

Manifestation of infrared instabilities in high energy processes in non-Abelian gauge theories

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We show that a high-frequency standing wave in SU(2) gauge theory is unstable against decay into long-wavelength modes. This provides a nonperturbative mechanism for energy transfer from initial high-momentum modes to final states with low momentum excitations. The Abelian case does not manifest such instability. Our analysis is supported by lattice simulations.

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Recently the intriguing problem of the possibly large rate for electroweak baryon-number-changing processes in high-energy collisions, originally based on the instanton approach [1], was considered from a completely different point of view, based on the classical analogue to high-energy particle collisions [2,3]. The hope was that one could observe the energy transfer from fast (high-frequency) classical wave modes, presumably corresponding to initial states containing few high-energy particles, to slow (low-frequency) classical wave modes, representing low-energy multiparticle final states. The studies involved numerical simulations of the (1+1)-dimensional Abelian Higgs model [2] and the ϕ^4 theory in 3+1 space-time dimensions [3]. In both cases no indication for the existence of such a mechanism was found. Instead, the classical high-energy scattering behavior was found to be fully consistent with expectations from lowest-order perturbation theory. The nonlinearity of the investigated theories apparently does not furnish a mechanism for the formation of final states containing many low-energy particles from the initial states with few high-energy particles.

In these approaches the main object of study was the nature of the transfer of energy from high-energy to low-energy modes, which is thought to be in some sense analogous to the high-energy scattering situation. We believe that the observed absence of an efficient energy transfer mechanism is intimately connected with the integrable nature of the considered classical systems. On the other hand, it is well known that non-Abelian gauge theories are nonintegrable in the classical limit [4–6]. So in these theories there might exist nonperturbative classical mechanisms for the coupling between fast and slow modes. In this paper we want to investigate the existence of such a mechanism in the framework of a simple instructive example involving a non-Abelian gauge theory.

Let us model a high-energy particle collision as the interaction between two counterpropagating Yang-Mills

plane waves. For simplicity let us work in SU(2) gauge theory. The simplest ansatz satisfying the free Yang-Mills equation is an Abelian standing wave

$$A_i^c(x, t) = \delta_{i3}\delta_{c3}A \cos k_0 x \cos \omega_0 t \quad (1)$$

with $k_0 = \omega_0$ assumed to be large. (We work in the temporal gauge $A_0^c = 0$.) We now consider small perturbations $a_i^c(\mathbf{x}, t)$ around this solution and study their stability properties. We will show that there are unstable low-energy modes for arbitrarily small amplitude A of the driving wave. The amplitude of these modes grows exponentially with time with a growth rate proportional to A . We will later confirm the presence of the instability by a numerical calculation and demonstrate that it leads to rapid total thermalization of the field energy, once the perturbation has grown to the same strength as the driving wave.

It is easy to see that fluctuations in the same color direction as the background field ($c = 3$) are stable; hence, we will consider only color fluctuations transverse in color space $a_i = a_i^1 + ia_i^2$. The linearized field equations and Gauss's law for the field perturbation are

$$\partial_t^2 \mathbf{a} = (\mathbf{D} \cdot \mathbf{D})\mathbf{a} - 2i(\mathbf{B} \times \mathbf{a}) - \mathbf{D}(\mathbf{D} \cdot \mathbf{a}) \quad (2)$$

and

$$\partial_t(\mathbf{D} \cdot \mathbf{a}) = 0, \quad (3)$$

where boldface type indicates three-dimensional vectors in position space. $\mathbf{B}$ is the magnetic background field generated by the standing wave, and $\mathbf{D} = \nabla + i\mathbf{a}$ is the gauge covariant derivative. Gauss's law (3) ensures that the physical states are invariant under time-independent gauge transformations. To fix the residual time-independent gauge, we impose the background field Coulomb gauge constraint:

$$\mathbf{D} \cdot \mathbf{a} = 0. \quad (4)$$

Now the field equation simplifies, allowing for a decomposition of the orientations of the polarization vector of the perturbation. Modes polarized in the direction of the

