

Dimensionally continued black holes

Máximo Bañados*

*Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile
and Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile*

Claudio Teitelboim*

*Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile
and Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540*

Jorge Zanelli*

*Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile
and Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile*

(Received 23 July 1993)

Static, spherically symmetric solutions of the field equations for a particular dimensional continuation of general relativity with a negative cosmological constant are studied. The action is, in odd dimensions, the Chern-Simons form for the anti-de Sitter group and, in even dimensions, the Euler density constructed with the Lorentz part of the anti-de Sitter curvature tensor. Both actions are special cases of the Lovelock action, and they reduce to the Hilbert action (with a negative cosmological constant) in the lower dimensional cases $\mathcal{D} = 3$ and $\mathcal{D} = 4$. Exact black hole solutions characterized by mass (M) and electric charge (Q) are found. In odd dimensions a negative cosmological constant is necessary to obtain a black hole, while in even dimensions both asymptotically flat and asymptotically anti-de Sitter black holes exist. The causal structure is analyzed and the Penrose diagrams are exhibited. The curvature tensor is singular at the origin for all dimensions greater than three. In dimensions of the form $\mathcal{D} = 4k, 4k - 1$, the number of horizons may be zero, one, or two, depending on the relative values of M and Q , while for a negative mass there is no horizon for any real value of Q . In the other cases, $\mathcal{D} = 4k + 1, 4k + 2$, both naked and dressed singularities with a positive mass exist. As in three dimensions, in all odd dimensions anti-de Sitter space appears as a "bound state" of mass $M = -1$, separated from the continuous spectrum ($M \geq 0$) by a gap of naked curvature singularities. In even dimensions anti-de Sitter space has zero mass. The analysis is Hamiltonian throughout, considerably simplifying the discussion of the boundary terms in the action and the thermodynamics. The Euclidean black hole has the topology $\mathbb{R}^2 \times S^{\mathcal{D}-2}$. Evaluation of the Euclidean action gives explicit expressions for all the relevant thermodynamical parameters of the system. The entropy, defined as a surface term in the action coming from the horizon, is shown to be a monotonically increasing function of the black hole radius, different from the area for $\mathcal{D} > 4$.

PACS number(s): 97.60.Lf, 04.20.Jb

I. INTRODUCTION

The black hole is a central element of general relativity, but its properties, especially at the quantum level, are far from being completely understood. It is of interest, therefore, to search for black holes in the extension of Einstein's gravitation theory to higher and lower dimensions. Here we consider traditional black-hole solutions with gravitational and electromagnetic fields only. Other systems, including two-dimensional (2D) black holes with a dilaton field, and string-inspired holes, are not considered [1].

The most general extension to higher dimensions of general relativity, which keeps the field equations for the metric of second order, is the so-called Lovelock action

[2] which may be regarded as formed by the dimensional continuation of the Euler characteristics of lower dimensions [3]. Although very similar in content and structure to the usual theory, the Lovelock theory has some unwanted features. First, for $\mathcal{D} > 4$, the time evolution of the fields appears to be nonunique: given an initial value surface at some time $t = t_0$, the fields at $t > t_0$ are not completely determined by the equations of motion. This is due to the presence of high powers of the velocities in the Lagrangian [3, 4]. Second, in addition to Newton's constant and the cosmological constant, the action has $n = [(D + 1)/2]$ arbitrary dimensionful parameters [5]. This arbitrariness makes the analysis of the black hole properties rather complicated.

In this work, we will deal neither with the pathological time evolution problem nor with the most general set of coefficients. Instead, we will assume from the very beginning a static metric and we will, in what we believe is a natural manner, restrict the coefficients to a subfam-

*Electronic address: cecsphy@lascar.puc.cl

ily of two parameters related to Newton's constant and to the cosmological constant. The chosen action *is not* the Hilbert action with a cosmological constant, but it reduces to it in the lower dimensional cases $\mathcal{D} = 3$ and $\mathcal{D} = 4$. We will deal only with a negative cosmological constant related to a length scale l by $\Lambda = -al^{-2}$ where a is a positive number. (Most of the results remain valid for positive Λ by the replacement $l \rightarrow il$. In that case, however, the spatial sections of spacetime are closed, and no conserved quantities such as mass and electric charge can be defined.) The construction of the action is carried out in Sec. II.

Static spherically symmetric solutions for the Lovelock action have been studied by several authors [6–8]. However, in those references no choice of the Lovelock coefficients is made, making it difficult to extract physical information from the solution. This is so because the metric components g_{rr} and g_{00} are the roots of a polynomial of degree $n - 1$ which cannot be solved explicitly. Apart from the problem of solving this algebraic equation, some physical consequences of a generic choice of the coefficients should be remarked: (i) There exist up to $n - 1$ different solutions, (ii) all of them may have horizons for both signs of the energy, and (iii) the entropy is not a monotonically increasing polynomial of the horizon radius r_+ , so that the second law of thermodynamics does not hold. All of this suggests that a special choice of the arbitrary coefficients should be made. The choice proposed here produces a unique solution for the metric with a dressed singularity only for positive masses, and the entropy is a monotonically increasing function of r_+ .

Our choice may not be the only one that has these properties. But, as shown in Sec. II, it is somewhat natural in that it is the Euler-Chern-Simons form in odd dimensions, and it may be regarded as the gravitational analogue of the Born-Infeld electrodynamics in even dimensions.

In Sec. III, the equations of motion are solved and explicit expressions for the metric are obtained. It is observed that two branches of black holes emerge: one for even dimensions, with strong similarities to the Schwarzschild metric in 3+1 dimensions, and another for odd dimensions, with many features in common with the 2+1 black hole [9, 10]. The Penrose diagrams displaying the causal structure are exhibited.

In Sec. IV the black hole thermodynamics is analyzed. It is shown that the entropy is a surface term in the action coming from the horizon [11], while the internal energy and electric charge are given by flux integrals at infinity. The Hamiltonian method allows us to compute the entropy by direct evaluation of the Euclidean action with an automatic regularization. As was previously noted in the four-dimensional case [12], the Hamiltonian symplectic structure is preserved by the thermodynamical description of the higher dimensional black holes. That is, conjugate variables in the sense of mechanics are also thermodynamical conjugates.

The immersion of the Lorentz Lie algebra in the anti-de Sitter algebra is reviewed in Appendix A. The Hamiltonian structure of the Lovelock action is worked out in Ref. [3] and it is reviewed in Appendix B.

II. ACTION

A. Lovelock action

The most general action that generalizes Einstein's gravity, while keeping the same degrees of freedom when spacetime has a dimension $\mathcal{D} \geq 3$, is a sum of the dimensionally continued Euler characteristics of all dimensions below $\mathcal{D}$ [2, 3, 13]. The action takes the form

$$I = \kappa \sum_{p=0}^n \alpha_p I_p \quad (1)$$

with

$$I_p = \int \epsilon_{a_1 \dots a_{\mathcal{D}}} R^{a_1 a_2} \wedge \dots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \dots \wedge e^{a_{\mathcal{D}}}. \quad (2)$$

Here e^a is the local frame one-form, $R_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c$ is the curvature two-form, and ω_b^a is the spin connection, $a_i = \{0, 1, \dots, \mathcal{D} - 1\}$. The coefficients α_p are arbitrary constants with dimensions $[\text{length}]^{-(\mathcal{D}-2p)}$ and κ has units of action.

For even $\mathcal{D}$, the last term in the sum ($p = \mathcal{D}/2$) is the Euler characteristic and does not contribute to the equations of motion. In quantum theory, however, this term assigns different phases to different topologies.

Varying (1) with respect to e^a yields the field equations, which are more complicated than Einstein's equations, involving high powers of the curvature (they are nonlinear in the velocities and accelerations). These equations, however, are still second order in the metric and the system has a standard constrained Hamiltonian formulation [3]. The variation with respect to the spin connection ω vanishes identically by the assumption of zero torsion.

