

Spontaneous symmetry breaking of (1+1)-dimensional ϕ^4 theory in light-front field theory. II

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We discuss spontaneous symmetry breaking of (1+1)-dimensional ϕ^4 theory in light-front field theory using a Tamm-Dancoff truncation. We show that, even though light-front field theory has a simple vacuum state which is an eigenstate of the full Hamiltonian, the field can develop a nonzero vacuum expectation value. This occurs because the zero mode of the field must satisfy an operator-valued constraint equation. In the context of (1+1)-dimensional ϕ^4 theory we present solutions to the constraint equation using a Tamm-Dancoff truncation to a finite number of particles and modes. We study the behavior of the zero mode as a function of coupling and Fock space truncation. The zero mode introduces new interactions into the Hamiltonian which breaks the Z_2 symmetry of the theory when the coupling is stronger than the critical coupling. We investigate the energy spectrum in the symmetric and broken phases, show that the theory does not break down in the vicinity of the critical coupling, and discuss the connection to perturbation theory. Finally, we study the spectrum of the field ϕ and show that, in the broken phase, the field is localized away from $\phi = 0$ as one would expect from equal-time calculations. We explicitly show that tunneling occurs.

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I. INTRODUCTION

Two fundamentally different phenomenological pictures of hadronic matter have developed over the past twenty years. One is the constituent quark model where hadrons are pictured as being made of a few heavy quarks bound by a confining potential. Here, Zweig's rule implies that very little of a hadron's momentum is carried by gluons. The other is the quark-parton model which is valid at higher energy scales. In this picture, a hadron is composed of an indefinite number of light quarks and most of the hadron momentum is carried by gluons. We believe that the underlying theory for both of these disparate pictures is the non-Abelian gauge theory quantum chromodynamics (QCD). It is believed that the light-front formulation of QCD is our best hope of joining the constituent quark model with the parton model and QCD. In this elegant approach the field theory is quantized on the null plane surface $x^+ = (x^0 + x^3)/\sqrt{2} = 0$ instead of the usual equal-time surface $x^0 = 0$. This formulation avoids many of the difficult problems that appear in the equal-time formulation of field theory.

In the light-front formalism the energy operator does not have a square root operator, and, consequently, the vacuum structure is relatively simple. Dirac [1], in 1949, showed that a maximum number of Poincaré generators become independent of the dynamics in the "front form" formulation, including certain Lorentz boosts. The eigenstates of the light-front Hamiltonian have Lorentz scalars $\bar{M}^2 = P^2$ as eigenvalues and describe bound states of arbitrary four-momentum and invariant mass $\bar{M}$, allowing easy computation of scattering amplitudes and other dynamical quantities. However, the most remarkable feature of this formalism is the apparent simplicity of the

vacuum. The Fock space vacuum is an eigenstate of the full Hamiltonian. In other words, the bare vacuum is the physical vacuum. More detail is given in a recent review [2].

For the past several years an increasingly large group of physicists has attempted to combine Tamm-Dancoff [3] procedures with light-front quantization in order to perform nonperturbative calculations in quantum field theory. The assumptions, consistent with the constituent quark model picture, are that a hadron is well approximated by only a few particles and that adding more particles only refines this initial approximation. This is in stark contrast to the equal-time formulation of QCD where an infinite number of gluons are essential to construct even the vacuum [2]. If these efforts are successful, they could lead to procedures for calculating not only the hadron mass spectrum but all the quantities which depend on hadron wave functions: structure functions, fragmentation functions, etc.

The origin of this remarkably simple vacuum in the light-front formalism is that conservation of P^+ requires $\sum_i k_i^+ = 0$. However, $k_i^+ > 0$ for massive particles, and the total light-front Hamiltonian annihilates the perturbative vacuum. In contrast, the physical vacuum in equal-time quantization is a highly complex composite of pair fluctuations which is believed to produce all phenomena that require long range order: color confinement, chiral symmetry breaking, the Goldstone pion, etc. How can one obtain such nontrivial properties in the light-front formulation of field theory? These phenomena occur in light-front field theory because the field includes a zero mode operator [4] which is not an independent degree of freedom. This mode is a complicated operator-valued function of all other modes in the theory and, since it

carries zero momentum, it can provide long range order.

This problem has recently been attacked from several directions. The question of whether boundary conditions can be consistently defined in light-front quantization has been discussed by McCartor and Robertson [5] and Lenz [6]. They have shown that for massive theories the energy and momentum derived from light-front quantization are conserved and are equivalent to the energy and momentum one would normally write down in an equal-time theory. In the analyses of Lenz *et al.* [6] and Hornbostel [7] one traces the fate of the equal-time vacuum in the limit $P^3 \rightarrow \infty$ and equivalently in the limit $\theta \rightarrow \pi/2$ when rotating the evolution parameter $\tau = x^0 \cos \theta + x^3 \sin \theta$ from the equal-time frame to the light-front frame. Heinzl *et al.* [8] have considered ϕ^4 theory in 1+1 dimensions and solved the zero mode constraint equation by truncating the equation to one particle and retaining all modes. They implicitly retain a two particle contribution in order to obtain finite results. (We will discuss this in detail later.) They find a critical coupling for the onset of spontaneous symmetry breaking. Other authors [9, 10] find that, for theories allowing spontaneous symmetry breaking, there is a degeneracy of light-front vacua and the true vacuum state can differ from the perturbative vacuum through the addition of zero mode quanta. In addition to these approaches there are many others [11–13].

