Improved effective potential

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It is shown that improved potentials and corrected mass terms can be introduced by using a quadratic source term in the path integral construction for the effective action. The advantage of doing things this way is that we avoid ever having to deal with complex propagators in the loop expansion. The resulting effective action for electroweak phase transitions is similar to the usual results.

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I. INTRODUCTION

According to the standard models of cosmology and particle physics the early stages of the Universe were exceedingly hot and at least some of the forces of nature were unified in a single force. The symmetry governing this unified force would have been broken when a Higgs field developed a nonzero vacuum expectation value and the four separate fundamental forces we observe today came into being as a consequence of this. As the standard electroweak theory has been experimentally confirmed, it has become interesting to study electroweak phase transitions at high temperatures (see, for instance [1,2]), as a preliminary guide to understanding the physics of the early Universe.

An important tool in discussing the dynamics of such phase transitions in quantum field theory is the method of the effective potential. This was first introduced by Goldstone, Salam, and Weinberg [3] and by Jona-Lasinio [4], as a functional Legendre transform with an external source J coupled linearly to the field ϕ. It is possible to calculate the effective action for the field of interest in perturbation theory, by making an expansion in powers of Planck's constant.

Although the effective potential formalism provides an efficient way to get quantum corrections to the classical potential, it requires some careful handling. The first problem that can arise is that the perturbative expansion of the effective potential in powers of $\hbar$ is not real when there is symmetry breaking, and for some values of the fields is even ill defined. Second, the exact effective potential according to some definitions is convex. In a theory with spontaneous symmetry breaking at the classical level, this means that the exact effective potential cannot have a double-well shape and the local maximum of the system at the symmetric point at the origin would never appear, even though the classical theory implies the existence of a maximum at the origin.

An early suggestion for overcoming these problems was to introduce a "corrected mass" into the propagator which is used in expanding out the effective potential [12]. This is very similar to a resummation of subsets of graphs [5,13] to obtain a corrected propagator. A difficulty with this approach is that each individual diagram might itself be ill defined, and steps need to be taken to avoid this happening.

If, however, one introduced another source K coupled to a term which is quadratic in the fields in the definition of the effective action, one would obtain an effective potential with a proper loop expansion to each order which is not convex. The idea of having a quadratic source term for discussing the quantum field theory of the field ϕ in the early Universe was first put forward by [7] (see also [8]). This work was based on an important paper by Cornwall, Jackiw, and Tomboulis [9], who defined an effective action for composite fields in flat space as a double Legendre transform, with two sources $J(x)$ and $K(x,x')$ which were coupled, respectively, to $\phi(x)$ and $\phi(x)\phi(x')$. An application of these ideas was made recently in [10,11], where a new effective action was defined for scalar fields at finite temperature and for an $O(N)$ gauge theory at zero temperature.

The derivation of the effective action in [10] differs from the conventional effective action by having both sources, J and K present initially, but the source J is made to vanish in the limit of the spatial volume going to infinity. The key point of this procedure is the condition that the expectation values are dominated by field configurations that lie along the direction of J before the limit is taken. The expectation values are calculated with both sources present initially, but when the limit $J \to 0$ and volume $\Omega \to \infty$ is taken, the magnitude of the fields retain only their dependence on the source K and lead to real masses. The beauty of this procedure is that, whereas on the one hand the conventional effective action is regained by setting $K = 0$ when there is no symmetry breaking, on the other hand keeping K finite and setting $J \to 0$ at the end gives a well-defined effective action for broken symmetric theories. Moreover, this effective action is not necessarily convex, which means that it can have the double-well shape of its classical theory.

For a system which is confined to a finite volume, the configuration space is divided into subsets with

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equal probability measure which are related by symmetry transformations. In the infinite volume limit, if one of the configuration space subsets develops a higher probability for its measure than the others, then the system exhibits symmetry breaking. This is the standard procedure for setting up a pure state in a theory having only the linear source term. The problem with this from our point of view is that the volume of the system may always be finite, because of cosmological considerations, and that it does not address the issues concerned with dynamically evolving fields.

In the alternative procedure used here [10,11], the pure broken symmetric state is prepared in the same way as before, with $J = 0$ in the infinite volume limit, and also by setting $K = 0$. In addition, however, the system can also be prepared in a state located at the local symmetric maximum near the origin of field space by setting $J = 0$ as before but by keeping K nonzero and finite. This is why the exact effective potential is not necessarily convex in this approach. All this is valid at zero as well as at nonzero temperature.

Interestingly, the improved effective potential (IEP) described in [10] gives the same (positive) mass squared of the scalar field at equilibrium at high temperatures as that obtained by Kirzhnits and Linde [12].

In this paper we shall develop the method described in [10] and show in Sec. III that in this new approach, one can reproduce two important results of the background field method, namely, that the effective field equations are given by

$$\frac{\delta\Gamma[\phi]}{\delta\phi} = 0 \tag{1}$$

and that the effective action defined in [10] is invariant under Becchi-Rovet-Stora (BRS) transformations $\hat{s}$ of the fields, in the sense that

$$\langle \hat{s}\Gamma \rangle = 0. \tag{2}$$

Finally in Secs. IV and V, we shall use this definition to calculate the effective potential for the Higgs field in the standard SU(2)×U(1) electroweak theory. At zero temperatures the resulting potential is real, without any resummation of graphs, and gives the usual mass relations for the Higgs, W, and Z bosons in terms of the Higgs expectation value. At nonzero temperatures our results compare favorably with others [13,14].