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magnetic background field $\mathbf{B}$ (the y direction) are stable; hence, we only need to consider modes of the form

$$\mathbf{a}(\mathbf{x}, t) = (\hat{\mathbf{z}} + i\sigma\hat{\mathbf{x}})a_\sigma(\mathbf{x}, t) \quad (\sigma = \pm 1), \quad (5)$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ denote unit vectors in the direction of the wave vector and the polarization of the background field, respectively. These modes obey the equation

$$\partial_t^2 a_\sigma - \nabla^2 a_\sigma = (2i\mathbf{a} \cdot \nabla - \mathbf{a}^2 - 2\sigma B)a_\sigma. \quad (6)$$

We now note that the background field A_i^c oscillates very rapidly in space and time, whereas we are seeking slowly varying perturbative modes a_i^c . The dynamics of these modes is governed by the time- and space-averaged interaction with the fast background field. Since we consider the amplitude of the background wave to be small, i.e., $A \ll k_0$, we will find a solution perturbatively by calculating the self-energy of the fluctuating field to second order in the background field. After spacetime averaging, we obtain the following expression for the self-energy for a fluctuation with wave vector $|\mathbf{k}| = k \ll k_0$ and frequency $\omega \ll \omega_0$:

$$\Sigma(\mathbf{k}, \omega) = -\frac{A^2}{4} \left[1 - (\omega^2 - \mathbf{k}^2) \frac{\omega^2 + k_x^2}{(\omega^2 - k_x^2)^2} \right]. \quad (7)$$

Note that the expression for Σ is independent of the sign of the circular polarization σ of the perturbation (5). The dispersion relation for the fluctuation modes is obtained from the poles of the background field propagator

$$G(\mathbf{k}, \omega)^{-1} = -\omega^2 + \mathbf{k}^2 - \Sigma(\mathbf{k}, \omega). \quad (8)$$

We shall check that the perturbative solutions also satisfy the background Coulomb gauge constraint (4). Since the equation of motion (6) is solved to second order in A , it is consistent that we only require the solutions to satisfy Gauss's law to second order. For a particular momentum $\mathbf{k}$, we look for a solution of the form

$$\mathbf{a}(\mathbf{x}, t) = \sum_{\sigma=\pm 1} (\hat{\mathbf{z}} + i\sigma\hat{\mathbf{x}})a_\sigma(\mathbf{x}, t). \quad (9)$$

Inserting it into (4), disregarding the fast oscillating terms, we get a relation between a_+ and a_- , which is represented by their ratio as a function of momentum $\mathbf{k}$:

$$\frac{a_+}{a_-} = \frac{(ik_x - k_z)(1 + I_1) - iI_2}{(ik_x + k_z)(1 + I_1) - iI_2}, \quad (10)$$

where

$$I_1 = \frac{A^2}{4} \left(\frac{1}{\omega^2 - \mathbf{k}^2} - \frac{\omega^2 + k_x^2}{(\omega^2 - k_x^2)^2} \right), \quad I_2 = \frac{A^2}{4} \frac{k_x}{\omega^2 - k_x^2}. \quad (11)$$

Hence for any $\mathbf{k}$ there is a solution satisfying Gauss's law. The most unstable modes are those with $k_y = k_z = 0$ for

which, keeping only terms up to second order in A , we obtain from (8) the dispersion relation

$$\omega^2 = k_x^2 \pm \frac{i}{\sqrt{2}} Ak_x, \quad (12)$$

which is complex for any nonzero k_x and A . We conclude that the infrared instability exists for arbitrarily small values of the amplitude A of the background standing wave. When $k_x \gg A$, the imaginary part of the frequency, i.e., the exponential growth rate of a perturbation,

$$\text{Im}[\omega(k)] = \frac{A}{2\sqrt{2}}, \quad (13)$$

is independent of the wave vector k_x .