We present here a discussion of the zero mode constraint equation for (1+1)-dimensional ϕ^4 field theory $[(\phi^4)_{1+1}]$ with symmetric boundary conditions and show how spontaneous symmetry breaking occurs within the context of this model. This work builds upon the investigation of the one-mode case by Bender, Pinsky, and van de Sande in Ref. [14]. Our basic approach is to apply a Tamm-Dancoff truncation to the Fock space. We restrict the number of particles and the number of modes to be less than some finite limit. This means that we can represent any operator by a finite matrix and solve the operator-valued constraint equation numerically. The truncation assumes that states with a large number of particles or large momentum do not have an important contribution to the zero mode.

We find the following general behavior: for small coupling (large g , where $g \propto 1/\text{coupling}$) the constraint equation has a single solution and the field has no vacuum expectation value (VEV). As we increase the coupling (decrease g) to what we will call the “critical coupling” g_{critical} , two additional solutions which give the field a nonzero VEV appear. These solutions differ only infinitesimally from the first solution near the critical coupling, indicating the presence of a second-order phase transition. Above the critical coupling ($g < g_{\text{critical}}$), there are three solutions: one with zero VEV, the “unbroken phase,” and two with nonzero VEV, the “broken phase.” As shown in Fig. 1, one can plot the VEV as a function of g . We will call these the “critical curves.”

In the weak coupling limit (g large) the solution to the constraint equation can be obtained in perturbation theory. This solution does not break the Z_2 symmetry and is believed to simply insert the missing zero momentum contributions into internal propagators. This

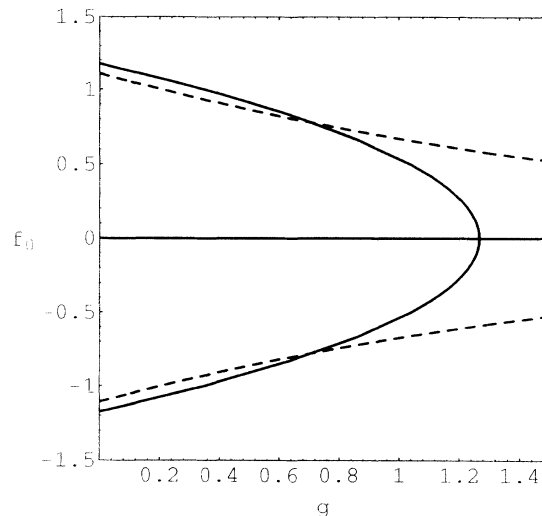


FIG. 1. $f_0 = \sqrt{4\pi}\langle 0|\phi|0\rangle$ vs $g = 24\pi\mu^2/\lambda$ in the one-mode case. The solid curves are the critical curves obtained from numerical solution of Eq. (3.3) with $N = 10$. The dashed curve is the critical curve obtained from the first-order δ expansion.

must happen if light-front perturbation theory is to agree with equal-time perturbation theory [15]. This has been shown for ϕ^3 theory in 1+1-dimensions [16] and we believe that it happens here as well.

Since the vacuum in this theory is trivial, all of the long range properties must occur in the operator structure of the Hamiltonian. Above the critical coupling ($g < g_{\text{critical}}$) quantum oscillations spontaneously break the Z_2 symmetry of the theory. In a loose analogy with a symmetric double well potential, we have two new Hamiltonians for the broken phase, each producing states localized in one of the wells. The structure of the two Hamiltonians is determined from the broken phase solutions of the zero mode constraint equation. We find that the two Hamiltonians have equivalent spectra. In a discrete theory without zero modes it is well known that, if one increases the coupling sufficiently, quantum correction will generate tachyons causing the theory to break down near the critical coupling. Here the zero mode generates new interactions that prevent tachyons from developing. In effect, what happens is that, while quantum corrections attempt to drive the mass negative, they also change the vacuum energy through the zero mode and the diving mass eigenvalue can never catch the vacuum eigenvalue. Thus, tachyons never appear in our spectra.

Another way to investigate the zero mode is to study the spectrum of the field operator ϕ . Here we find a picture that agrees with the symmetric double well potential analogy. In the broken phase, the field is localized in one of the minima of the potential and there is tunneling to the other minimum.

In Sec. II we discuss the derivation of the zero mode constraint equation and the formal operator constraint equation that relates the zero mode to all the other modes in the problem. In Sec. III we apply a truncation allowing only the first nonzero mode and many particles. In

Sec. IV we examine the truncation where we allow many modes and only one particle. Next, in Sec. V, truncations with several particles and several modes are examined. In Sec. VI, we examine the spectrum of the field operator $\sqrt{4\pi}\phi$. Finally, in Sec. VII we discuss our results and the remaining work needed on this problem.

II. QUANTIZATION

The details of the Dirac-Bergmann prescription and its application to the system considered in this paper are discussed elsewhere in the literature [4, 8, 14, 17]. In this section, we will summarize those results and introduce our notation. We define light-front coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$. For a classical field the $(\phi^4)_{1+1}$ Lagrangian is

$$\mathcal{L} = \partial_+ \phi \partial_- \phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (2.1)$$

We put the system in a box of length d and impose periodic boundary conditions. For most of our discussion we work in momentum space. We define q_k by

$$\phi(x) = \frac{1}{\sqrt{d}} \sum_n q_n(x^+) e^{ik_n^+ x^-}, \quad (2.2)$$

where $k_n^+ = 2\pi n/d$ and summations run over all integers unless otherwise noted. Following the Dirac-Bergmann prescription, we can identify first-class constraints which define the conjugate momenta and a secondary constraint which determines the “zero mode” q_0 in terms of the other modes in the theory. This result can also be obtained by integrating the equations of motion in position

space or differentiating the Hamiltonian with respect to the zero mode [14].