II. IMPROVED EFFECTIVE ACTIONS

First we shall very briefly review the method of the improved effective potential for a scalar field described in [10]. We shall use Riemannian signature (++++) for the metric throughout and work in units in which $c = 1$.

Consider a real scalar field Φ, which we will allow to have more than one internal component. The composite field partition function $Z[J, K]$ is defined by a path integral [9]

$$Z[J,K] = e^{-W[J,K]/\hbar} = \int d\mu[\Phi] \exp\{-I_{JK}[\Phi]\}, \tag{3}$$

where

$$I_{JK}[\Phi] = I[\Phi] + \int d\mu(x)\Phi(x)^T J(x) + \tfrac{1}{2}\int d\mu(x)d\mu(x')\,\Phi(x)^T K(x,x')\phi(x'), \tag{4}$$

and $I[\Phi]$ is the action. If Φ has N components, then K is an $N \times N$ matrix.

The expectation value of the field ϕ and the connected propagator G are defined in the presence of the external sources by

$$\phi(x) = \frac{\delta W[J,K]}{\delta J(x)},$$
$$\hbar G(x,x') = 2\frac{\delta W[J,K]}{\delta K(x,x')} - \frac{\delta W[J,K]}{\delta J(x)}\frac{\delta W[J,K]}{\delta J(x')}. \tag{5}$$

It is also possible to write $G(x, x')$ as

$$\hbar G(x,x') = \langle \Phi(x)\Phi^T(x')\rangle - \langle\Phi(x)\rangle\langle\Phi^T(x')\rangle. \tag{6}$$

The composite effective action $\Gamma[\phi, G]$ is then defined by a double Legendre transform

$$\Gamma[\phi,G] = W[J,K] - \int d\mu(x)\phi^T(x)J(x) - \tfrac{1}{2}\int d\mu(x)d\mu(x')\phi(x)^T K(x,x')\phi(x') - \tfrac{\hbar}{2}\int d\mu(x)d\mu(x')G(x,x')K(x',x). \tag{7}$$

The effective field equation in the presence of the two sources J and K follow from functional differentiation of the effective action:

$$\frac{\delta\Gamma[\phi,G]}{\delta\phi(x)} = -J^T(x) - \int d^4y\,\phi^T(y)K(y,x), \tag{8}$$

$$\frac{\delta\Gamma[\phi,G]}{\delta G(x,x')} = -\tfrac{1}{2}\hbar K(x,x'). \tag{9}$$

Physical on-shell solutions are obtained by setting the sources J and K to zero in (8) and (9). When K is zero, Eq. (9) can be solved for G in terms of $\bar{\phi}$, and this value of G can be substituted back in (8) to get the conventional effective action, $\Gamma[\phi] = \Gamma[\phi, G(\phi)]$.

However, G need not necessarily be obtained only in this manner. There are other possible solutions for G, ones with nonzero K, which can be substituted into $\Gamma[\phi]$ to get a different form of $\Gamma[\phi, G(\phi)]$. The conventional definition of the effective action for ϕ considers only the special case $K = 0$. The presence of K allows the effective action to give information not only about the ground state but also about the symmetric solution about the origin.

In order to proceed further it will be helpful to examine what the field configurations are that dominate the partition function. We will take the classical symmetry-breaking potential

$$V(\Phi) = -\tfrac{1}{2}\mu^2|\Phi|^2 + \tfrac{1}{4}\lambda|\Phi|^4. \quad (10)$$

In the first place consider what happens with a constant linear source

$$J(x) = j, \quad (11)$$

and diagonal quadratic source

$$K(x,y) = k\, I\, \delta(x-y). \quad (12)$$

The path integral is dominated by saddle points, obtained from the condition

$$\frac{\delta I_{JK}}{\delta \Phi} = 0, \quad (13)$$

which gives

$$V'(\phi_s) + j + k\phi_s = 0. \quad (14)$$

This is solved for small j by making the ansatz

$$\phi = \sum_n a_n |j|^n. \quad (15)$$

The first few terms in the series are

$$\phi_{s\pm} = \pm\frac{(\mu^2-k)^{1/2}}{\sqrt{\lambda}}\hat{\jmath} - \tfrac{1}{2}(\mu^2-k)^{-1} j + O(j^2). \quad (16)$$

There are therefore two solutions: one in the direction of j and one in the opposite direction. If $j = 0$, then instead of two solutions there is a continuum of saddle points with a fixed value of the modulus $|\phi|$.

