PHYSICAL REVIEW D VOLUME 5, NUMBER 4 15 FEBRUARY 1972

Gauge Transformation in the Thirring Model*

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(Received 23 August 1971)

It is shown that gauge transformations, which modify the dimension of the field $\psi(x)$ in the Thirring model, leave invariant the anomalous dimension of the compound fields $\phi_{\pm}(x)$. The modifications of the operator-product expansion of $T(\psi(x)\bar{\psi}(y))$ required by the gauge transformation are also presented.

In a recent paper,[1] it was pointed out that the dimension of the field $\psi(x)$ that occurs in the massive Thirring model is gauge-dependent. It is of interest to discuss the dimensions of compound fields which are, in principle, observable. In particular, the fields $\phi_{\pm}(x) = :\bar{\psi}(x)(1\pm\gamma_5)\psi(x):$ have been shown[2] to have anomalous dimension in Johnson's solution of the Thirring-model equations.[3] In this note we show that this dimension is invariant under the gauge transformation and so cannot be gauged into the naive dimension, as could be done for the dimension of $\psi(x)$. At the same time, we examine the behavior of the operator-product expansion[4,2] for $\psi(x)\bar{\psi}(0)$ under a gauge transformation. The composite fields $\phi_{\pm}$ and j_μ are taken to be defined by this expansion. A new term is found in the expansion which is absent only in the gauge corresponding to Johnson's solution. A simple interpretation of this term is given.

We effect the gauge transformation by the replacement

$$\psi(x) \to e^{ia\chi(x)}\psi(x)\,, \tag{1}$$

where $\chi(x)$ describes a free Hermitian scalar field of mass M. The relevant Green's functions transform according to

$$\langle T(\psi(x)\bar{\psi}(y))\rangle_0 \to e^{-ia^2D^c(x-y)}\langle T(\psi(x)\bar{\psi}(y))\rangle_0\,, \tag{2a}$$

$$\langle T(\psi(x')j_\mu(x)\bar{\psi}(y'))\rangle_0 \to e^{-ia^2D^c(x'-y')}\langle T(\psi(x')j_\mu(x)\bar{\psi}(y'))\rangle_0\,, \tag{2b}$$

$$\begin{aligned}\langle T(\psi(x')\psi(x)\bar{\psi}(y)\bar{\psi}(y'))\rangle_0 \to \exp\{&-ia^2[-D^c(x'-x) - D^c(y-y') + D^c(x-y) + D^c(x'-y) + D^c(x-y') + D^c(x'-y')]\}\\ &\times\langle T(\psi(x')\psi(x)\bar{\psi}(y)\bar{\psi}(y'))\rangle_0\,.\end{aligned} \tag{2c}$$

Here, the causal function $D^c(x-y)$ is given by

$$\begin{aligned}D^c(x-y) &= \frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{dk}{(k^2+M^2)^{1/2}}\exp[ik(y_1-x_1) - i(k^2+M^2)^{1/2}|x_0-y_0|]\\ &= \frac{i}{2\pi}K_0([-(y-x)^2M^2]^{1/2}) \quad \text{for} \quad (y-x)^2<0\\ &\sim \frac{i}{4\pi}\ln[-(y-x)^2M^2] + \cdots \quad \text{for} \quad (y-x)^2 \to 0.\end{aligned}$$

The operator-product expansion, in Johnson's gauge, is[4,2]

$$T(\psi(x)\bar{\psi}(y)) = -iG(x-y)I + C_1(x-y)\phi_+(y) + C_2(x-y)\phi_-(y) + C_3(x-y)j_+(y) + C_4(x-y)j_-(y) + \text{remainder}\,, \tag{3}$$

with

$$j_{\pm}(y) = j_1(y) \pm j_0(y)$$

and

$$G(x-y) = i\langle T(\psi(x)\bar{\psi}(y))\rangle_0\,.$$

Lowenstein[4] (see also Ref. 2) has determined the singularities of the functions C_i as $x\to y$: C_3 and C_4 are one power less singular than G; this is required in order for $j_\pm$ to have normal dimension. The singularity of