In the standard metric formulation, the action (1) is constructed by the same requirements as in $\mathcal{D} = 4$ [2]: general covariance and second-order field equations for the metric. In the language of forms, (1) can be obtained by the requirement that the Lagrangian be a local Lorentz invariant $\mathcal{D}$ -form (up to closed forms), constructed entirely out of e , ω , and their exterior derivatives, without using the Hodge dual ($*$ operation). This is equivalent, in turn, to defining the action by dimensional continuation of topological invariants (characteristic classes) of lower dimensions [14, 15].

These demands, however, do not restrict the values of the coefficients α_p . In order to select these coefficients we will consider embedding the Lorentz group $\text{SO}(\mathcal{D} - 1, 1)$ into a larger group. Since we are interested in open spaces, the minimal extension for $\text{SO}(\mathcal{D} - 1, 1)$ is the anti-de Sitter group $\text{SO}(\mathcal{D} - 1, 2)$.

In odd dimensions it is possible to construct a Lagrangian invariant under the anti-de Sitter group by making a certain choice of the Lovelock coefficients. That Lagrangian is the Chern-Simons form associated with the Euler density for one dimension above $\mathcal{D}$. In even dimensions, on the other hand, it is not possible to construct a nontrivial action principle invariant under $\text{SO}(\mathcal{D} - 1, 2)$ and it is necessary to break the symmetry down to the Lorentz group. This symmetry breaking is similar to that

used by MacDowell and Mansouri in four dimensions [16].

In what follows, we will consider a particular choice of the Lovelock coefficients α_p in (1) given by

$$\alpha_p = \begin{cases} \frac{1}{\mathcal{D} - 2p} \binom{n-1}{p} l^{-\mathcal{D}+2p}, & \mathcal{D} = 2n - 1, \\ \binom{n}{p} l^{-\mathcal{D}+2p}, & \mathcal{D} = 2n, \end{cases} \quad (3a)$$

$$(3b)$$

where l is a length. As we will see in the next sections, this choice provides an action principle with the above properties.

B. Action in odd dimensions

In order to construct the action for $\mathcal{D} = 2n - 1$, we consider the Euler density in one dimension above $\mathcal{D}$. This density is an exact form and can be written as

$$\mathcal{E}_{2n} = \kappa \epsilon_{A_1 \dots A_{2n}} \hat{R}^{A_1 A_2} \wedge \dots \wedge \hat{R}^{A_{2n-1} A_{2n}} = d \wedge \mathcal{L}_{2n-1}. \quad (4)$$

$$\mathcal{L}_{2n-1} = \kappa \sum_{p=0}^{n-1} \alpha_p \epsilon_{a_1 \dots a_{\mathcal{D}}} R^{a_1 a_2} \wedge \dots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \dots \wedge e^{a_{\mathcal{D}}},$$

where α_p is given in (3a).

Under $\text{SO}(\mathcal{D} - 1, 2)$ gauge transformations, the density $\mathcal{L}_{2n-1}$ is changed by the addition of a closed form. This makes its integral invariant under gauge transformations that do not change the boundary data.

The construction given here for the action in odd dimensions is very similar to the Chern-Simons action in three dimensions. One may ask whether the equivalence between local gauge transformations and diffeomorphisms in three dimensions is also valid here. The answer is in the negative because this equivalence requires the curvature two-form to vanish as a consequence of the equations of motion, which is true in three dimensions only (see Sec. IID).

Note that in the limit $l \rightarrow \infty$ (zero cosmological constant), only the last term $p = n - 1$, and not the Hilbert term, in the sum (7) contributes to the action.

C. Action in even dimensions

Now the situation is different: there is no analogue of the Chern-Simons action, invariant under the enlarged gauge group up to surface terms. It is necessary to break the full anti-de Sitter symmetry in order to produce a nontrivial action principle [16].

For $\mathcal{D} = 2n$ the Lagrangian must be of the form

For later convenience, the units are chosen so that

$$\kappa = \frac{l}{(\mathcal{D} - 2)! \Omega_{\mathcal{D}-2}} \quad (\mathcal{D} \text{ odd}), \quad (5)$$

where $\Omega_{\mathcal{D}-2}$ is the area of the $(\mathcal{D} - 2)$ unit sphere.

The curvature tensor $\hat{R}^{AB}$ is constructed with the $\text{SO}(\mathcal{D} - 1, 2)$ connection W^{AB} and the capital Latin indices run from 0 to $2n$ (see Appendix A). The form $\mathcal{E}_{2n}$ cannot be used as a Lagrangian in $2n$ dimensions since it is a total derivative, but $\mathcal{L}_{2n-1}$ is a Lagrangian in $2n - 1$ dimensions. In analogy with the Chern-Simons forms constructed from the Pontryagin classes, we call $\mathcal{L}_{2n-1}$ the Euler-Chern-Simons $(2n - 1)$ -form.

Using the decomposition of W^{AB} into Lorentz rotations and “inner translation” described in Appendix A which expresses the anti-de Sitter curvature form $\hat{R}$ in terms of the Lorentz curvature R as

$$\hat{R}^{ab} = R^{ab} + l^{-2} e^a \wedge e^b, \quad (6)$$

the following expression for $\mathcal{L}_{2n-1}$ as a function on the spin connection w^{ab} and the local frame e^a is obtained:

$$\mathcal{L}_{2n} = \kappa \hat{R}^{A_1 A_2} \wedge \hat{R}^{A_3 A_4} \wedge \dots \wedge \hat{R}^{A_{\mathcal{D}-1} A_{\mathcal{D}}} Q_{A_1 A_2 \dots A_{\mathcal{D}}}, \quad (8)$$

where $Q_{A_1 A_2 \dots A_{\mathcal{D}}}$ is a tensor of rank $\mathcal{D}$ under the group and $\hat{R}^{AB}$ is the anti-de Sitter curvature [see Eq. (A3) and (A5)]. For later convenience, we choose the units so that

$$\kappa = \frac{l^2}{2\mathcal{D}(\mathcal{D} - 2)! \Omega_{\mathcal{D}-2}} \quad (\mathcal{D} \text{ even}) \quad (9)$$

which reproduces the standard units used in $\mathcal{D}=4$ and provides a convenient normalization for the black hole mass in higher dimensions.

If we choose $Q_{A_1 A_2 \dots A_{\mathcal{D}}}$ as an invariant tensor of the anti-de Sitter group then, by virtue of the Bianchi identity, the Lagrangian (8) gives no equations of motion. If, instead, $Q_{A_1 A_2 \dots A_{\mathcal{D}}}$ is chosen as an invariant tensor under the Lorentz group only, namely,

$$Q_{A_1 A_2 \dots A_{\mathcal{D}}} = \begin{cases} \epsilon_{a_1 \dots a_{\mathcal{D}}} & \text{for } A_i = a_i \quad (i = 1, \dots, \mathcal{D}), \\ 0 & \text{otherwise,} \end{cases} \quad (10)$$

then (8) becomes

$$\mathcal{L}_{2n} = \kappa (R^{a_1 a_2} + l^{-2} e^{a_1} \wedge e^{a_2}) \wedge \dots \wedge (R^{a_{\mathcal{D}-1} a_{\mathcal{D}}} + l^{-2} e^{a_{\mathcal{D}-1}} \wedge e^{a_{\mathcal{D}}}) \epsilon_{a_1 a_2 \dots a_{\mathcal{D}}} \quad (11)$$

which corresponds to the Lagrangian in (1) with the choice (II Ab). This Lagrangian may be regarded as the gravitational analogue of the Born-Infeld Lagrangian [17]. The analogy is more clearly brought out by rewriting (11) as

$$\mathcal{L}_{2n} = \kappa \text{Pf}[R^{ab} + l^{-2}e^a \wedge e^b], \quad (12)$$

where Pf denotes the Pfaffian (a multilinear function whose square is the determinant) in the exterior product.

$$(R^{a_1 a_2} + l^{-2}e^{a_1} \wedge e^{a_2}) \wedge \dots \wedge (R^{a_{2n-3} a_{2n-2}} + l^{-2}e^{a_{2n-3}} \wedge e^{a_{2n-2}}) \epsilon_{a_1 a_2 \dots a_{2n-1}} = 0 \quad (13)$$

in odd dimensions ($\mathcal{D} = 2n - 1$), and

$$(R^{a_1 a_2} + l^{-2}e^{a_1} \wedge e^{a_2}) \wedge \dots \wedge (R^{a_{2n-3} a_{2n-2}} + l^{-2}e^{a_{2n-3}} \wedge e^{a_{2n-2}}) \wedge e^{a_{2n-1}} \epsilon_{a_1 a_2 \dots a_{2n}} = 0 \quad (14)$$

in even dimensions ($\mathcal{D} = 2n$).