Quantizing, we define creation and annihilation operators $a_k^\dagger$ and a_k by

$$q_k = \sqrt{\frac{d}{4\pi|k|}} a_k, \quad a_k = a_{-k}^\dagger, \quad k \neq 0, \quad (2.3)$$

which satisfy the usual commutation relations

$$\begin{aligned} [a_k, a_l] &= 0, \quad [a_k^\dagger, a_l^\dagger] = 0, \\ [a_k, a_l^\dagger] &= \delta_{k,l}, \quad k, l > 0. \end{aligned} \quad (2.4)$$

Likewise, we define the zero mode operator

$$q_0 = \sqrt{\frac{d}{4\pi}} a_0. \quad (2.5)$$

It is useful to define the quantity,

$$\Sigma_n = \frac{1}{n!} \sum_{i_1, i_2, \dots, i_n \neq 0} \delta_{i_1+i_2+\dots+i_n, 0} \frac{a_{i_1} a_{i_2} \dots a_{i_n}}{\sqrt{|i_1 i_2 \dots i_n|}}. \quad (2.6)$$

General arguments suggest that the Hamiltonian should be symmetrically ordered [18]. However, it is not clear how one should treat the zero mode since it is not a dynamical field. As an ansatz we will treat a_0 as an ordinary field operator when symmetrically ordering the Hamiltonian.

Rescaling P^- , the quantum Hamiltonian is [14]

$$\begin{aligned} H = \frac{96\pi^2}{\lambda d} P^- &= \frac{g}{2} a_0^2 + \frac{a_0^4}{4} + g \Sigma_2 + 6 \Sigma_4 \\ &+ \frac{1}{4} \sum_{n \neq 0} \frac{1}{|n|} (a_0^2 a_n a_{-n} + a_n a_{-n} a_0^2 + a_n a_0^2 a_{-n} + a_n a_0 a_{-n} a_0 + a_0 a_n a_0 a_{-n} + a_0 a_n a_{-n} a_0 - 3a_0^2) \\ &+ \frac{1}{4} \sum_{k, l, m \neq 0} \frac{\delta_{k+l+m, 0}}{\sqrt{|klm|}} (a_0 a_k a_l a_m + a_k a_0 a_l a_m + a_k a_l a_0 a_m + a_k a_l a_m a_0) - C. \end{aligned} \quad (2.7)$$

where $g = 24\pi\mu^2/\lambda$. We have removed tadpoles from the symmetrically ordered Hamiltonian by normal ordering the third and fourth terms and subtracting:

$$-\frac{3}{4} a_0^2 \sum_{n \neq 0} \frac{1}{|n|}. \quad (2.8)$$

In addition, we have subtracted a constant C so that the VEV of H is zero. Note that this renormalization prescription is equivalent to a conventional mass renormalization and does not introduce any new operators (aside from the constant) into the Hamiltonian. The constraint equation for the zero mode can be obtained by taking a derivative of P^- with respect to a_0 . Consequently, we symmetrically order the constraint equation:

$$0 = g a_0 + a_0^3 + \sum_{n \neq 0} \frac{1}{|n|} \left(a_0 a_n a_{-n} + a_n a_{-n} a_0 + a_n a_0 a_{-n} - \frac{3a_0}{2} \right) + 6 \Sigma_3. \quad (2.9)$$

Using the constraint equation, we can rewrite H as:

$$H = g \Sigma_2 + 6 \Sigma_4 - \frac{a_0^4}{4} + \frac{1}{4} \sum_{n \neq 0} \frac{1}{|n|} (a_n a_0^2 a_{-n} - a_0 a_n a_{-n} a_0) + \frac{1}{4} \sum_{k, l, m \neq 0} \frac{\delta_{k+l+m, 0}}{\sqrt{|klm|}} (a_k a_0 a_l a_m + a_k a_l a_0 a_m) - C. \quad (2.10)$$

It is clear from the general structure of (2.9) that a_0 as a function of the other modes is not necessarily odd under the transform $a_k \rightarrow -a_k$, $k \neq 0$ associated with the Z_2 symmetry of the system. Consequently, the zero mode can induce Z_2 symmetry breaking in the Hamiltonian. Along with the zero mode and the Hamiltonian, Σ_n commutes with the longitudinal momentum operator:

$$[a_0, P^+] = 0, \quad [H, P^+] = 0, \quad [\Sigma_n, P^+] = 0. \quad (2.11)$$

In order to render the problem tractable, we impose a Tamm-Dancoff truncation on the Fock space. Define M to be the number of nonzero modes and N to be the maximum number of allowed particles. Thus, each state in the truncated Fock space can be represented by a vector of length $S = (M + N)! / (M!N!)$ and operators can be represented by $S \times S$ matrices. One can define the usual Fock space basis,

$$|n_1, n_2, \dots, n_M\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \cdots \frac{(a_M^\dagger)^{n_M}}{\sqrt{n_M!}} |0\rangle, \quad (2.12)$$

where $n_1 + n_2 + \cdots + n_M \leq N$. Thus, for example,

$$P^+ |n_1, n_2, \dots, n_M\rangle = \frac{2\pi}{d} (n_1 + 2n_2 + \cdots + Mn_M) \times |n_1, n_2, \dots, n_M\rangle, \quad (2.13)$$

$$\Sigma_2 |n_1, n_2, \dots, n_M\rangle = \left(\frac{n_1}{1} + \frac{n_2}{2} + \cdots + \frac{n_M}{M} \right) \times |n_1, n_2, \dots, n_M\rangle. \quad (2.14)$$