We will proceed now without assuming that $J(x)$ is constant. The total partition function $Z[J,K]$ should be constructed by summing the contributions from all of the saddle points separately. For example,

$$Z \sim e^{-W_+/\hbar} + e^{-W_-/\hbar}, \quad (17)$$

where

$$e^{-W_+[J,K]/\hbar} = e^{-I_{JK}[\phi_+]/\hbar}\int d\mu[\tilde{\phi}]\exp(-I^{(1)}/\hbar). \quad (18)$$

The action $I^{(1)}$ consists of the terms from the expansion of $I_{JK}[\Phi]$, where $\Phi = \phi_+ + \tilde{\phi}$, that are linear or higher order in $\tilde{\phi}$. This can be written

$$I^{(1)} = \int d\mu(x)\tilde{\phi}^T(x)J^{\rm int}(x) + \int d\mu(x,x')\tilde{\phi}^T(x)[\Delta(x,x') + K(x,x')]\tilde{\phi}(x') + I^{(3)}, \quad (19)$$

with

$$J^{\rm int}(x) = \frac{\delta I_{JK}}{\delta\Phi(x)}, \quad \Delta(x,x') = \frac{\delta^2 I}{\delta\Phi^T(x)\delta\Phi(x')}. \quad (20)$$

The partition function is obtained by summing the two series obtained from each of the two saddle points. In practice, it proves even better to expand about background fields $\phi_\pm$ shifted from the saddle points by $O(\hbar)$ quantum corrections. Let

$$\langle\Phi_\pm(x)\rangle = \frac{\delta W_\pm}{\delta J(x)}, \quad (21)$$

then we solve

$$\phi_\pm = \langle\Phi_\pm(x)\rangle \quad (22)$$

to get $\phi_\pm$. Because of (13), $\phi = \phi_{s\pm}$ to leading order in $\hbar$.

By definition, the expectation value ϕ of the field Φ is

$$\phi = -\frac{\delta}{\delta J}\ln Z. \quad (23)$$

This can be rewritten using (17) as

$$\phi = \frac{e^{-W_+/\hbar}}{Z}\phi_+ + \frac{e^{-W_-/\hbar}}{Z}\phi_- \quad (24)$$

or alternatively

$$\phi = \tfrac{1}{2}(\phi_+ + \phi_-) - \tfrac{1}{2}(\phi_+ - \phi_-)\tanh\left(\frac{W_+ - W_-}{2\hbar}\right). \quad (25)$$

For constant sources, the expectation value ϕ of the field Φ can be shown to be

$$\phi = -\tfrac{1}{2}(\mu^2 - k)^{-1} j - \frac{(\mu^2 - k)^{\frac{1}{2}}}{\sqrt{\lambda}} \tanh\left(\frac{W_+ - W_-}{2\hbar}\right) \hat{\jmath} + O(j^2). \tag{26}$$

As for the connected propagator $G(x, x')$, from Eq. (6)

$$\hbar G(x, x') = \langle \Phi(x) \Phi^T(x') \rangle - \langle \Phi(x) \rangle \langle \Phi^T(x') \rangle. \tag{27}$$

The asymptotic expansions about $\phi_\pm$ would then be

$$\hbar G(x, x') \sim Z^{-1} e^{-W_+/\hbar} \langle \Phi(x) \Phi^T(x') \rangle_+ + Z^{-1} e^{-W_-/\hbar} \langle \Phi(x) \Phi^T(x') \rangle_- - \phi(x) \phi^T(x'). \tag{28}$$

Some rearrangements of this result give

$$\hbar G(x, x') = \tfrac{1}{2}\hbar(G_+ + G_-) - \tfrac{1}{2}\hbar(G_+ - G_-) \tanh\left(\frac{W_+ - W_-}{2\hbar}\right) + \rho_c(x, x'), \tag{29}$$

where

$$G_\pm = \langle \tilde{\phi}(x) \tilde{\phi}(x')^T \rangle_\pm \tag{30}$$

and

$$\rho_c(x, x') = \tfrac{1}{4} \mathrm{sech}^2\left(\frac{W_+ - W_-}{2\hbar}\right) [\phi_+(x) - \phi_-(x)][\phi_+(x') - \phi_-(x')]^T. \tag{31}$$

In order to obtain an effective potential $\Gamma[\phi]$ it is necessary to eliminate the propagator G. If we set the source $K = 0$ the result should be the usual effective action. There are two possible cases to consider depending on whether we keep both of the series expansions about $\phi_\pm$ or just one. With both series the result is a convex potential [10]. Taking the infinite volume limit with nonzero J leads to the dominance of only one of the two series, but in this case the propagator becomes

$$G \sim G_+ \sim -\nabla^2 + V''(\phi) \tag{32}$$

to order $\hbar$. The mass term in this propagator is imaginary for small values of ϕ.

The alternative way of eliminating G is to take

$$K(x, x') = k(x) \delta(x, x') \tag{33}$$

and $J \to 0$. However, when $J = 0$ there is a continuum of saddle points in the path integral. This would cause problems with the asymptotic expansions and therefore we arrange that only one saddle point should dominate the result. This can be done by taking the limit $\Omega \phi . J \to -\infty$ as the volume $\Omega \to \infty$. This limit can be imposed simultaneously with the limit $J \to 0$.