As we will see later, the factorized form of these equations, which is a consequence of the particular choice of the coefficients, leads to a considerable simplification in the study of the physical properties of its solutions.

Equation (13) represents the extension of the Chern-Simons equation to higher dimensions. For $\mathcal{D} > 3$ and $l \neq 0$ the curvature tensor does not necessarily vanish as a consequence of the equation of motion. That means that the anti-de Sitter connection is not pure gauge. Of course $\tilde{R}^{ab} \equiv R^{ab} + l^{-2}e^a \wedge e^b = 0$ is a solution in both cases and it is locally anti-de Sitter space.

III. SPHERICALLY SYMMETRIC SOLUTIONS

A. Reduced action and equations of motion

Consider first the case of zero electric charge. Spherically symmetric solutions for the Einstein-Lovelock equations have been studied by several authors [6–8]. Our goal here is the study of those solutions in the specialized case in which the Lovelock coefficients are chosen as in (3). For our purpose here it is enough to consider a reduced action principle where the allowed metrics are static and spherically symmetric. In an appropriate coordinate system the metric can be written as

$$ds^2 = -N^2(r)g^2(r)dt^2 + g^{-2}(r)dr^2 + r^2 d\Omega^2, \quad (15)$$

where $d\Omega^2 = \gamma_{mn}dx^m dx^n$ is the metric on the $(\mathcal{D} - 2)$ unit sphere [we will also use later on the notation $\gamma = \det(\gamma_{mn})$], and $N(r)$ and $g(r)$ are functions to be varied.

Working in Hamiltonian form, it is a simple matter to evaluate the action for the metric (15). By direct application of the results of Ref. [3] one finds, for the reduced action (see Appendix B),

$$\begin{aligned} I &= (t_2 - t_1) \int dr N \left(\frac{\partial F}{\partial g} g' + \frac{\partial F}{\partial r} \right) + B \\ &= (t_2 - t_1) \int dr N F' + B. \end{aligned} \quad (16)$$

In the limit $l \rightarrow \infty$, the only nonvanishing contributions to the Lagrangian are the Euler density and the term with the second highest power in the curvature tensor (R^{n-1}). The Euler density (R^n), which does not contribute to the field equations, acquires in this limit an infinite coupling constant.

D. Equations of motion

The equations of motion derived from the Lagrangians (7) and (11) are

Here B is a surface term that will be adjusted below, the prime denotes derivative with respect to the radial coordinate r , and the function $F[r, g(r)]$ is given by

$$F[r, g(r)] = \begin{cases} \frac{1}{2}r [1 + (r/l)^2 - g^2(r)]^{n-1}, & \mathcal{D} = 2n, \\ [1 + (r/l)^2 - g^2(r)]^{n-1}, & \mathcal{D} = 2n - 1. \end{cases} \quad (17)$$

The equations of motion derived from (16) are

$$\frac{dN}{dr} = 0, \quad \frac{dF}{dr} = 0 \quad (18)$$

having the solutions

$$N(r) = N_\infty = \text{const}, \quad F[r, g(r)] = C = \text{const}. \quad (19)$$

The two parameters C and N_∞ are the constants of integration of the problem. By adjusting the time scale, N_∞ can be set equal to one, but it will be conveniently kept as an independent parameter for the analysis of the energy and the thermodynamical properties of the solution. The parameter C is, up to an additive constant, the black hole mass:

$$C = M + C_0 \quad (20)$$

(see Sec. III F). We will fix the constant C_0 in Sec. III B.

B. Black holes

From (17) and (19) we can express the metric coefficient $g^2(r)$ as a function of the mass M and r as

$$g^2(r) = \begin{cases} 1 - (2M/r)^{\frac{1}{n-1}} + (r/l)^2, & \mathcal{D} = 2n, \\ 1 - (M+1)^{\frac{1}{n-1}} + (r/l)^2, & \mathcal{D} = 2n - 1. \end{cases} \quad (21)$$

Choosing $N_\infty = 1$ the metric takes the form

$$ds^2 = - \left[1 - (2M/r)^{\frac{1}{n-1}} + (r/l)^2 \right] dt^2 + \frac{dr^2}{1 - (2M/r)^{\frac{1}{n-1}} + (r/l)^2} + r^2 d\Omega^2 \quad (22)$$

for even dimensions, and

$$ds^2 = - \left[1 - (M+1)^{\frac{1}{n-1}} + (r/l)^2 \right] dt^2 + \frac{dr^2}{1 - (M+1)^{\frac{1}{n-1}} + (r/l)^2} + r^2 d\Omega^2 \quad (23)$$

for odd dimensions.

Note that for even $\mathcal{D}$ the mass has dimensions of length whereas for odd $\mathcal{D}$, it is dimensionless. This is because in the even case the constant κ in the action has dimensions of length squared [Eq. (9)] whereas in the odd case it is a length to the first power [Eq. (5)].

The criterion we adopt here to fix C_0 is that for zero energy the horizon should disappear. This yields

$$C_0 = \begin{cases} 0, & \mathcal{D} = 2n, \\ 1, & \mathcal{D} = 2n - 1. \end{cases} \quad (24)$$

The usual cases $\mathcal{D}=4$ and $\mathcal{D}=3$ are obtained from (22) and (23) in the special case $n = 2$.

The metrics (22) and (23) describe a black hole if the function $g^2(r)$ has at least one root for a real positive value r_+ :

$$g^2(r_+) = 0, \quad r_+ > 0. \quad (25)$$

From (21) it follows that for odd n ($\mathcal{D} = 4k+1, 4k+2$), the following happens: (i) there is a sign ambiguity in the $(n-1)$ th root that appears in g^2 and (ii) the mass M must be positive in order to have a real solution. If the plus sign is chosen, then the solution has an event horizon. If the minus sign is chosen, the solution has a naked singularity. Note that one could have a naked singularity with positive mass in this case.

For even n ($\mathcal{D} = 4k-1, 4k$), there is no sign ambiguity and it is a simple matter to check that a horizon exists if and only if the mass is positive.

We will take as a fundamental requirement that there should be no naked singularities with positive mass. This is a form of cosmic censorship. Hence we will exclude odd n from the physical spectrum and we are only left with the spacetime dimensions

$$\begin{aligned} \mathcal{D} &= 4k-1 = 3, 7, 11, \dots, \\ \mathcal{D} &= 4k = 4, 8, 12, \dots \end{aligned}$$

C. Causal structure in even dimensions

The metric (22) for $\mathcal{D} = 4k$ represents a natural extension of the Schwarzschild geometry to higher dimensions. The metric diverges at the origin as $r^{\frac{1}{2k-1}}$ and the curvature tensor is also singular there. The asymptotic region, on the other hand, approaches anti-de Sitter space. The Penrose diagram of that solution does not depend on $\mathcal{D}$, and it is shown in Fig. 1(a). There are two asymptotically anti-de Sitter regions (vertical lines) and two singularities, past and future (horizontal lines), where the

geodesics end. The horizon is drawn as 45° lines.