In matrix form, a_0 is real and symmetric. Moreover, it is block diagonal in states of equal P^+ eigenvalue. Because of the Fock space truncation, the commutator $[a_k, a_l^\dagger]$ is somewhat different than its full Fock space counterpart. We will use the full Fock space commutators to move creation operators to the left as much as possible in a product before performing any matrix multiplications.

III. ONE MODE, MANY PARTICLES

In this section we review the results of Ref. [14]. Consider the case of one mode $M = 1$ and many particles. In this case, the zero mode is diagonal and can be written as

$$a_0 = f_0 |0\rangle \langle 0| + \sum_{k=1}^N f_k |k\rangle \langle k|. \quad (3.1)$$

Note that a_0 in (3.1) is even under $a_k \rightarrow -a_k$, $k \neq 0$ and any nonzero solution breaks the Z_2 symmetry of the original Hamiltonian. The VEV is given by

$$\langle 0|\phi|0\rangle = \frac{1}{\sqrt{4\pi}} \langle 0|a_0|0\rangle = \frac{1}{\sqrt{4\pi}} f_0. \quad (3.2)$$

Substituting (3.1) into the constraint equation (2.9) and sandwiching the constraint equation between Fock states, we get a recursion relation for $\{f_n\}$:

$$0 = g f_n + f_n^3 + (4n - 1) f_n + (n + 1) f_{n+1} + n f_{n-1}, \quad (3.3)$$

where $n \leq N$, and we define f_{N+1} to be unknown. Consequently, $\{f_1, f_2, \dots, f_{N+1}\}$ is uniquely determined by a given choice of g and f_0 . In particular, if $f_0 = 0$ all the f_k 's are zero independent of g . This is the unbroken phase.

Our first objective is to determine which solutions of the recursion relation are consistent with the Tamm-Dancoff truncation. If f_n is large for large n then the high energy levels will be strongly affected. The paradigm for spontaneous symmetry breaking is the symmetric double well potential in ordinary quantum mechanics which has a ground state centered in either well rather than at the symmetry point. This paradigm indicates that the behavior of the system is unaffected by the barrier for energies far above the barrier separating the wells. Hence, we only seek solutions where f_n is small for large n . We caution here that this paradigm does not translate entirely to the light-front formulation since the symmetry breaking occurs in the Hamiltonian and the vacuum itself is unaffected.

We begin our analysis of Eq. (3.3) by dropping the cubic term and finding solutions to the resulting linear system. Assume for now that $N = \infty$ and that $|f_n|$ is decreasing with n . For large n , the terms linear in n dominate, and Eq. (3.3) becomes

$$f_{n+1} + 4f_n + f_{n-1} = 0. \quad (3.4)$$

There are two solutions to this equation:

$$f_n \propto (\sqrt{3} \pm 2)^n. \quad (3.5)$$

We reject the plus solution because it grows with n . Dropping the cubic term from Eq. (3.3) we define the generating function

$$F(z) = \sum_{n=0}^{\infty} f_n z^n. \quad (3.6)$$

If f_n goes like $(\sqrt{3} - 2)^n$ then the radius of convergence of $F(z)$ is $2 + \sqrt{3}$ and we expect $F(z)$ to be singular at $|z| = 2 + \sqrt{3}$. Similarly, if $f_n \sim (\sqrt{3} + 2)^n$, then we expect $F(z)$ to be singular at $|z| = 2 - \sqrt{3}$.

The function $F(z)$ satisfies a differential equation whose solution is

$$\frac{F(z)}{F(0)} = \left(\frac{z + 2 - \sqrt{3}}{2 - \sqrt{3}} \right)^{-\frac{\sqrt{3}-3+g}{2\sqrt{3}}} \left(\frac{z + 2 + \sqrt{3}}{2 + \sqrt{3}} \right)^{-\frac{\sqrt{3}+3-g}{2\sqrt{3}}} \quad (3.7)$$

Note that this solution for $F(z)$ has singularities at the expected values of z . If we want f_n to have the asymptotic behavior $(\sqrt{3} - 2)^n$ for large n , then we must eliminate the branch point of $F(z)$ at $|z| = 2 - \sqrt{3}$. This gives the condition

$$-\frac{\sqrt{3}-3+g}{2\sqrt{3}} = K, \quad K = 0, 1, 2, \dots \quad (3.8)$$

Concentrating on the $K = 0$ case, we find a critical coupling

$$g_{\text{critical}} = 3 - \sqrt{3} \quad (3.9)$$

or

$$\lambda_{\text{critical}} = 4\pi (3 + \sqrt{3}) \mu^2 \approx 60\mu^2, \quad (3.10)$$

consistent with equal-time calculations [19].

The solution to the linearized equation is an approximate solution to the full equation (3.3) for f_0 sufficiently small. We need to determine solutions of the full nonlinear equation which converge for large n . We will use both the δ expansion and numerical methods to do this.