The effective action can now be evaluated with only one saddle-point contribution corresponding to $\phi = \phi_+$. The composite field effective action is defined by

$$\begin{aligned} \Gamma[\phi, G] = W[J, K] &- \int d\mu(x) \phi^T(x) J(x) \\ &- \tfrac{1}{2} \int d\mu(x) d\mu(x') \phi^T(x) K(x, x') \phi(x') \\ &- \tfrac{1}{2}\hbar \int d\mu(x) d\mu(x') G(x, x') K(x, x'). \end{aligned} \tag{34}$$

The asymptotic series for W can be expressed in terms of Feynman diagrams with scalar field propagator $G_0 = (\Delta + K)^{-1}$. This has the form

$$W[K] = I[\phi_+] - \tfrac{1}{2}\hbar \ln \det G_0 + W^{(2)}, \tag{35}$$

where $W^{(2)}$ contains all of the connected diagrams in the series that have two or more loops. The full connected propagator $G(x, x')$ is given by

$$\hbar G(x, x') = \langle \tilde{\phi}(x) \tilde{\phi}^T(x') \rangle_+. \tag{36}$$

Substituting W into definition of the action gives

$$\Gamma[\phi] \sim I[\phi] - \tfrac{1}{2} \mathrm{tr}(G_0 K) - \tfrac{1}{2}\hbar \ln \det G_0 + W^{(2)}. \tag{37}$$

Note that $K[\phi]$ is given by the condition (22), which corresponds to the vanishing of the tadpole diagrams.

Following Cornwall *et al.* [9], it is possible to rewrite the expression on the right in terms of the full propagator:

$$\Gamma[\phi] \sim I[\phi] - \tfrac{1}{2}\hbar \, \mathrm{tr}\,(1 - G\Delta) - \tfrac{1}{2}\hbar \ln \det G + \Gamma^{(2)}, \tag{38}$$

where $\Gamma^{(2)}$ consists of two-particle irreducible diagrams. The form of $G[\phi]$ is determined by equating the tadpole diagram series to zero, with vertices representing the shifted theory including J^{int} [Eq. 20].

III. GAUGE THEORIES

In this section we will set up the improved effective action for Higgs fields in gauge theories. Our aim is to prove two results. First of all we will show that the effective field equations are given by variation of the effective action. Then we will show that the effective action has the usual behavior under BRS symmetry. This is important because of the way that BRS symmetry underlies proofs of renormalizability. Higgs and gauge fields are denoted by Φ and A.

We will fix the gauge symmetry of the Lagrangian density by adding gauge fixing and ghost terms:

$$\mathcal{L}_{\mathrm{tot}} = \mathcal{L}_\phi + \mathcal{L}_A + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{gh}}. \tag{39}$$

The approach that we take is essentially a background field expansion where the gauge fixing Lagrangian density is likely to have an explicit background field dependence. We denote these background fields by $\hat{\phi}$ and $\hat{A}$, and we have dependencies

$$\mathcal{L}_{\rm GF}[\phi, A, \hat{\phi}, \hat{A}], \tag{40}$$

$$\mathcal{L}_{\rm gh}[\phi, A, c, \bar{c}, \hat{\phi}, \hat{A}], \tag{41}$$

where c and $\bar{c}$ are the ghost fields.

Fixing the gauge leaves a residual BRS symmetry s, whose action is given by

$$\begin{aligned} sA_\mu &= D_\mu c, \\ s\Phi &= ic\Phi, \\ sc &= -\tfrac{1}{2}[c, c], \\ s\bar{c} &= \xi^{-1}\mathcal{F}, \\ s\hat{A}_\mu &= [c, A_\mu], \\ s\hat{\phi} &= ic\hat{\phi}. \end{aligned} \tag{42}$$

The gauge covariant derivative with connection A is denoted by D and $\mathcal{F}$ is a gauge fixing functional.

External sources will be introduced for each of the quantum fields, but for simplicity only the Higgs field will have a quadratic source. We define,

$$I_{JK} = \int d\mu(x)\left\{\mathcal{L}_{\rm tot} + \Phi^T J + J^\mu A_\mu + \bar{\theta}c + \bar{c}\theta\right\} + \tfrac{1}{2}\int d\mu(x,x')\Phi(x)^T K(x,x')\Phi(x'). \tag{43}$$

The generating functional is now

$$e^{-W/\hbar} = \int d\mu[\Phi, A, c, \bar{c}]e^{-I_{JK}} \tag{44}$$

with connected diagrams generated by $W[J, K, \theta, \bar{\theta}, \hat{\phi}, \hat{A}]$. The path integral is evaluated perturbatively, expanding the Higgs field about a background ϕ_+. Later we will set ϕ_+ to equal the other background field $\hat{\phi}$. We will also arrange that the other saddle points never contribute to the path integral in the same way as in the previous section.

Construction of the effective action for composite fields proceeds as before:

$$\begin{aligned} \Gamma[\phi, A, c, \bar{c}, g; \hat{\phi}, \hat{A}] = W &- \int \left\{J\phi + J^\mu A_\mu + \bar{\theta}c + \bar{c}\theta\right\} \\ &- \tfrac{1}{2}\int \phi^T K\phi - \tfrac{1}{2}\,{\rm tr}(KG). \end{aligned} \tag{45}$$

Variation of this action gives back the sources:

$$\frac{\delta\Gamma}{\delta\phi(x)} = -J^T(x) - \int d\mu(x')\phi^T(x')K(x',x), \tag{46}$$

$$\frac{\delta\Gamma}{\delta G(x,x')} = -\tfrac{1}{2}K(x,x'), \tag{47}$$

$$\frac{\delta\Gamma}{\delta A_\mu(x)} = -J^\mu(x). \tag{48}$$

At this stage we would like to set $\hat{\phi} = \phi$ and eliminate $G(x,x')$ to obtain an effective action that is a function of the field expectation values and nothing else. The obstacle to doing this is that there is an explicit dependence on $\hat{\phi}$ from the gauge fixing terms in the action. This would not be so if the action were truly independent of the gauge fixing term, but this is possible only in some approaches [18]. A simple way to fix up this problem is to subtract off the offending term:

$$\bar{\Gamma} = \Gamma - \int \left\{\langle\mathcal{L}_{\rm GF}\rangle + \langle\mathcal{L}_{\rm gh}\rangle\right\}. \tag{49}$$