In the case of zero mass one obtains anti-de Sitter space [Fig. 1(b)]. The points i^+ and i^- represents future and past infinity and are disjoint points in the figure. Further properties of that space can be found in Ref. [18].

D. Causal structure in odd dimensions

The higher dimensional black hole in odd dimensions has a strong resemblance with the 2+1 black hole solution [9]. However, for $\mathcal{D} > 3$ the curvature tensor is no longer constant as in $\mathcal{D} = 3$ and, therefore, those solutions cannot be obtained from anti-de Sitter space by an identification process [10]. The nonvanishing components of the curvature tensor are

$$\begin{aligned} R_{0r}^{0r} &= -l^{-2}, \\ R_{0n}^{0n} &= -l^{-2} \delta_n^m, \\ R_{rn}^{rn} &= -l^{-2} \delta_n^m, \\ R_{n_1 n_2}^{m_1 m_2} &= \left[-l^{-2} + \frac{(M+1)^{\frac{1}{n-1}}}{r^2} \right] \delta_{[n_1 n_2]}^{[m_1 m_2]}, \end{aligned} \quad (26)$$

where the antisymmetrized δ function is normalized by $\delta_{[m_1 m_2]}^{[m_1 m_2]} = (\mathcal{D}-2)(\mathcal{D}-3)$.

The scalar curvature is

$$R = -\frac{\mathcal{D}(\mathcal{D}-1)}{l^2} + \frac{(M+1)^{\frac{1}{n-1}}(\mathcal{D}-2)(\mathcal{D}-3)}{r^2} \quad (\mathcal{D} = 2n-1). \quad (27)$$

As we showed before, for negative values of M there are no horizons and a naked singularity is obtained. However, when one reaches the point $M = -1$ the singularity in the curvature tensor vanishes and the geometry is regular; the space obtained has constant negative curvature (anti-de Sitter space). This is quite similar to the lower dimensional case of the black hole in three dimensions. In that case, a naked conical singularity develops for negative masses, but when one reaches $M = -1$ the singu-

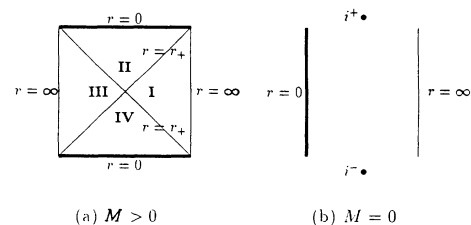


FIG. 1. Penrose diagrams for $\mathcal{D} = 2n$.

larity disappears and the solution also becomes anti-de Sitter space. In this sense anti-de Sitter space emerges as a “bound state” in the mass spectrum [9].

At large distances from $r = 0$, $R^{\alpha\beta}_{\mu\nu} \rightarrow -l^{-2}\delta^{\alpha\beta}_{[\mu\nu]}$ and the spacetime approaches anti-de Sitter space.

Note also that for $\mathcal{D} = 3$ the curvature singularity disappears for any value of M , and one obtains a space of constant negative curvature. As shown in [10], the black-hole spacetime may then be obtained from anti-de Sitter space through identifications. The surface $r = 0$ is then a singularity in the causal structure only, in that closed timelike lines appear for $r < 0$. It was argued in [10] that the metric smoothness at $r = 0$ is unstable under matter couplings. The present results show that it is also unstable upon dimensional continuation.

The Penrose diagram in odd dimensions is shown in Fig. 2. Three different types of states exist in this case, Fig. 2(a) the generic case $M \geq 0$, Fig. 2(b) the vacuum state $M = 0$ and Fig. 2(c) anti-de Sitter space of mass $M = -1$.

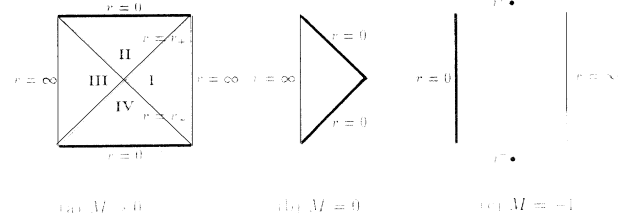


FIG. 2. Penrose diagrams for $\mathcal{D} = 2n + 1$.

E. Charged black holes

Electric charge can be incorporated by coupling the electromagnetic field to the gravitational part of the action.

The Hamiltonian action for the electromagnetic field in a curved background is

$$I_{\text{em}} = \int dt \int d^{D-1}x \left[p^i \dot{A}_i - \frac{1}{2} N^\perp \left(\alpha g^{-1/2} p^i p_i + \frac{g^{1/2}}{2\alpha} F^{ij} F_{ij} \right) + \varphi p^i{}_{,i} \right] + B_{\text{em}}, \quad (28)$$

where P^i is the momentum conjugate to the spatial components of the gauge field A_i , $\varphi \equiv A_0$, B_{em} is a surface term that depends on the boundary conditions, and α may be conveniently taken to be equal to the area of the $(\mathcal{D} - 2)$ unit sphere. We will only be interested in solutions without magnetic charge, static and spherically symmetric, that is,

$$\begin{aligned} F_{ij} &= 0 & (\text{no magnetic field}), \\ p^i &= (0, p^r, 0, \dots, 0) & (\text{radial field}), \\ \dot{A}_i &= 0 = \dot{p}^i & (\text{static field}). \end{aligned} \quad (29)$$

We may impose conditions (29) and then proceed to vary the action. The reduced action for the Coulomb field takes the form

$$I_{\text{em}}^{\text{red}} = (t_2 - t_1) \int dr \left[-\frac{1}{2} N r^{D-2} p^2 + \varphi (r^{D-2} p)^r \right] + B_{\text{em}}, \quad (30)$$

where $N = N^\perp g_{rr}^{1/2}$ and p is the rescaled radial component of p^i :

$$p^r = r^{D-2} \frac{\gamma^{1/2}}{\alpha} p. \quad (31)$$

The total reduced action of the gravitational plus electromagnetic system is then

$$\begin{aligned} I &= I_G + I_{\text{em}} \\ &= (t_2 - t_1) \int dr \left[N \left(F' - \frac{1}{2} r^{D-2} p^2 \right) + \varphi (r^{D-2} p)^r \right] + B, \end{aligned} \quad (32)$$

where the function $F[r, g(r)]$ is defined by (17) and B is a boundary term that will be fixed in next section.

Varying the action (32) with respect to N, g, P , and φ , the following equations are found:

$$F' = \frac{1}{2} r^{D-2} p^2, \quad (33)$$

$$(r^{D-2} p)^r = 0, \quad (34)$$

$$\varphi' = -Np, \quad (35)$$

$$N' = 0, \quad (36)$$

whose solutions are

$$p = \frac{Q}{r^{D-2}}, \quad (37)$$

$$\varphi = \frac{N_\infty Q}{(\mathcal{D} - 3) r^{D-3}} + \varphi_\infty, \quad (38)$$

$$F = -\frac{\frac{1}{2} Q^2}{(\mathcal{D} - 3) r^{D-3}} + C, \quad (39)$$

$$N = N_\infty. \quad (40)$$

The parameters $Q, C = M + C_0, \varphi_\infty$ and N_∞ are the integration constants of the problem; φ_∞ and N_∞ are the values of φ and N at infinity and are conjugates to Q and M , respectively (see Sec. IIIF). From (39) and (17), choosing $N_\infty = 1$, one can obtain the form of the metric,

$$ds^2 = -g^2(r) dt^2 + g^{-2}(r) dr^2 + r^2 d\Omega^2 \quad (41)$$

with

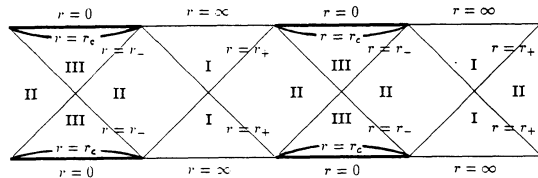


FIG. 3. Penrose diagram for the charged black hole (generic case).