The δ expansion is a powerful perturbative technique for linearizing nonlinear problems. It has been shown to be an accurate technique for solving problems in differential equations, quantum mechanics, and quantum field theory [20].

We rewrite Eq. (3.3) as

$$(g - 1 + 4n) f_n + f_n^{1+2\delta} + (n+1) f_{n+1} + n f_{n-1} = 0. \quad (3.11)$$

Setting $\delta = 0$ gives the linear finite difference equation which is the zeroth-order approximation in the δ expansion. One then expands in powers of δ about $\delta = 0$. One recovers the problem of interest at $\delta = 1$.

We find to first order the critical curve for $\delta = 1$:

$$g = (2 - \sqrt{3}) \left(1 + \frac{1}{\sqrt{3}} \ln(2 + \sqrt{3}) \right) - \ln f_0^2. \quad (3.12)$$

These results are plotted as the dashed curve in Fig. 1. The expansion behaves badly near $f_0 = 0$ because of the $\ln f_0^2$ term in the expansion. The δ -expansion analysis clearly shows that there is a critical curve and not merely a critical point.

We can also study the critical curves by looking for numerical solutions to Eq. (3.3). The method used here is to find values of f_0 and g such that $f_{N+1} = 0$. Since we seek a solution where f_n is decreasing with n , this is a good approximation. We find that for $g > 3 - \sqrt{3}$ the only real solution is $f_n = 0$ for all n . For g less than $3 - \sqrt{3}$ there are two additional solutions. Near the critical point $|f_0|$ is small and

$$f_n \approx f_0 (2 - \sqrt{3})^n. \quad (3.13)$$

The critical curves are indicated by the solid lines in Fig. 1. These solutions converge quite rapidly with N . The critical curve for the broken phase is approximately parabolic in shape:

$$g \approx 3 - \sqrt{3} - 0.9177 f_0^2. \quad (3.14)$$

It is instructive to study the behavior of the constraint equation (3.3) away from the critical curves. In Fig. 2 we plot $|f_n|$ as a function of n and f_0 for $g = 1.2$. We see

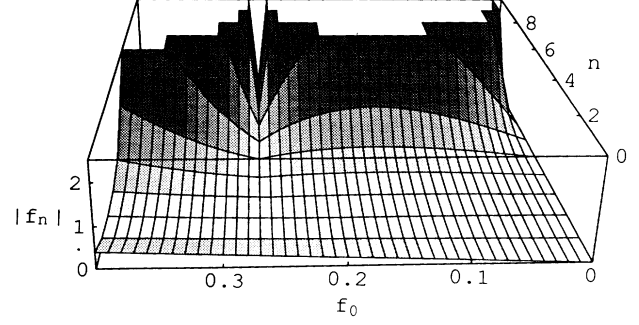


FIG. 2. $|f_n|$ vs n and f_0 for $g = 1.2$ using Eq. (3.3).

that, as n becomes large, all the $|f_n|$ increase and as f_0 approaches the critical curve, which is at $f_0 \approx 0.2700$ for $g = 1.2$, all the $|f_n|$'s decrease rapidly. As f_0 increases beyond the critical curve the $|f_n|$'s increase rapidly once again. The fact that $|f_n|$ increases rapidly on both sides of the critical curve is a manifestation of the nonlinearity in (3.3).

In Fig. 3 we extend the critical curves to the negative g region. This region corresponds to negative values of μ^2 in the Hamiltonian. We have not studied the spectrum for $\mu^2 < 0$ ($g < 0$); however, it is easy to see from the figures that nothing unusual happens, at least for small g . As predicted by Eq. (3.8), there are additional solutions near the critical curve for the unbroken phase. The curves shown are independent of N for N large. It is not clear how to interpret these solutions.

We can also study the eigenvalues of the Hamiltonian for the one-mode case. The Hamiltonian is diagonal for this Fock space truncation and,

$$\begin{aligned} \langle n | H | n \rangle = & \frac{3}{2} n(n-1) + ng - \frac{f_n^4}{4} - \frac{2n+1}{4} f_n^2 \\ & + \frac{n+1}{4} f_{n+1}^2 + \frac{n}{4} f_{n-1}^2 - C. \end{aligned} \quad (3.15)$$

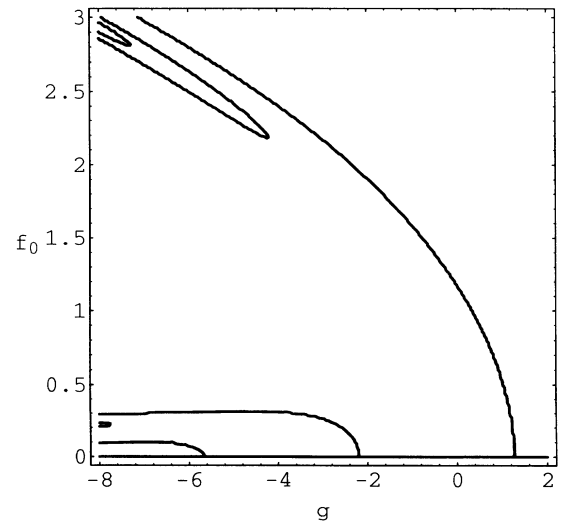


FIG. 3. Critical curves for a larger range of g in the one mode case with $N = 10$. The curves shown are independent of N , $N \geq 10$.