Now the variation of Γ with $\hat{\phi}$ is canceled by the variation of the other two terms, and we may set

$$\Gamma[\phi, A, c, \bar{c}] = \bar{\Gamma}[\phi, A, c, \bar{c}, G[\phi]; \phi, A] \tag{50}$$

after solving for $G[\phi]$ by using vanishing tadpole diagrams. Finally, it follows from (46) that the effective field equations becomes $\delta\Gamma = 0$.

In low-order perturbation theory we can simplify the subtraction of $\mathcal{L}_{\rm GF}$ by making use of the ξ dependence,

$$\int \langle\mathcal{L}_{\rm GF}\rangle = \xi\frac{dW}{d\xi}. \tag{51}$$

In particular, at order $\hbar$ the ξ dependence is linear and therefore the subtraction is not necessary in the Landau gauge $\xi = 0$.

The other issue of importance is gauge invariance. In a background field approach it should be possible to retain covariance under gauge transformations of the background fields explicitly at each stage of the calculation. Therefore the nontrivial issue is the residual BRS invariance left after the gauge invariance of the quantum fields has been fixed.

It should be possible to change the variables of integration in the path integral to their images under a BRS transformation without changing the generating functions. This implies that the path integral of the BRS variation vanishes, or

$$\langle s\, I_{JK}\rangle = 0. \tag{52}$$

The BRS symmetry of the action implies that

$$\begin{aligned} s\, I_{JK} = \int &\left\{Js\Phi + J^\mu sA_\mu + \bar{\theta}sc + s\bar{c}\theta\right\} \\ &+ \tfrac{1}{2}\,{\rm tr}\left[Ks(\Phi\Phi^T)\right]. \end{aligned} \tag{53}$$

The sources can be eliminated by expressing them as functional derivatives of the (composite) effective action. We also define a new transformation $\hat{s}$ that acts on expectation values and produces quantum fields. For field expectation values it acts in the same way as s, but the action of $\hat{s}$ on the propagator is

$$\hat{s}\,G = s\,\Phi\,\Phi^T + \Phi\,s\Phi^T - 2\phi\,\hat{s}\Phi. \quad (54)$$

Using the transformation $\hat{s}$ it is possible to write (52) in the form

$$\langle \hat{s}\,\Gamma \rangle = 0. \quad (55)$$

This equation is an expression of the Ward-Slavnov-Taylor identities for the *composite* effective action.

IV. IMPROVED EFFECTIVE ACTION FOR THE STANDARD ELECTROWEAK THEORY

We now compute the effective action at high temperature in SU(2)×U(1) theory using the procedure developed in the earlier sections, starting with the bosonic sector.

The classical Lagrangian density for the Glashow-Salam-Weinberg model [15–17] is given by

$$\mathcal{L}_{\rm GSW} = \mathcal{L}_s + \mathcal{L}_g + \mathcal{L}_{\rm GF} + \mathcal{L}_f + \mathcal{L}_y + \mathcal{L}_{\rm gh}. \quad (56)$$

Leaving aside the fermion terms $\mathcal{L}_f$ and Yukawa couplings $\mathcal{L}_y$ for the moment, we have

$$\mathcal{L}_s = (D_\mu\Phi)^\dagger(D^\mu\Phi) - \mu^2\Phi^\dagger\Phi + \tfrac{1}{2}\lambda(\Phi^\dagger\Phi)^2,$$
$$\mathcal{L}_g = -\tfrac{1}{2}\delta^{ab}F_{a\mu\nu}F_b^{\mu\nu}. \quad (57)$$

The conventions for the field strengths and covariant derivatives which will be used are as follows. The field strength tensor $F_{\mu\nu} = F_{\mu\nu a}T^a$, with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]. \quad (58)$$

The covariant derivative D_μ is defined as

$$D_\mu = \nabla_\mu - igA_{\mu\ a}T^a \quad (59)$$

where

$$T^a = \sigma^a \quad \text{for} \quad a = 1,2,3,$$
$$T^a = \frac{g'}{g}\mathrm{I} \quad \text{for} \quad a = 4. \quad (60)$$

The first three generators are Pauli matrices and the fourth is proportional to the unit matrix I. Group indices can be raised with the metric $2\delta^{ab}$ or lowered with $\frac{1}{2}\delta_{ab}$.