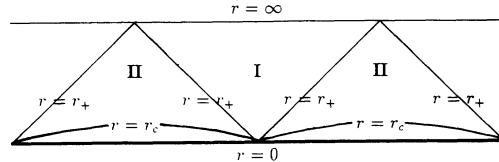


FIG. 4. Penrose diagram from the charged black hole (extreme case).

$$g^2(r) = \begin{cases} 1 + \frac{r^2}{l^2} - \left[\frac{2M}{r} - \frac{Q^2}{(\mathcal{D}-3)r^{\mathcal{D}-2}} \right]^{\frac{1}{n-1}} & (\mathcal{D} = 2n), \\ 1 + \frac{r^2}{l^2} - \left[M + 1 - \frac{Q^2}{2(\mathcal{D}-3)r^{\mathcal{D}-3}} \right]^{\frac{1}{n-1}} & (\mathcal{D} = 2n-1). \end{cases} \quad (42)$$

For each of the two functions given by (42), the equation $g^2 = 0$ has two, one, or zero solutions depending on whether for a given mass M the squared charge is greater, equal, or less than the extremal value Q_{ext}^2 . It is not possible to obtain an expression for Q_{ext}^2 in closed form but it may be shown to exist by a graphical analysis. Thus, charged black holes only exist for $Q^2 < Q_{\text{ext}}^2$. Of the two roots r_+ and r_- that exist when $Q^2 < Q_{\text{ext}}^2$, the greater root r_+ is the black hole horizon. Both roots coalesce when $Q^2 = Q_{\text{ext}}^2$ corresponding to the extremal black hole.

The charged black-hole geometry has curvature singularities: $r = 0$, and $r = r_c$, where the expression whose $(n-1)$ th root appears in (42) vanishes. This may be seen by looking at the curvature scalar of the metric (41) which reads

$$\tilde{R} = \frac{1}{r^{\mathcal{D}-2}} [r^{\mathcal{D}-2}(1-g^2)]'' \quad (43)$$

The curvature singularity $r = r_c$, which only exists in the charged case, is also hidden by the horizon. Indeed one has

$$0 < r_c < r_- < r_+$$

as may be again verified by a graphical analysis of $g^2(r)$.

In the generic case, when two horizons are present, the manifold splits into three different regions: region I or outer region ($r > r_+$), the intermediate region II ($r_- < r < r_+$), and the inner region III ($r < r_-$). The causal connection between these regions is shown in the Penrose diagram in Fig. 3.

For the extreme case (one horizon), there exist only two regions, inner (I), $r < r_+$ and outer (II), $r > r_+$; the Penrose diagram is shown in Fig. 4.

F. Surface integrals: mass and electric charge

The black-hole mass and electric charge can be identified in a simple manner by using the Hamiltonian approach. The surface term B present in (32) must be chosen so that the action has an extremum under variations

of the fields with appropriate boundary conditions. One demands that the fields approach the classical solutions [Eqs. (37)–(40)] at infinity [19]. Varying the action (32), one obtains, for the boundary term B ,

$$\delta B = (t_2 - t_1)(-N_\infty \delta M - \varphi_\infty \delta Q). \quad (44)$$

The term B is the conserved charge associated to the “improper gauge transformations” produced by time evolution [20]. These transformations are time displacements, whose charge is the mass (M), and asymptotically constant gauge transformations of the electromagnetic field, whose conserved charge is the electric charge (Q). From Eq. (44) one learns that (N_∞, M) and (φ_∞, Q) are conjugate pairs. Therefore, if M and Q are varied, their conjugates, N_∞ and φ_∞ , must be fixed. Thus, the boundary term is

$$B = (t_2 - t_1)(-N_\infty M - \varphi_\infty Q) + B_0, \quad (45)$$

where B_0 is an arbitrary constant without variation. The freedom in the value of this constant was used in Secs. III B and III E to fix the zero-point energy when the horizon disappears.

In asymptotically anti-de Sitter spaces, the rescaled lapse function $N \equiv N^\perp g^{-1}$, measures time displacements at infinity along the Killing vector $\partial/\partial t$, instead of the usual lapse function $N^\perp$. This is so because the normal vector $\mathbf{n} = N^\perp \partial/\partial t$ does not approach the Killing vector $\partial/\partial t$ at infinity since $N^\perp$ diverges there. The rescaled vector $\tilde{\mathbf{n}} = g^{-1} N^\perp \partial/\partial t = N \partial/\partial t$ does approach the Killing vector up to a constant factor.

G. Asymptotically flat black hole in even dimensions

So far we have only dealt with asymptotically anti-de Sitter spaces. In odd dimensions one is forced to deal with this asymptotic form since no black hole horizons are found in the absence of cosmological constant, even in the presence of an electric charge [21].

In even dimensions, on the other hand, one can take the limit $l \rightarrow \infty$ in the solution (42) obtaining the asymptotically flat line element

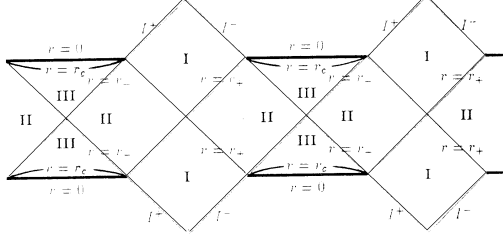


FIG. 5. Penrose diagram of the asymptotically flat charged black hole in even dimensions (non-extreme case).

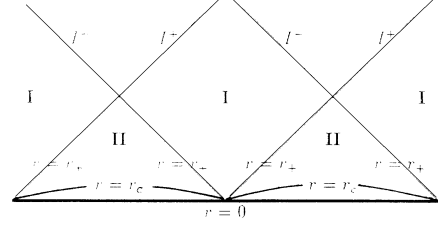


FIG. 6. Penrose diagram for the asymptotically flat charged black hole in even dimensions (extreme case).

$$ds^2 = - \left\{ 1 - \left[\frac{2M}{r} - \frac{Q^2}{(\mathcal{D}-3)r^{\mathcal{D}-2}} \right]^{\frac{1}{n-1}} \right\} dt^2 + \frac{dr^2}{1 - \left[\frac{2M}{r} - \frac{Q^2}{(\mathcal{D}-3)r^{\mathcal{D}-2}} \right]^{\frac{1}{n-1}}} + r^2 d\Omega^2. \quad (46)$$

In this limit, M becomes the usual mass conjugate to asymptotic displacements in proper time.

The black hole radius is related to the mass and electric charge by the equation

$$r_+^{\mathcal{D}-2} - 2Mr_+^{\mathcal{D}-3} + \frac{Q^2}{\mathcal{D}-3} = 0, \quad (47)$$

which has zero, one, or two solutions, depending on the relative values of the mass and charge. The Penrose diagram for the generic case (two horizons) is shown in Fig. 5.

The extreme case (one horizon) occurs for the value

$$Q_{\text{ext}}^2 = a_n M^n, \quad (48)$$

where

$$a_n = (2n-3)(n-1)^{n-1} \left(\frac{2}{n} \right)^n.$$

For a given M two horizons exist if $Q^2 < Q_{\text{ext}}^2$. They coalesce if $Q^2 = Q_{\text{ext}}^2$ and if $Q^2 > Q_{\text{ext}}^2$ there is no horizon.

The Penrose diagram for the extreme case is shown in Fig. 6.