The invariant mass eigenvalues are given by

$$P^2|n\rangle = 2P^+P^-|n\rangle = \frac{n\lambda\langle n|H|n\rangle}{24\pi}|n\rangle. \quad (3.16)$$

In Fig. 4 the dashed lines show the first few eigenvalues as a function of g without the zero mode. When we include the broken phase of the zero mode, the energy levels shift as shown by the solid curves. For $g < g_{\text{critical}}$ the energy levels increase above the value they had without the zero mode. The higher levels change very little, as our paradigm would suggest, because f_n is small for large n .

IV. MANY MODES, ONE PARTICLE

Now, let us look at the case of one particle $N = 1$ and many modes. In this case, the zero mode is diagonal and can be written as

$$a_0 = b_0|0\rangle\langle 0| + \sum_{k=1}^M b_k a_k^\dagger|0\rangle\langle 0|a_k. \quad (4.1)$$

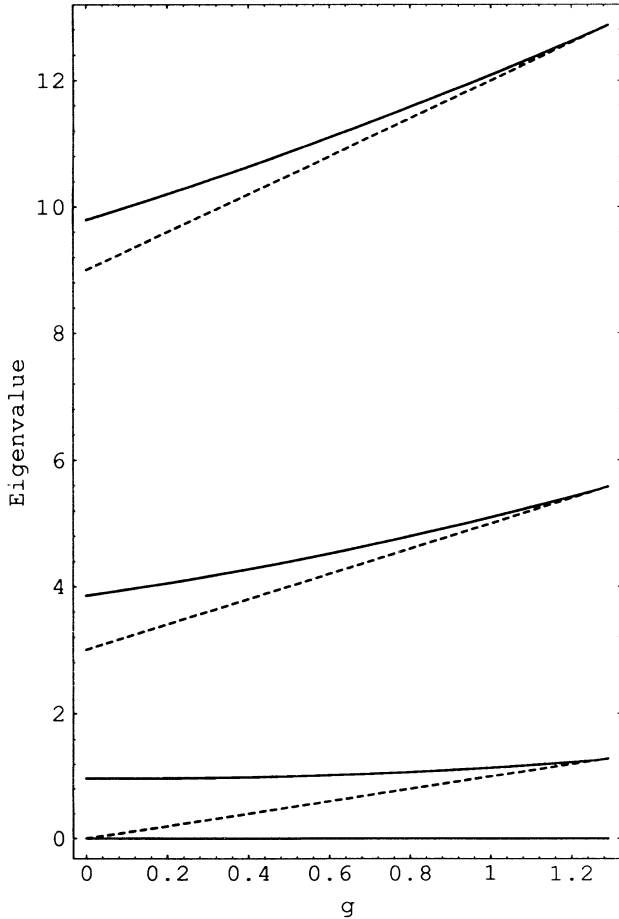


FIG. 4. The lowest three energy eigenvalues for the one-mode case as a function of g from the numerical solution of Eq. (3.15) with $N = 10$. The dashed lines are for the unbroken phase $f_0 = 0$ and the solid lines are for the broken phase $f_0 \neq 0$.

Substituting this into the constraint equation (2.9) and sandwiching the constraint equation between various states, one obtains a system of equations for the coefficients $\{b_k\}$:

$$0 = b_0^3 + gb_0 + \sum_{k=1}^M \frac{b_k - b_0}{k}, \quad (4.2)$$

$$0 = b_m^3 + gb_m + \frac{4}{m}b_m + \frac{b_0}{m} - b_m \sum_{k=1}^M \frac{1}{k} + \sum_{k=1}^M \frac{\langle 0|a_m a_k a_0 a_k^\dagger a_m^\dagger|0\rangle}{k}. \quad (4.3)$$

The last term of (4.3) couples the one and two particle sectors of a_0 . Assuming that the Fock space truncation is a good approximation, we set this term equal to zero. Note that $b_k = 0$, $k = 0, 1, \dots, M$, is always a solution of this system of equations; this is the unbroken phase. Evaluating the system of equations numerically, we find the critical curves shown in Fig. 5.

Let us calculate the critical coupling. As with the one mode case, we drop the cubic terms from (4.2) and (4.3) and find a solution to the resulting linear system. Defining $\theta = g - \sum_{k=1}^M 1/k$, we find

$$b_m = -\frac{b_0}{m\theta + 4}, \quad (4.4)$$

$$\theta = \sum_{m=1}^M \frac{1}{m(m\theta + 4)}, \quad (4.5)$$

or, in the limit of large M ,

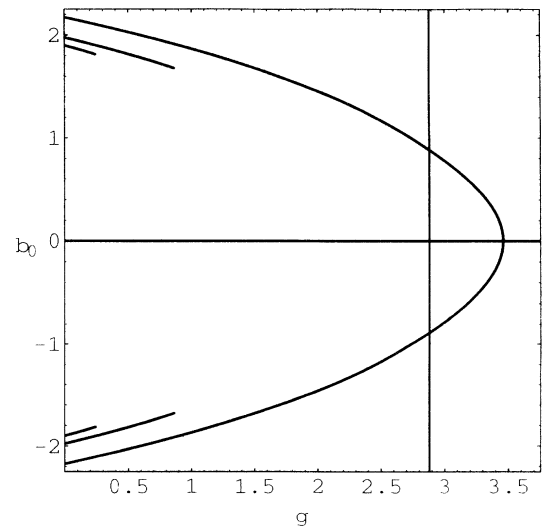


FIG. 5. $b_0 = \sqrt{4\pi}\langle 0|\phi|0\rangle$ vs g for the one particle case. The critical curves are obtained from numerical solution of Eqs. (4.2) and (4.3) with $M = 10$. The vertical line represents g_0 . Note the presence of additional solutions for g small.

$$4\theta = \gamma_E + \Psi(1 + 4/\theta) + O(1/M), \quad (4.6)$$

$$\theta = 0.626\,7537 \dots + O(1/M)$$

where γ_E is Euler's constant and $\Psi(x)$ is the digamma function. Thus, the critical coupling is logarithmically divergent,

$$g_{\text{critical}} = \ln M + \gamma_E + \theta + O(1/M). \quad (4.7)$$

We compare this expression with numerical results in Fig. 6.