Unfortunately, the convergence of the above expansion in increasing powers of A^2 is not evident. An explicit calculation shows that the fourth order contribution to self-energy has the same form as the second-order one (7) near the pole, but with a smaller numerical factor. In order to verify the existence of this instability beyond perturbation theory, and to find out what happens when the fluctuation begins to absorb a significant fraction of the driving background wave, we have studied the evolution of a slightly perturbed low-amplitude standing plane wave in $SU(2)$ gauge theory on a three-dimensional lattice in the classical limit. The numerical aspects of such simulations have been described in detail elsewhere [6,7]. Here we have initialized the gauge field as an Abelian standing wave with an amplitude corresponding to less than 5% of the maximal magnetic energy density on the 16^3 lattice. When we apply a small *Abelian* perturbation, restricted to the same direction in color space as the standing wave, the field oscillations remain stable, as shown by the dashed line in Fig. 1. When we add a general *non-Abelian* perturbation, pointing in a random color direction, the field oscillations develop a visible instability around time $t = 20$, as shown by the solid curve in Fig. 1.

It is instructive to compare the Fourier spectrum of the field energy density

$$E(\mathbf{k}) = \frac{1}{2} \int (\mathbf{B}(\mathbf{x})^2 + \mathbf{E}(\mathbf{x})^2) e^{i\mathbf{k} \cdot \mathbf{x}} d^3x, \quad (14)$$

at the end of our simulation ($t = 50$) with the energy spectrum of the initial configuration, as shown in Fig. 2, in both cases. The initial spectrum is almost completely concentrated in the mode with $|\mathbf{k}| = \pi/a$, where a is the lattice spacing. The case with a *non-Abelian* perturbation is shown in the upper part of Fig. 2, where the final spectrum is distributed over all modes. On the other hand, in the case with an *Abelian* perturbation, the final distribution remains concentrated in the original mode, as shown in the lower part of Fig. 2. In order to check whether the available field energy has been thermalized in the case with non-Abelian perturbation, we have

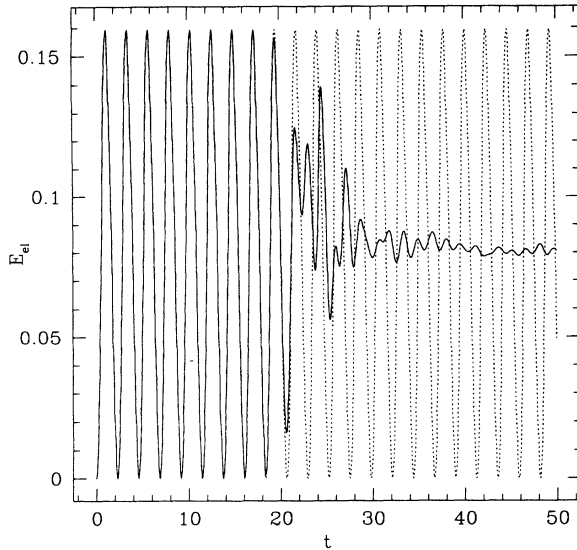


FIG. 1. The total electric energy as a function of time. The initial state is a standing wave with a small perturbation. The solid line shows the case when the perturbation is non-Abelian, which results in the destruction of the plane wave beginning at $t \approx 20$. The dotted line is for an Abelian perturbation, in which case the standing wave is stable. The computation is performed on a 16^3 lattice and energy per plaquette is about 0.16 in lattice units.

calculated the probability distribution of the magnetic plaquette energy $P(E)$, divided by the single plaquette phase space $\sigma(E)$. The distribution $P(E)/\sigma(E)$ falls exponentially with E , indicating thermalization [7].

The quantitative feature of the instability can be characterized by the associated largest Lyapunov exponent λ_0 . Numerically we find, in dimensionless form,

$$\lambda_0 a = k a f(\alpha), \quad \alpha = \frac{g A a}{k a}, \quad (15)$$

where g is the coupling constant. Obviously, λ_0 survives in the continuum limit $a \rightarrow 0$. The function f is obtained numerically and shown in Fig. 3. For small α it is a linear function $f(\alpha) \approx 0.5\alpha$. Inserting into the above relation, we have

$$\lambda_0 \approx \frac{g A}{2}, \quad (16)$$

which resembles our result (13) from second-order perturbation but involves a different coefficient. We note that the scaling behavior (15) is different from the scaling behavior of a Lyapunov exponent of a random trajectory [7], which scales with the energy rather than the amplitude of the background field. The unstable mode described above is more directly related to the instability of a uniform chromomagnetic background field [8], and can be traced back to the magnetic moment interaction of a spin-1 particle, expressed by the form $2\sigma B$ in (6).

