_2|e^{\delta}J_I^+(0)|p_0^+, \vec{r}_\perp; \delta_1, \beta_1\rangle = \lim_{p^+\to\infty}\sqrt{2}\,\frac{p^+}{m}F(\vec{s}_\perp-\vec{r}_\perp, \delta_1, \delta_2, \beta_1, \beta_2)e^{i\theta(\delta_1-\delta_2)}.$$

Defining the matrix $\mathfrak{F}(\vec{s}_\perp-\vec{r}_\perp)$ by

$$[\mathfrak{F}(\vec{s}_\perp-\vec{r}_\perp)]_{(\delta_2,\beta_2);(\delta_1,\beta_1)} = F(\vec{s}_\perp-\vec{r}_\perp, \delta_1, \delta_2, \beta_1, \beta_2),$$

one can write

$$\begin{aligned}\lim_{p^+\to\infty;\,p^2=m^2}&\Phi(x;p,\lambda,\alpha;A)\\ &=\lim_{p^+\to\infty;\,p^2=m^2}\sum_\delta\sum_\beta f(p,\delta,\beta)e^{-ip\cdot x}\left\{\exp\left[\int_{-\infty}^0 dz^+ i\int d^2z_\perp A_+(x^++z^+, \vec{x}_\perp+\vec{z}_\perp, x^-)\int\frac{d^2q_\perp}{(2\pi)^2}e^{-i\vec{z}_\perp\cdot\vec{q}_\perp}\mathfrak{F}(\vec{q}_\perp)\right]\right\}_{(\delta,\beta);(\lambda,\alpha)}\\ &=\lim_{p^+\to\infty;\,p^2=m^2}\sum_\delta\sum_\beta f(p,\delta,\beta)e^{-ip\cdot x}\left\{\exp\left[i\int_{-\infty}^0 dz^+\int\frac{d^2q_\perp}{(2\pi)^2}e^{i\vec{x}_\perp\cdot\vec{q}_\perp}a_+(x^++z^+, \vec{q}_\perp, x^-)\mathfrak{F}(-\vec{q}_\perp)\right]\right\}_{(\delta,\beta);(\lambda,\alpha)},\end{aligned}$$

where

$$a_+(x^++z^+, \vec{q}_\perp, x^-) \equiv \int d^2z_\perp e^{i\vec{q}_\perp\cdot\vec{z}_\perp}A_+(x^++z^+, \vec{z}_\perp, x^-).$$

This result is the eikonal approximation for the wave function. In the general case this result becomes

$$\begin{aligned}\lim_{p^+\to\infty;\,p^2=m^2}\Phi(x;p,\lambda,\alpha;A) &= \lim_{p^+\to\infty;\,p^2=m^2}\sum_\delta\sum_\beta f(p,\delta,\beta)e^{-ip\cdot x}\\ &\times\left\{\exp\left[i(p^+)^{n-1}\int_{-\infty}^0 dz^+\int\frac{d^2q_\perp}{(2\pi)^2}e^{i\vec{x}_\perp\cdot\vec{q}_\perp}a_{+\cdots+}^{\alpha_1\cdots\alpha_m}(x^++z^+, \vec{q}_\perp, x^-)\,\mathfrak{F}_{\alpha_1\cdots\alpha_m}(-\vec{q}_\perp)\right]\right\}_{(\delta,\beta);(\lambda,\alpha)},\end{aligned}$$

where

$$f(p,\delta,\beta) \equiv \frac{1}{\sqrt{2}\,m}\langle 0|\phi_I(0)|p,\delta,\beta\rangle,$$

$$[\mathfrak{F}_{\alpha_1\cdots\alpha_m}(\vec{q}_\perp)]_{(\delta_2,\beta_2);(\delta_1,\beta_1)}\frac{\sqrt{2}\,(p^+)^n}{m}e^{i(\delta_1-\delta_2)\theta} = \langle p_0^+, \vec{q}_\perp; \delta_2, \beta_2|J_{I\alpha_1\cdots\alpha_m}^{+\cdots+}(0)|p_0^+, \vec{0}_\perp; \delta_1, \beta_1\rangle(e^{\delta})^n \text{ as } p^+\to\infty,$$

and

$$a_{+\cdots+}^{\alpha_1\cdots\alpha_m}(x^++z^+, \vec{q}_\perp, x^-) \equiv \int d^2z_\perp e^{i\vec{q}_\perp\cdot\vec{z}_\perp}A_{+\cdots+}^{\alpha_1\cdots\alpha_m}(x^++z^+, \vec{z}_\perp, x^-).$$