A 't Hooft gauge fixing term will be used:

$$\mathcal{L}_{\rm GF} = -\tfrac{1}{2}\xi^{-1}\mathcal{F}_a\mathcal{F}^b, \quad (61)$$

where

$$\mathcal{F} = \nabla\cdot A + i\xi g\hat{\phi}^\dagger T_a\Phi\, T^a. \quad (62)$$

The corresponding ghost action is then

$$\mathcal{L}_c = \bar{c}^a(\nabla^2\delta_a{}^b - \xi g^2\hat{\phi}^\dagger T_a T^b\Phi)c_b. \quad (63)$$

Calculation of the effective potential begins by expanding the action about a background in powers of the perturbed fields. The only nonvanishing background field which we will consider will be the Higgs field ϕ:

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}. \quad (64)$$

Perturbed fields will be denoted by a tilde. The total action is then

$$I[\phi+\tilde{\phi},\tilde{A},\tilde{c},\tilde{\bar{c}}] = \int d\mu(x)\,\{\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\}, \quad (65)$$

where $\mathcal{L}_i$ denotes the part of $\mathcal{L}$ that has the ith power of the perturbed fields.

The quadratic terms determine fluctuation operators Δ, given explicitly by

$$\Delta_A = -\delta_\mu{}^\nu\nabla^2 + (1-\xi^{-1})\nabla_\mu\nabla^\nu + X_A\delta_\mu{}^\nu, \quad (66)$$
$$\Delta_\phi = -\nabla^2 + X_\phi, \quad (67)$$
$$\Delta_c = -\nabla^2 + \xi X_A, \quad (68)$$

where

$$X_A = 2g^2\phi^\dagger T_a T^b\phi, \quad (69)$$
$$X_\phi = \xi g^2 T_a\phi\phi^\dagger T^a - \mu^2 + \lambda\phi^\dagger\phi + 2\lambda\phi\phi^\dagger. \quad (70)$$

The mass matrices can all be decomposed with respect to the broken symmetry subgroup. A general way to do this is described in the Appendix. This gives

$$X_A = \tfrac{1}{2}g^2v^2P_{\|a}{}^b, \quad (71)$$
$$X_\phi = (-\mu^2 + \lambda v^2 + \tfrac{1}{2}\xi g^2v^2)I + \lambda v^2P_\|. \quad (72)$$

The effective action given in the previous section was

$$\Gamma = I - \tfrac{1}{2}\hbar\,\mathrm{tr}(1 - G_\phi\Delta_\phi) + \Gamma^{(1)} + \Gamma^{(2)} \quad (73)$$

with

$$\Gamma^{(1)} = -\tfrac{1}{2}\hbar\ln\det G_\phi - \tfrac{1}{2}\hbar\ln\det G_A + \hbar\ln\det G_c. \quad (74)$$

The propagators are given by Feynman diagram expansions with internal lines representing $(\Delta_\phi + K)^{-1}$, etc.

For constant background fields we can find the effective potential from $V(\phi) = \Omega^{-1}\Gamma[\phi]$. We will use Landau gauge $\xi \to 0$, in which the ghosts are independent of the background field. We also define corrected masses in terms of the full propagators by

$$(G_\phi)^{-1} = -\nabla^2 + X_\phi(T) + O(\hbar),$$
$$(G_A)^{-1} = -\nabla^2\delta_\mu{}^\nu + X_A(T) + O(\hbar). \quad (75)$$

The one-loop contribution to the effective potential is then

$$V^{(1)} = -\tfrac{1}{2}\hbar\left(G_\phi^{-1} - \Delta_\phi\right)G_\phi(x,x)$$
$$+\tfrac{1}{2}\hbar\Omega^{-1}\ln\det[-\nabla^2 + X_\phi(T)]$$
$$+\tfrac{1}{2}\hbar\Omega^{-1}\ln\det{}_T[-\nabla^2 + X_A(T)]. \quad (76)$$

The vector field determinant is over transverse modes.

The value of k which appears in G_0 is found by equating the tadpole diagram series to zero,

$$k = \mu^2 - \tfrac{1}{2}\lambda v^2 + O(\hbar^{-1}\lambda T^2) + O(\hbar). \tag{77}$$

The leading order terms in the propagator are given by

$$X_\phi(T) = X_\phi - k\,\mathrm{I} + O(\hbar^{-1}\lambda T^2), \tag{78}$$

with X_ϕ given in (72). As in Ref. [10], the one-loop contribution to the propagator cancels with the one-loop contribution to k. Thus,

$$X_\phi(T) = m_\phi^2 P_\| + O(\hbar),$$
$$X_A(T) = m_W^2 P_{\|a}{}^b + O(\hbar^{-1}g^2T^2), \tag{79}$$

where

$$m_\phi^2 = \lambda v^2, \quad m_W^2 = \tfrac{1}{2}g^2v^2, \quad m_Z^2 = \tfrac{1}{2}(g^2+g'^2)^{1/2}v^2. \tag{80}$$

The gauge loop corrections to the vector boson masses can be found in the literature, e.g., [2].

The effective potential now becomes

$$V^{(1)} \sim -\tfrac{1}{2}\hbar\left(\mu^2 - \tfrac{1}{2}m_\phi^2\right)G_\phi(x,x) + \tfrac{1}{2}\hbar\Omega^{-1}\ln\det(-\nabla^2+m_\phi^2) + \tfrac{3}{2}\hbar\Omega^{-1}\left[2\ln\det{}_T(-\nabla^2+m_W^2) + \ln\det{}_T(-\nabla^2+m_Z^2)\right]. \tag{81}$$

This result is valid for any temperature. At zero temperature the main feature of this result is the simple form of the Higgs boson mass terms, which are generally different from those in the standard one-loop result. The results agree, however, when the Higgs field takes its vacuum expectation value.