In the case of zero electric charge one obtains the Schwarzschild-like line element

$$ds^2 = - \left\{ 1 - \left[\frac{2M}{r} \right]^{\frac{1}{n-1}} \right\} dt^2 + \frac{dr^2}{1 - \left[\frac{2M}{r} \right]^{\frac{1}{n-1}}} + r^2 d\Omega^2 \quad (49)$$

which has only one horizon at $r_+ = 2M$. The Penrose diagram for this case is shown in Fig. 7.

It should be stressed here that the metric (49) is not an extremum for the Hilbert action but, rather, for the action

$$I_{l \rightarrow \infty} = \frac{1}{4(\mathcal{D}-2)! \Omega_{\mathcal{D}-2}} \int \epsilon_{a_1 \dots a_{\mathcal{D}}} R^{a_1 a_2} \wedge \dots \wedge R^{a_{\mathcal{D}-3} a_{\mathcal{D}-2}} \wedge e^{a_{\mathcal{D}-1}} \wedge e^{a_{\mathcal{D}}} \quad (50)$$

in which only the term with the highest nontrivial power of the curvature tensor in the Lovelock action is retained.

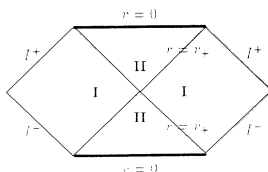


FIG. 7. Penrose diagram for the asymptotically flat uncharged black hole in even dimensions.

IV. THERMODYNAMICS

A. Temperature

The black-hole temperature can be calculated by imposing regularity at the horizon of the Euclidean continuation of the manifold [22]. In imaginary time $t = -i\tau$, the black-hole metric (15) takes the form

$$ds_E^2 = N^2(r) g^2(r) d\tau^2 + g^{-2}(r) dr^2 + r^2 d\Omega^2, \quad (51)$$

where $g^2(r)$ is given by (21) and $N(r) = N_\infty$. The Euclidean section is defined for $r > r_+$. Since $g^2(r_+) = 0$,

the Euclidean black hole has the topology $\mathbb{R}^2 \times S^{D-2}$.

The Euclidean coordinate τ is periodic and one may fix its range in the interval $[0,1]$. If this is done, N_∞ represents the Euclidean time period whose inverse is the temperature: $N \equiv \beta = T^{-1}$. The standard formula

$$T = \frac{1}{4\pi} \left(\frac{dg^2(r)}{dr} \right)_{r=r_+} \quad (52)$$

relates the inverse Euclidean period with the black hole parameters (mass and electric charge) so that no conical singularity appears at the horizon. In the case $Q = 0$ one obtains

$$T = \begin{cases} \frac{1+(2n-1)(r_+/l)^2}{4\pi(n-1)r_+}, & \mathcal{D} = 2n, \\ \frac{r_+}{2\pi l^2}, & \mathcal{D} = 2n-1. \end{cases} \quad (53a)$$

$$(53b)$$

This last relation shows that evaporating black holes would behave very differently for even and odd $\mathcal{D}$. As $M \rightarrow 0$, $r_+ \rightarrow 0$ in both cases, whereas $T \rightarrow \infty$ for $\mathcal{D} = 2n$ and $T \rightarrow 0$ for $\mathcal{D} = 2n-1$. In the limit $l \rightarrow \infty$, (53a) reproduces the standard result for $\mathcal{D}=4$, $T = (8\pi m)^{-1}$. For $n \rightarrow \infty$, (53a) approaches (53b).

B. Euclidean action and entropy

A deeper insight into black-hole thermodynamics is gained from the identification of the Euclidean path integral in the saddle point approximation around the black-hole solution with the partition function for a thermodynamic ensemble [22]. In this approximation,

$$I_E = \frac{\text{free energy}}{\text{temperature}} = \frac{M}{T} - S + \sum \frac{\mu_i}{T} Q^i, \quad (54)$$

where the μ_i are the chemical potentials associated with the charges Q^i .

Consider the reduced action of the coupled system (32):

$$I = I_G + I_Q = (t_2 - t_1) \int dr \left[N \left(F' - \frac{1}{2} r^{D-2} p^2 \right) + \varphi (r^{D-2} p)' \right] + B. \quad (55)$$

The Euclidean and Minkowskian actions are related by

$$e^{iI_M} = e^{-I_E}, \quad \tau = it. \quad (56)$$

For the thermodynamical description of the black hole, one needs the Euclidean action evaluated on the physical values of the fields (i.e., those which solve the field equations with Minkowskian signature). Thus, one obtains

$$I_E = - \int_{r_+}^{\infty} dr \left[N \left(F' - \frac{1}{2} r^{D-2} p^2 \right) + \varphi (r^{D-2} p)' \right] + B_E, \quad (57)$$

where we have set $\tau_2 - \tau_1 = 1$.

Since the volume piece of the reduced action is a linear combination of the constraints, the on-shell value of the

action is equal to the boundary term B_E . The value of B_E , which is fixed by the boundary conditions, gives then the free energy of the system [23]:

$$B_E = \frac{M}{T} + \sum \frac{\mu_i}{T} Q^i - S. \quad (58)$$

Let us compute B_E . For $r \rightarrow \infty$, we have

$$\begin{aligned} N &\rightarrow N_\infty \equiv \beta, \\ \varphi &\rightarrow \varphi_\infty, \\ \delta F &\rightarrow \delta M, \\ \delta p &\rightarrow r^{-D+2} \delta Q. \end{aligned} \quad (59)$$

At the horizon, we impose the regularity condition of no conical singularities (52),

$$N(r_+) \left(\frac{dg^2(r)}{dr} \right)_{r=r_+} = 4\pi \quad (60)$$

and the condition

$$\varphi(r_+) = 0. \quad (61)$$

This last condition is just a matter of convention. One can assume a nonzero value of φ at the horizon and impose instead $\varphi(\infty) = 0$. The difference between these choices is that the chemical potential associated with the electric charge is, in the first case, the value of φ at infinity, while in the second case, it is minus the value of φ at the horizon. Both descriptions are, however, equivalent [24].

Once the boundary conditions have been chosen, it remains to choose B_E so that it cancels all the boundary terms that appear by partial integration in the variation of the Euclidean action.

The variation of the Euclidean action is

$$\delta I_E = - [N \delta F + \varphi r^{-D+2} \delta p]_{r_+}^{\infty} + \delta B_E(\infty) + \delta B_E(r_+) + \text{terms vanishing on shell.} \quad (62)$$

Let us compute first the term coming from the horizon. As the value of φ is zero there, the only contribution to B_E from r_+ is the gravitational term

$$\delta B_E(r_+) = -N(r_+) (\delta F)_{r_+}. \quad (63)$$

The variation of F at the horizon is given by

$$(\delta F)_{r=r_+} = \left(\frac{\partial F}{\partial g^2} \right)_{r=r_+} [\delta g^2(r)]_{r=r_+} \quad (64)$$

and, from the definition of the horizon $g^2(r_+) = 0$, it follows that

$$[\delta g^2]_{r_+} + \left(\frac{dg^2}{dr} \right)_{r=r_+} \delta r_+ = 0, \quad (65)$$

then, one finds,

$$\begin{aligned}\delta B_E(r_+) &= N(r_+) \left(\frac{dg^2}{dr} \right)_{r=r_+} \left(\frac{\partial F}{\partial g^2} \right)_{r=r_+} \delta r_+ \\ &= 4\pi \left(\frac{\partial F}{\partial g^2} \right)_{r=r_+} \delta r_+, \end{aligned} \quad (66)$$

where we have used condition (60).