It has thus been shown that the "exponentiation" of the eikonal approximation occurs provided that

$$\int_{-\infty}^\infty dx^-\int_{-\infty}^\infty dy^-[J_{\alpha_1\cdots\alpha_m}^{+\cdots+}(0, \vec{x}_\perp, x^-), J_{\beta_1\cdots\beta_m}^{+\cdots+}(0, \vec{y}_\perp, y^-)] = 0.$$

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[13]The assumption that we can take the limits inside the integration is the criterion for the validity of the eikonal approximation in field theory. We do not attempt to justify this assumption in this paper.

[14]See, e.g., R. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966); S. Fubini and G. Furlan, Physics 1, 229 (1964). See also Ref. 5.

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Derivation of Iwasaki's Propagators from Lagrangians

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Using Schwinger's formulation of the Green's function and the assumption that the covariant propagator is equal to the time-ordered Green's function, we derive Iwasaki's covariant propagators from Lagrangians introduced by Chang.

I. INTRODUCTION

Iwasaki[1,2] has recently shown that high-spin covariant propagators satisfy certain symmetry and consistency conditions. He writes down expressions which satisfy these constraints, but he claims that his propagators cannot be derived from a Lagrangian.

It seemed to us an unsatisfactory situation that Iwasaki's covariant propagators should not be derivable from field theory, so we critically examined the basis for his claim. We found that implicit in his claim are the following two assumptions: (i) In field theory the covariant propagator is given by the time-ordered Green's function, and (ii) the Green's function is given by the Takahashi-Umezawa (T-U) prescription.[3] Using Schwinger's formulation of the Green's function[4] we show below that the T-U Green's function follows from a specific choice of the form for the free-field Lagrangian and that Iwasaki's covariant propagator is identical to the Green's function corresponding to a different but equally valid type of free-field Lagrangian introduced by Chang.[5]

II. LAGRANGIANS, GREEN'S FUNCTIONS, AND COVARIANT PROPAGATORS

A. Free-Field Lagrangians

Massive spin-S systems with definite parity can be described by a covariant tensor or tensor-spinor ϕ satisfying certain differential equations of motion as well as certain nondifferential constraints. All the equations to be satisfied by ϕ can be summarized in a single equation of the form[3,6]

$$\Lambda_{\mu\cdots,\alpha\cdots}\phi^{\alpha\cdots}=0, \tag{1}$$

where Λ is a local differential operator of second order for S an integer (first order for $S+\frac{1}{2}$ an integer). Equation (1) is the Euler-Lagrange equation for the following Lagrangian density[7]:

$$\mathcal{L}_0=\lambda\bar{\phi}\Lambda\phi, \tag{2}$$

where $\lambda=1$ unless $\phi_{\mu}\ldots=\phi^{\dagger}_{\mu}\ldots$, in which case $\lambda=\frac{1}{2}$. We will see below that the T-U prescription for determining the Green's function is only applicable when the free-field Lagrangian density is of the form given in Eq. (2).

All the nonderivative constraint equations satisfied by ϕ can be summarized by an equation of the form

$$\phi=Z\phi, \tag{3}$$

where $Z=Z^{(S)}$ is an orthogonal covariant projection operator.[8] Chang[5] has introduced a type of $\mathcal{L}_0$ in which Eq. (3) is assumed to hold in advance so that only the differential equations of motion follow from the Euler-Lagrange equations. For $Z\neq I$ Chang's $\mathcal{L}_0$'s contain auxiliary fields as well as ϕ and hence are not of the form given in Eq. (2). Chang's Lagrangians can be modified so that they contain complete information about the system by treating Eq. (3) as a holonomic constraint and introducing a Lagrange multiplier.

It is often possible to write down Lagrangians in which some of the symmetry conditions are assumed while others are derived along with the differential equations of motion from the Euler-Lagrange equations.[9] As with the Chang Lagrangians, the assumed symmetry constraints can be derived from the Lagrangian by introducing Lagrange multipliers.