In the high temperature limit, when $T \gg m_\phi$ and $T \gg m_W$, we have the expansions [6,5]

$$G(x,x) \sim \hbar^{-2}\left(\tfrac{1}{12}T^2 - \tfrac{1}{2\pi}\hbar m T - \tfrac{1}{8\pi^2}\hbar^2 m^2 \ln(\mu_R/T)\right) \tag{82}$$

and

$$\ln\det G \sim \Omega\hbar^{-4}\left(\tfrac{\pi^2}{45}T^4 - \tfrac{1}{12}\hbar^2m^2T^2 + \tfrac{1}{6\pi}\hbar^3m^3T + \tfrac{1}{16\pi^2}\hbar^4m^4\ln(\mu_R/T)\right). \tag{83}$$

The quantity μ_R is the renormalization scale.

In this limit the effective potential becomes

$$V^{(1)} \sim \tfrac{1}{12\pi}\mu^2 m_\phi T + \tfrac{1}{16}\hbar^{-1}m_\phi^2T^2 - \tfrac{1}{8\pi}m_\phi^3 T + \hbar\tfrac{1}{16\pi^2}(\mu^2 - m_\phi^2)m_\phi^2\ln(\mu_R/T) + \tfrac{1}{8}\hbar^{-1}(2m_W^2+m_Z^2)T^2 - \tfrac{1}{4\pi}(2m_W^3+m_Z^3)T - \hbar\tfrac{3}{32\pi^2}(2m_W^4+m_Z^4)\ln(\mu_R/T). \tag{84}$$

The terms involving vector boson masses have the standard form [2,13,14]. We have not included the ring corrections to the vector boson masses, but these would be identical to those in the references. The other terms have some resemblance to results that use a corrected mass for the Higgs field, the main difference with some of these is the term linear in v. Although the authors of Ref. [13] argue strongly against such a term, their arguments are only valid for their definition of the effective action.

V. FERMION CONTRIBUTIONS

As in the paper by Carrington [2], we work in the approximation in which the only nonvanishing Yukawa coupling is the top quark coupling constant, h^t. The fermion part of the Lagrangian is given by

$$\mathcal{L}_f = \bar\psi_R\gamma^\mu(\nabla_\mu - i\tfrac{1}{2}g'YB_\mu)\psi_R + \bar\psi_L\gamma^\mu(\nabla_\mu - ig\tfrac{1}{2}\sigma^a A_{\mu\ a} - i\tfrac{1}{2}g'YB_\mu)\psi_L, \tag{85}$$

where the left-handed fermions have been denoted by ψ_L:

$$\psi_L = \tfrac{1}{2}(1-\gamma_5)\begin{pmatrix}\nu_\alpha\\ e_\alpha\end{pmatrix}, \quad \tfrac{1}{2}(1-\gamma_5)\begin{pmatrix}q_{u\alpha}\\ q_{l\alpha}\end{pmatrix}_{3\text{ colors}}, \tag{86}$$

and the right-handed fermions by ψ_R :

$$\psi_R = \tfrac{1}{2}(1+\gamma_5)e_\alpha, \quad \tfrac{1}{2}(1+\gamma_5)q_{u\alpha}, \quad \tfrac{1}{2}(1+\gamma_5)q_{l\alpha}, \tag{87}$$

and $B = A_4$.

The subscript α denotes the three generations of quarks (u,d), (c,s), and (t,b) or for the leptons (ν_e,e), (ν_μ,μ), (ν_τ,τ). We have denoted the hypercharge by Y and we follow the notation in which the charge Q of the field is defined by $Q = T^3 + \frac{1}{2}Y$, where T^3 is the third component of isospin. In this notation, $Y = -1$ for the left-handed leptons, $Y = \frac{1}{3}$ for the left-handed quarks,

$Y = -2$ for the right-handed electrons, $Y = \frac{4}{3}$ for the right-handed quarks $q_{u\alpha}$ and $Y = -\frac{2}{3}$ for $q_{l\alpha}$.

The Yukawa term $\mathcal{L}_Y$ is given by

$$\mathcal{L}_Y = -h^t \bar{q}_L \phi_c t_R, \tag{88}$$

where ϕ_c has been defined as

$$\phi_c = i\tau^2 \phi^*. \tag{89}$$

In our notation, the Dirac matrices γ^a are all Hermitian:

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & i\tau^i \\ -i\tau^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}, \tag{90}$$

with $i = 1, \ldots, 3$.

In terms of the components of the Higgs field, the Yukawa term can be rewritten as

$$\begin{aligned} \mathcal{L}_Y = & -\frac{h^t}{\sqrt{2}} \bar{t}(\tilde{\phi}_3 + v)t - \frac{h^t}{2\sqrt{2}} \{ \bar{b}(1+\gamma_5)(-\tilde{\phi}_1 + i\tilde{\phi}_2)t, \\ & - \bar{t}(1-\gamma_5)(\tilde{\phi}_1 + i\tilde{\phi}_2)b \} + i\frac{h^t}{\sqrt{2}} \bar{t}\gamma_5 \tilde{\phi}_4 t, \end{aligned} \tag{91}$$

where we have performed the shift in the Higgs field .