Let us now compute the term coming from the asymptotic region $B_E(\infty)$. Using the boundary conditions (59) and (62) one easily finds

$$\delta B_E(\infty) = \beta \delta M + \varphi_\infty \delta Q. \quad (67)$$

Thus, as the value of the boundary term $B_E \equiv B_E(\infty) + B_E(r_+)$ is equal to the free energy, one finds

$$\text{free energy} = \beta M + \varphi_\infty Q + 4\pi \int dr_+ \left(\frac{\partial F}{\partial g^2} \right)_{r=r_+} - S_0, \quad (68)$$

where S_0 is a constant without variation. Comparing (68) with (58) one learns that M is the internal energy of the system, β is the inverse temperature and $\beta^{-1}\varphi_\infty$ is the chemical potential conjugate to the electric charge. The entropy as a function of r_+ is given by

$$S(r_+) = -4\pi \int dr_+ \left(\frac{\partial F}{\partial g^2(r)} \right)_{r=r_+} + S_0 \quad (69)$$

and the black hole radius r_+ depends on the mass and electric charge through the equation

$$g^2(r_+) = 0 \rightarrow r_+ = r_+(M, Q). \quad (70)$$

In even dimensions, integral (69) can be computed in closed form obtaining

$$S(r_+) = \pi l^2 \left[\left(1 + \frac{r_+^2}{l^2} \right)^{n-1} - 1 \right]. \quad (71)$$

Here, we have fixed $S_0 = -\pi l^2$ so that

$$S(r_+ = 0) = 0.$$

This provides the usual result in $\mathcal{D}=4$ ($n=2$) and also a finite value for the entropy in the limit $l \rightarrow \infty$, $S(r_+) \rightarrow \pi(n-1)r_+^2$.

In odd dimensions, the integral

$$S(r_+) = 4\pi(n-1) \int_0^{r_+} dr \left(1 + \frac{r^2}{l^2} \right)^{n-2} + S_0 \quad (72)$$

cannot be evaluated in closed form for arbitrary n .

Choosing $S_0 = 0$ gives zero entropy for the vacuum solution ($r_+ = 0$) and one recovers the value $S = 4\pi r_+$ for the special case $\mathcal{D}=3$ [9]. Since the energy is dimensionless for odd $\mathcal{D}$, the length parameter l plays a key role in the black hole properties. The limit $l \rightarrow \infty$ is not well defined due to the dependence on l of r_+ (for $Q = 0$). When $l \rightarrow \infty$, r_+ also goes to infinity and one is left only with the black hole interior. In even dimensions on the

other hand, $r_+ \rightarrow 2M$ for $l \rightarrow \infty$.

The entropy in the higher dimensional black holes described here is no longer proportional to the area of the horizon. It is, however, an increasing function of r_+ and therefore the second law of thermodynamics holds. It should be stressed nevertheless, that for a generic choice of the Lovelock coefficients the entropy is not necessarily a monotonically increasing function of r_+ [7]. Our result for the entropy is a particular case of the expression given in [7]. The Lagrangian method used in that reference, however, involves the subtraction of a divergent term at infinity that obscures the fact that the entropy is associated with the horizon.

Note added. After this paper was written we learned that Jacobson and Myers had obtained a general expression for the entropy of dimensionally continued black holes [25]. Our results agree with their formula for the special case of spherical symmetry.

ACKNOWLEDGMENTS

We are grateful to Dr. T. Jacobson and Dr. R. Myers for their careful critical reading of this paper. We thank them for that and, in particular, for their suggestion that the singularity at $r = r_c$ in the charged case might be a curvature singularity, which proved to be correct. This work was partially supported by Grant Nos. 0862/91 and 193.0910/93 from FONDECYT (Chile), by a European Communities research contract, and by institutional support to the Centro de Estudios Científicos de Santiago, provided by SAREC (Sweden) and a group of Chilean private companies (COPEC, CMPC, ENERSIS). M.B. would like to thank Fundación Andes for financial support. J.Z. wishes to express his gratitude to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

APPENDIX A: LORENTZ LIE ALGEBRA AS A SUBALGEBRA OF ANTI-DE SITTER ALGEBRA

The anti-de Sitter group in $\mathcal{D}$ dimensions is isomorphic to the orthogonal group in $\mathcal{D}+1$ dimensions $\text{SO}(\mathcal{D}-1, 2)$. Let $\eta_{AB} = \text{diag}(-1, 1, \dots, 1, -l^{-2})$ ($A, B = 0, 1, \dots, \mathcal{D}$) and J_{AB} are the generators of the group,

$$[J_{AB}, J_{CD}] = -J_{AC}\eta_{BD} + J_{AD}\eta_{BC} + J_{BC}\eta_{AD} - J_{BD}\eta_{AC}. \quad (A1)$$

The generators J_{AB} can be split into rotations in $\mathcal{D}$ dimensions (generated by J_{ab}) and “inner translations” (generated by $J_{a\mathcal{D}} \equiv J_a$). The commutation relations of the split generators defines the anti-de Sitter algebra

$$\begin{aligned} [J_a, J_b] &= l^{-2} J_{ab}, \\ [J_{ab}, J_c] &= J_a \eta_{bc} - J_b \eta_{ac}, \\ [J_{ab}, J_{cd}] &= -J_{ac} \eta_{bd} + J_{ad} \eta_{bc} + J_{bc} \eta_{ad} - J_{bd} \eta_{ac}. \end{aligned} \quad (A2)$$

Let W^{AB} be the connection one-form for the group and $D = d + J_{AB} W^{AB}$ the exterior covariant derivative. The curvature two-form is defined as usual:

$$\hat{R}^{AB} = dW^{AB} + W_C^A \wedge W^{CB}. \quad (\text{A3})$$

The connection W^{AB} can also be decomposed into the connection under $\mathcal{D}$ rotations, w^{ab} , and inner translations, e^a , in the form

$$W^{AB} = \begin{pmatrix} w^{ab} & -e^a \\ e^b & 0 \end{pmatrix}. \quad (\text{A4})$$

In this splitting, the curvature two-form reads

$$\hat{R}^{AB} = \begin{pmatrix} R^{ab} + l^{-2} e^a \wedge e^b - T^a & \\ T^b & 0 \end{pmatrix}, \quad (\text{A5})$$

where $T^a \equiv de^a + w_b^a e^b$ is the torsion two-form. Note that e^a transforms as a vector under rotations generated by J_{ab} , but as a connection under those generated by J_a .

APPENDIX B: HAMILTONIAN FORMALISM FOR THE LOVELOCK ACTION

The Hamiltonian form of the action (1) is discussed in [3]. In that approach, the second-order formalism is used. The torsion tensor is set to zero and the connection is solved in terms of the local frame and their derivatives. Just as in $D=4$, the canonical coordinates are the spatial components of the metric g_{ij} , and their conjugate momenta π^{ij} . The time components $g_{0\mu}$ are Lagrange multipliers associated with the generators of surface deformations, $\mathcal{H}_\mu = (\mathcal{H}, \mathcal{H}_i)$. The action (1) takes the form

$$I = \int (\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i) d^{D-1}x dt + B. \quad (\text{B1})$$

The momenta can be explicitly given in terms of the velocities:

$$\pi_j^i = -\frac{1}{4} \sqrt{g} \sum_{p=0}^{n-1} \frac{\alpha_p}{2p!} \frac{(D-2p)!}{2^p} \sum_{s=0}^{p-1} C_{s(p)} \delta_{[j_1 \dots j_{2p-1} j]}^{[i_1 \dots i_{2p-1} i]} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{2s-1} i_{2s}}^{j_{2s-1} j_{2s}} K_{i_{2s+1}}^{j_{2s+1}} \dots K_{i_{2p-1}}^{j_{2p-1}}, \quad (\text{B2})$$

where $C_{s(p)} = \frac{(-4)^{p-s}}{s! [2(p-s)-1]!!}$, and

$$\delta_{[j_1 \dots j_l]}^{[i_1 \dots i_l]} = \begin{vmatrix} \delta_{j_1}^{i_1} & \dots & \delta_{j_l}^{i_1} \\ \vdots & & \vdots \\ \delta_{j_1}^{i_l} & \dots & \delta_{j_l}^{i_l} \end{vmatrix}. \quad (\text{B3})$$