Next we need to examine whether the Fock space truncation is consistent, i.e., whether b_m converges for large m . If we look at the linearized solutions, Eq. (4.4), we see that b_m does converge near the critical coupling. Now, examine the full nonlinear equations. From Eq. (4.3), we see that b_m is independent of m if $b_m = -b_0/4$. Substituting this into Eqs. (4.2) and (4.3), one can solve for g . Define g_0 to be this value of g :

$$g_0 = \frac{59}{60} \sum_{k=1}^M \frac{1}{k}. \quad (4.8)$$

Note that $g_0 < g_{\text{critical}}$ for all M and that g_0 is logarithmically divergent in M . For $g = g_0$, b_m is independent of m . For $g < g_0$, b_m diverges with m , and, for $g > g_0$, b_m converges with m . This means that the solution to the constraint equation is inconsistent with the Fock space truncation for $g \leq g_0$.

Elsewhere it has been suggested that one use a somewhat different ansatz for the zero mode [8]:

$$\tilde{a}_0 = c_0 + \sum_{k=1}^M c_k a_k^\dagger a_k. \quad (4.9)$$

How does it compare with the definition in Eq. (4.1)? Let us look at some matrix elements in each case:

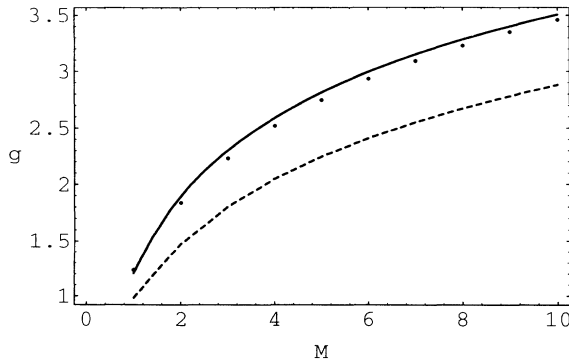


FIG. 6. g_{critical} vs M in the one particle case. The solid curve is from Eq. (4.7) and the points are from numerical solution of (4.2) and (4.3). The dashed curve is g_0 from (4.8).

$$\begin{aligned} \langle 0|a_0|0\rangle &= b_0, \quad \langle 0|\tilde{a}_0|0\rangle = c_0, \\ \langle 0|a_k a_0 a_k^\dagger|0\rangle &= b_k, \quad \langle 0|a_k \tilde{a}_0 a_k^\dagger|0\rangle = c_0 + c_k, \end{aligned}$$

$$\begin{aligned} \langle 0|a_k a_l a_0 a_l^\dagger a_k^\dagger|0\rangle &= 0, \\ \langle 0|a_k a_l \tilde{a}_0 a_l^\dagger a_k^\dagger|0\rangle &= c_0 + c_k + c_l, \quad k \neq l, \end{aligned} \quad (4.10)$$

$$\begin{aligned} \frac{1}{2} \langle 0|(a_k)^2 a_0 (a_k^\dagger)^2|0\rangle &= 0, \\ \frac{1}{2} \langle 0|(a_k)^2 \tilde{a}_0 (a_k^\dagger)^2|0\rangle &= c_0 + 2c_k. \end{aligned}$$

The important point is that this ansatz makes very different assumptions about the two particle sector. If we identify b_0 with c_0 and b_k with $c_0 + c_k$, it would be equivalent to a different choice for the last term in Eq. (4.3):

$$\begin{aligned} \sum_{k=1}^M \frac{\langle 0|a_m a_k a_0 a_k^\dagger a_m^\dagger|0\rangle}{k} &= \sum_{k=1}^M \frac{b_k - b_0}{k} + b_m \sum_{k=1}^M \frac{1}{k} \\ &\quad + \frac{2b_m - b_0}{m}. \end{aligned} \quad (4.11)$$

Using this assumption for the two particle sector gives us a different system of equations for the coefficients,

$$0 = b_0^3 + g b_0 + \sum_{k=1}^M \frac{b_k - b_0}{k}, \quad (4.12)$$

$$0 = b_m^3 + g b_m + \frac{6}{m} b_m + \sum_{k=1}^M \frac{b_k - b_0}{k}. \quad (4.13)$$

These are equivalent to Eqs. (3.6a) and (3.6b) in Heinzl *et al.* [8]. In this case, the zero mode matrix elements along with the critical coupling converge in the limit of large M .

How do we interpret this ansatz in terms of a Fock space truncation? We have assumed, in essence, that the zero mode has a contribution from the two particle sector that cannot be neglected. Is this the correct way to truncate the Fock space? We will answer this question in Sec. V.