The fermionic contribution to the one-loop effective action is given formally by

$$\Gamma_f^{(1)} = -\ln\ \det\ D_f = -\ln\ \det\ [\gamma^\mu \nabla_\mu - m_t], \tag{92}$$

where we have denoted the top quark mass by

$$m_t = \frac{h^t v}{\sqrt{2}}. \tag{93}$$

Since this determinant is ill defined we rewrite it in terms of the determinant of $D_f^\dagger D_f$ which is well defined as

$$\begin{aligned} \Gamma_f^{(1)} &= -\tfrac{1}{2} \ln \det\ D_f^\dagger D_f \\ &= -\tfrac{1}{2} \ln \det\ (-\nabla^2 + m_t^2). \end{aligned} \tag{94}$$

The high temperature effective potential can be calculated as in [6], or by using the result for fermions that

$$\begin{aligned} [\ln \det \Delta]_{\text{antiperiod } \beta} &= [2 \ln \det \Delta]_{\text{period } 2\beta} \\ &\quad - [\ln \det \Delta]_{\text{period } \beta}, \end{aligned} \tag{95}$$

where $\beta = 1/T$, and we work in units in which Boltzmann's constant, $k = 1$. This gives

$$\begin{aligned} \ln \det G \sim \Omega \hbar^{-4} \Bigg(& -\tfrac{7}{8}\tfrac{\pi^2}{45} T^4 + \tfrac{1}{24} \hbar^{-2} m^2 T^2 \\ & + \tfrac{1}{16\pi^2} \hbar^{-4} m^4 \ln(\mu_R/T) \Bigg) \end{aligned} \tag{96}$$

and therefore

$$V^{(1)}_{\text{fermions}} = \hbar \left[\tfrac{1}{4} \hbar^{-2} T^2 m_t^2 + \tfrac{3}{8} m_t^4 \ln(\mu_R/T) \right], \tag{97}$$

where we have neglected all the terms of order T^{-1} at high temperatures. We have also performed the trace over the color and the spin indices. It should be noted that the term linear in T is absent for the case of fermions.

VI. CONCLUSIONS

We have shown that improved potentials and corrected mass terms can be introduced by using a quadratic source term in the path integral construction. The advantage of doing things this way is that we avoid ever having to deal with complex propagators in the loop expansion. We are also able to include the possibility of multiple saddle points in the path integral, and in a cosmological context where the volume might be finite this should not be overlooked.

The resulting effective potential at zero temperatures is different from the usual results. However, at finite temperatures, where it has become standard to use corrected mass terms, the results are mostly in agreement with the literature. Differences in the Higgs contribution may be due to differences in truncation of the series. If the linear terms that we have found are really present, then the phase transition from the symmetric to the broken symmetry phase is affected.

The phase transition takes place by bubble nucleation. These bubbles satisfy the effective field equations generated by $\delta\Gamma = 0$. We have been careful to ensure that these equations are a property of the effective action construction, even to the extent of correcting for the gauge fixing terms.

APPENDIX: DECOMPOSITIONS OF MASS MATRICES

The generators of the Lie algebra of the gauge group are represented by matrices T^a. These define a metric $g^{ab} = \mathrm{tr}(T^a T^b)$. For product groups this metric is not unique because the relative scaling of some generators is not fixed. Therefore another metric $\tilde{g}^{ab}$ will be used to raise or lower indices.

The background Higgs fields ϕ can be used to define a projection $P_\parallel$ by $\phi^\dagger \phi P_\parallel = \phi\phi^\dagger$, and an orthogonal projection $P_\perp = 1 - P_\parallel$. The matrices $P_\perp T^a P_\perp$ generate another Lie algebra. We define

$$g_\perp^{ab} = \mathrm{tr}(P_\perp T^a P_\perp T^a P_\perp), \tag{A1}$$

$$g_1^{ab} = \mathrm{tr}(P_\parallel T^a P_\parallel T^a P_\parallel), \tag{A2}$$

$$g_\parallel^{ab} = 2\mathrm{tr}(P_\parallel T^a P_\perp T^a P_\parallel). \tag{A3}$$

These are not independent because $g^{ab} = g_\perp^{ab} + g_1^{ab} + g_\parallel^{ab}$. We also define operators that act on the Lie algebra

$$P_{\perp a}{}^{b} = \tilde{g}_{ac}(g_{\perp}^{cb} - g_{1}^{cb}), \quad P_{\| a}{}^{b} = \tilde{g}_{ac}(g_{\|}^{cb} + 2g_{1}^{cb}). \quad \text{(A4)}$$

The significance of these operators is that the vector mass matrix in the Higgs model is $2g^2\phi^\dagger T_a T^b \phi = g^2 \phi^\dagger \phi P_{\| a}{}^{b}$.

For the group SU(2)×U(1) with doublet Higgs fields the generators can be chosen to be $T^a = \sigma^a$ for $a = 1, \ldots, 3$ and $T^4 = I\,t$, where $t = g'/g$. We take $\tilde{g}^{ab} = 2\,\delta^{ab}$. For $\phi = (0, v)^T/\sqrt{2}$, the corresponding projections are

$$P_{\perp a}{}^{b} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & t \\ & & t & 0 \end{pmatrix}, \qquad P_{\| a}{}^{b} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & -t \\ & & -t & t^2 \end{pmatrix}. \quad \text{(A5)}$$

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