C_1 and C_2 depends on the coupling constant so $\phi_\pm$ has anomalous dimension. Using the coupling constant λ as defined in Ref. 2, the dimension d_ϕ is

$$d_\phi = \frac{(1-\lambda/2\pi)}{(1+\lambda/2\pi)}\,.$$

We now insert this expansion in $\langle T(\psi(x')\psi(x)\overline{\psi}(y)\overline{\psi}(y'))\rangle_0$, perform the gauge transformation, and let $x \to y$. Because $C_{3,4}$ are one power less singular than G, it is necessary to keep the second term in the expansion of the gauge transformation as $x \to y$ when it multiplies G. For the other terms, it is sufficient to keep the leading term in the expansion of the gauge transformation. Thus we find

$$\begin{aligned}\langle T(\psi(x')\psi(x)\overline{\psi}(y)\overline{\psi}(y'))\rangle_0 \to{}& \exp\{-ia^2[D^c(x-y)+D^c(x'-y')]\}\\ &\times\Big(-G(x-y)G(x'-y')+C_1(x-y)\langle T(\psi(x')\phi_+(y)\overline{\psi}(y'))\rangle+C_2(x-y)\langle T(\psi(x')\phi_-(y)\overline{\psi}(y'))\rangle\\ &\quad+C_3(x-y)\langle T(\psi(x')j_+(y)\overline{\psi}(y'))\rangle+C_4(x-y)\langle T(\psi(x')j_-(y)\overline{\psi}(y'))\rangle\\ &\quad-\frac{a^2}{4\pi}G(x-y)G(x'-y')(x-y)^\mu\frac{\partial}{\partial y^\mu}\ln\frac{(x'-y)^2}{(y'-y)^2}+\text{remainder}\Big)\,. \end{aligned} \tag{4}$$

From this we see that

$$\langle T(\psi(x')\phi_\pm(y)\overline{\psi}(y'))\rangle \to e^{-ia^2D^c(x'-y')}\langle T(\psi(x')\phi_\pm(y)\overline{\psi}(y'))\rangle$$

and hence the dimension of $\phi_\pm(y)$ is unchanged by the gauge transformation. The same is obviously true for $j_\pm(y)$. The singular functions are changed in the same way as the propagator:

$$G(x-y)\to e^{-ia^2D^c(x-y)}G(x-y)\,,\quad C_j(x-y)\to e^{-ia^2D^c(x-y)}C_j(x-y)\,.$$

There appears in the expansion now a new term which requires the presence of another term in the operator-product expansion with the same dimension as $j_\pm(y)$. There is a ready interpretation of this: Consider, for small $x-y$,

$$\begin{aligned}\psi(x)\overline{\psi}(y)\to\psi(x)e^{ia\chi(x)}e^{-ia\chi(y)}\overline{\psi}(y)&=e^{ia[\chi_+(x)-\chi_+(y)]}\psi(x)e^{a^2[\chi_-(x),\chi_+(y)]}\overline{\psi}(y)e^{ia[\chi_-(x)-\chi_-(y)]}\\ &=e^{-ia^2D^c(x-y)}[1+ia(x-y)^\mu\partial_\mu\chi(y)+\cdots]\psi(x)\overline{\psi}(y)\,.\end{aligned}$$

Thus, the operator-product expansion should contain a new term:

$$T(\psi(x)\overline{\psi}(y))=\cdots+ae^{-ia^2D^c(x-y)}G(x-y)(x-y)^\mu\partial_\mu\chi(y)\,.$$

The matrix element of the new term is easily calculable. It is

$$\langle T(\psi(x')\overline{\psi}(y')\partial_\mu\chi(y))\rangle_0=-\frac{a}{4\pi}e^{-ia^2D^c(x'-y')}G(x'-y')\frac{\partial}{\partial y^\mu}\ln\frac{(y-x')^2}{(y-y')^2}\,,$$

which is precisely the required term in Eq. (4). The gauge corresponding to the Johnson solution is then distinguished by the absence of the term $\partial_\mu\chi$ in the operator-product expansion.

It was shown in Ref. 1 that the Feynman graphs admit a more general gauge transformation than can be effected by Eq. (1). It is easy to carry through the above calculation for these more general transformations and show, as above, that d_ϕ is invariant. However, we have not determined the corresponding modification of the operator-product expansion.

Dell'Antonio, Frishman, and Zwanziger[5] and Schroer and Seiler[6] have also noted that the dimension of $\phi_\pm(y)$ is gauge-independent.

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*Work supported, in part, by the U. S. Atomic Energy Commission.

[1]A. H. Mueller and T. L. Trueman, Phys. Rev. D 4, 1635 (1971).

[2]Kenneth G. Wilson, Phys. Rev. D 2, 1473 (1970).

[3]K. Johnson, Nuovo Cimento 20, 773 (1961).

[4]J. Lowenstein, Commun. Math. Phys. 16, 265 (1970).

[5]G. F. Dell'Antonio, Y. Frishman, and D. Zwanziger, Weizmann Institute report (unpublished).

[6]B. Schroer (private communication).