Let us make one final remark connected with the high-energy limit of Eq. (6) for the perturbation $\mathbf{a}(\mathbf{x}, t)$. If we consider the behavior of this equation under rescaling of the longitudinal coordinates x and t (or that of the

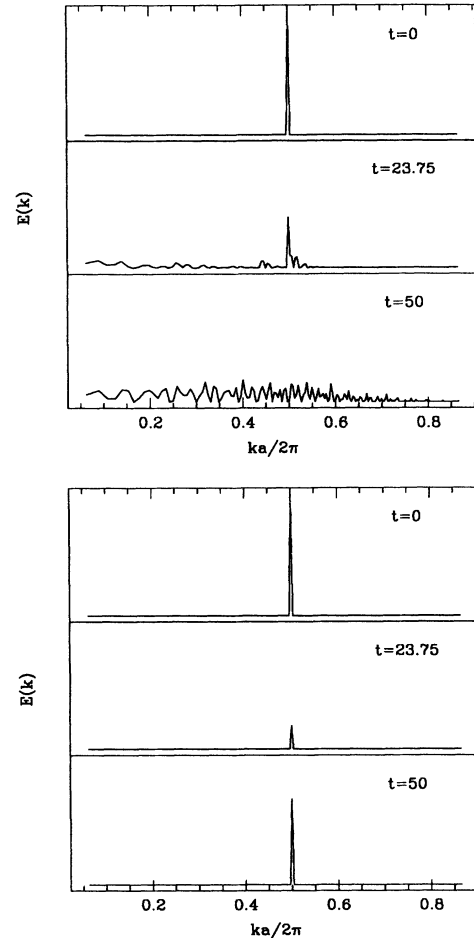


FIG. 2. Time evolution of $|E_k|$, the absolute value of energy spectrum defined in eq.(14). E_0 is set to zero by hand. The upper part is for the case of a non-Abelian perturbation, against which the original standing wave is unstable, while the lower part is for an Abelian perturbation, where we observe no instability.

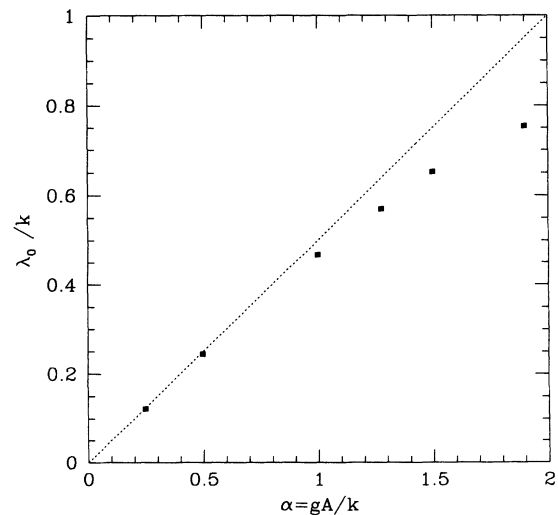


FIG. 3. The scaling function $f(\alpha)$ defined in Eq. (15). The dotted line is a linear fit with slope 0.5. The results are obtained by varying the amplitude A with a fixed wave number $ka = 0.49$ on a 256×4^2 lattice.

light-cone coordinates $x \pm t$) by

$$t \rightarrow \lambda t, \quad x \rightarrow \lambda x, \quad y \rightarrow y, \quad z \rightarrow z, \quad (17)$$

one will see that after taking the high-energy limit [9] $\lambda \rightarrow 0$ the terms with magnetic field B in (6), which caused the above described instability, will disappear. One is left with the trivial wave equation

$$(\partial_t^2 - \partial_x^2)\mathbf{a}(\mathbf{x}, t) = 0. \quad (18)$$

This example shows that the transverse directions must be handled with care because sometimes they can be the origins of singularities associated with the gauge sector of non-Abelian gauge theories. In our example the color field does not behave like an Abelian electromag-

netic field, but shows the complexity of the transverse dynamics that cannot be ignored.

In conclusion, we have shown that a standing wave of high frequency in SU(2) gauge theory is unstable against decay into long-wavelength modes. This implies that interactions between high-energy particles can result in a final state with many low-energy particles, which might be relevant in baryon-number changing process.

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