The generators of reparametrizations of the surfaces $t = \text{const}$,

$$\mathcal{H}_i = -2\pi_{i/l}^l, \quad (\text{B4})$$

do not depend on the action but only on the transformation laws of g_{ij} and π^{ij} . Here the slash denotes covariant differentiation in the spatial metric, and $\pi_{j/i}^i = \pi_{j,i}^i - \Gamma_{ji}^n \pi_n^i$.
The normal generator $\mathcal{H}$ is

$$\mathcal{H} = -\sqrt{\det(g_{ij})} \sum_{p=0}^{n-1} \frac{(D-2p)!}{2^p} \alpha_p \delta_{[j_1 \dots j_{2p}]}^{[i_1 \dots i_{2p}]} \tilde{R}_{i_1 i_2}^{j_1 j_2} \tilde{R}_{i_3 i_4}^{j_3 j_4} \dots \tilde{R}_{i_{2p-1} i_{2p}}^{j_{2p-1} j_{2p}}, \quad (\text{B5})$$

where $\tilde{R}_{kl}^{ij}$ are the spatial components of the space-time curvature tensor. They depend on the velocities through the Gauss-Codazzi equations

$$\tilde{R}_{ijkl} = R_{ijkl} + K_{ik} K_{jl} - K_{il} K_{jk}, \quad (\text{B6})$$

where R_{ijkl} are the components of the intrinsic curvature tensor of the spatial sections, constructed from g_{ij} and its spatial derivatives, and $K_{ij} = \frac{1}{2N} (-\dot{g}_{ij} + N_{i/j} + N_{j/i})$.

In the static, spherically symmetric case one can ignore the distinction between $\tilde{R}_{ijkl}$ and R_{ijkl} [see Eq. (B6)]. Also, the only nonvanishing components of the spatial curvature are

$$\begin{aligned} R_{n_1 n_2}^{m_1 m_2} &= \frac{f(r)}{r^2} \delta_{[n_1 n_2]}^{[m_1 m_2]}, \\ R_{rn}^m &= \frac{f'(r)}{2r} \delta_n^m, \end{aligned} \quad (\text{B7})$$

where $f(r) \equiv 1 - g^{rr}(r)$.

The normal generator $\mathcal{H}$ can be computed by direct substitution of (B7) into (B5) obtaining

$$\mathcal{H} = -(D-2)! \sqrt{g} g^{-1} \frac{d}{dr} \left[r^{D-1} \sum \alpha_p (D-2p) \left(\frac{1-g^2}{r^2} \right)^p \right], \quad (\text{B8})$$

Replacing in (B8) the choices (3) for the coefficients α_p , the expression (17) for the function $F[r, g(r)]$ is obtained.

The tangential generator $\mathcal{H}_i$ is identically zero for the metrics considered here.

In the transition from (B5) to (B8) the following identities between the antisymmetrized Kronecker δ 's are useful. For $m < p$,

$$\delta_{[j_1 \dots j_p]}^{[i_1 \dots i_p]} \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \dots \delta_{i_m}^{j_m} = \frac{(r-p+m)!}{(r-p)!} \delta_{[j_{m+1} \dots j_p]}^{[i_{m+1} \dots i_p]} \quad (\text{B9})$$

$$\delta_{[j_1 \dots j_{2p}]}^{[i_1 \dots i_{2p}]} \delta_{[i_1 i_2]}^{[j_1 j_2]} \dots \delta_{[i_{2m-1} i_{2m}]}^{[j_{2m-1} j_{2m}]} = \frac{2^m (r-2[p-m])!}{(r-2p)!} \delta_{[j_{2m+1} \dots j_{2p}]}^{[i_{2m+1} \dots i_{2p}]}, \quad (\text{B10})$$

where r is defined as the range of the indices.

-
- [1] E. Witten, Phys. Rev. D **44**, 314 (1991); C.G. Callan, S.B. Giddings, J.A. Harvey, and A. Strominger, *ibid.* **45**, 1005 (1992); G. Horowitz, in 1992 Trieste Spring School on Strings and Quantum Gravity (unpublished).
- [2] D. Lovelock, J. Math. Phys. **12**, 498 (1971).
- [3] C. Teitelboim and J. Zanelli, Class. Quantum Grav. **4**, L125 (1987); in *Constraint Theory and Relativistic Dynamics*, edited by G. Longhi and L. Lussana (World Scientific, Singapore, 1987).
- [4] M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. A **36**, 4417 (1987).
- [5] Throughout this work we define the integer n by $\mathcal{D} = 2n$ for even dimensions, and $\mathcal{D} = 2n - 1$ for odd dimensions.
- [6] J.T. Wheeler, Nucl. Phys. **B268**, 737 (1986); **B273**, 732 (1986).
- [7] B. Whitt, Phys. Rev. D **38**, 3001 (1988).
- [8] R.C. Myers and J. Simon, Phys. Rev. D **38**, 2434 (1988); D.L. Wiltshire, *ibid.* **38**, 2445 (1988).
- [9] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
- [10] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
- [11] J.D. Brown, E.A. Martínez, and J.W. York, Phys. Rev. Lett. **66**, 2281 (1991).
- [12] J.D. Brown, G.L. Comer, E.A. Martínez, J. Melmed, B.F. Whiting, and J.W. York, Class. Quantum Grav. **7**, 1433 (1990).
- [13] Here we will assume that the torsion tensor is not explicitly contained in the action; the extension of Lovelock theory for this case is worked out in Ref. [15].
- [14] T. Regge, Phys. Rep. **137**, 31 (1986); B. Zumino, *ibid.* **137**, 109 (1986).
- [15] A. Mardones and J. Zanelli, Class. Quantum Grav. **8**, 1545 (1991).
- [16] S.W. MacDowell and F. Mansouri, Phys. Rev. Lett. **38**, 739 (1977).
- [17] M. Bañados, C. Teitelboim, and J. Zanelli, in *J.J. Giambiagi Festschrift*, edited by H. Falomir, R.E. Gamboa, P. Leal, and F.A. Schaposnik (World Scientific, Singapore, 1991).
- [18] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973).
- [19] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).
- [20] R. Benguria, P. Cordero, and C. Teitelboim, Nucl. Phys. **B122**, 61 (1977).
- [21] In odd dimensions without a cosmological constant, a cosmological horizon appears when $Q \neq 0$. However, the singularity is then naked so we will not consider that geometry here.
- [22] G.W. Gibbons and S.W. Hawking, Phys. Rev. D **15**, 2753 (1977).
- [23] If one considers the unreduced action, this argument still holds. In fact, the general form of the action of the coupled system is $I = \int (\pi^A \dot{\Phi}_A + \lambda^A G_A) + B$, where B is a boundary term, the generators G_A vanish on shell, and π^A, Φ_A denote the gravitational and electromagnetic fields. If one expects to have a system in thermal equilibrium, the solution must be static ($\dot{\Phi}_A = 0$) and, hence, the value of the action is given by the boundary term B alone.
- [24] The one-form $A_\mu dx^\mu = \varphi d\tau$ is singular at $r = r_+$ if $\varphi(r_+) \neq 0$. This singularity is a gauge artifact and is removed by the (singular) gauge transformation $\varphi \rightarrow \varphi - \varphi(r_+) \tau$ leading to (61). Conceptually it is preferable to use (61) which makes it explicit that the electric charge is related to the gauge invariance at infinity. One may also consider $\varphi(r_+) = 0$ as fixing the “initial coordinate” $\varphi(r_+)$, and the “final momentum” $p(\infty)$. This amounts to setting $\delta Q = 0$ and hence no surface integral proportional to Q comes in. The free energy thus obtained is that of the canonical ensemble $F = U - TS$, and not the grand canonical one $F = U - TS - \mu Q$.
- [25] T. Jacobson and R. Myers, Phys. Rev. Lett. **70**, 3684 (1993).