We turn our attention to the Hamiltonian. For this Fock space truncation the Hamiltonian is diagonal. Substituting Eq. (4.1) into Eq. (2.10) and sandwiching between states, we obtain

$$\langle 0|H|0\rangle = \frac{1}{4} \sum_{k=1}^M \frac{b_k^2 - b_0^2}{k} - \frac{b_0^4}{4} - C, \quad (4.14)$$

$$\langle 0|a_m H a_m^\dagger|0\rangle = \frac{g}{m} - \frac{b_m^4}{4} - \frac{b_m^2}{2m} - b_m^2 \sum_{k=1}^M \frac{1}{k} + \frac{b_0^2}{4m} - C. \quad (4.15)$$

We choose C such that the VEV of H is zero and plot a spectrum in Fig. 7. Note the vertical line representing the value of g_0 . Presumably, to the left of this line, the Fock space truncation breaks down. Also, note that states with larger energy have smaller longitudinal momentum P^+ . Clearly, the zero mode has a large effect on the spectrum.

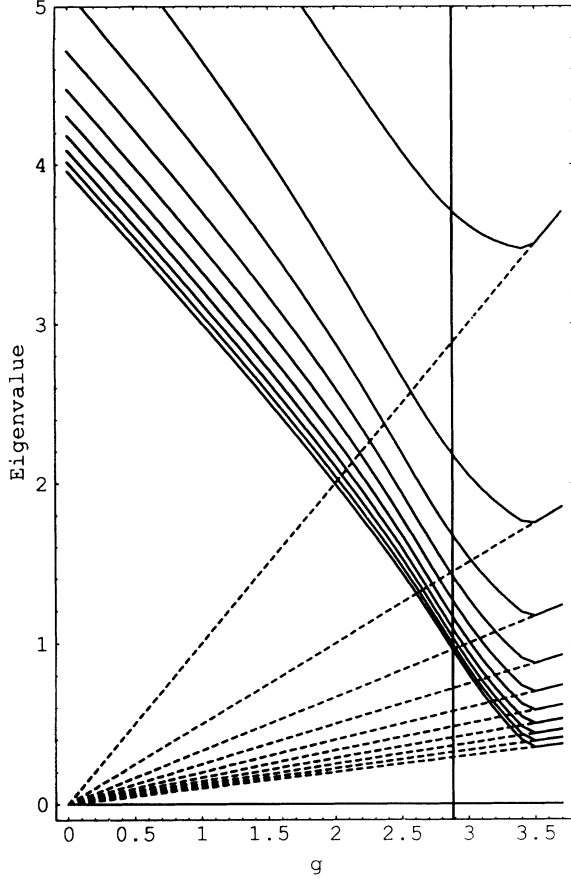


FIG. 7. Eigenvalues as a function of g in the one particle case using Eq. (4.15) with $M = 10$. The solid lines are for the broken phase and the dashed lines are for the unbroken phase. The vertical line represents g_0 .

V. MANY MODES AND MANY PARTICLES

We turn our attention to the more general case of many modes and many particles. Many of the features that were seen in the one-mode and one particle cases remain. Plotting the VEV vs g , one finds that the critical curves have the same form with an unbroken phase, a broken phase, and a critical coupling. There is rapid convergence as one increases N and a logarithmic divergence as one increases M . However, now one can have more than one state of a given momentum. Thus, H and a_0 are no longer strictly diagonal and Σ_3 is nonzero.

Consequently, the zero mode for the unbroken phase is no longer zero. We can use Eq. (2.9) and solve for the unbroken phase of a_0 perturbatively in λ or, equivalently, in $1/g$:

$$a_0 = -\frac{6}{g}\Sigma_3 + \frac{6}{g^2}\left(2\Sigma_2\Sigma_3 + 2\Sigma_3\Sigma_2 + \sum_{k=1}^M \frac{a_k\Sigma_3 a_k^\dagger + a_k^\dagger\Sigma_3 a_k - \Sigma_3}{k}\right) + O(1/g^3). \quad (5.1)$$

For each order in $1/g$, one can see that a_0 is odd under the transform $a_k \rightarrow -a_k$, $k \neq 0$. Consequently,

$$\langle \alpha | a_0 | \beta \rangle = 0 \text{ if } \langle \alpha | \hat{N} | \alpha \rangle - \langle \beta | \hat{N} | \beta \rangle \text{ is even,} \quad (5.2)$$

where $\hat{N}$ is the number operator and $|\alpha\rangle$ and $|\beta\rangle$ are Fock space basis states. In particular, the diagonal matrix elements of a_0 are zero. It is important to note that, as expected, this expansion does not produce the broken phase [10]. When substituted into the Hamiltonian, the perturbative expansion of the zero mode produces new interactions in perturbation theory. It is generally believed that equal-time and light-front perturbation theories are equivalent [15] and that the zero modes will change the internal propagator to be the full propagator that one would have if there were a dynamical zero mode. Maeno has, in fact, shown that this is true for ϕ^3 theory in 1+1-dimensions [16].

In order to calculate the zero mode for a given value of g one converts the constraint equation (2.9) into an $S \times S$ matrix equation in the truncated Fock space. This becomes a set of S^2 coupled cubic equations and one can solve for the matrix elements of a_0 numerically [21]. Considerable simplification occurs because a_0 is symmetric and is block diagonal in states of equal momentum. For example, in the case $M = 3$, $N = 3$, the number of coupled equations is 34 instead of $S^2 = 400$. In order to find the critical coupling, we take $\langle 0 | a_0 | 0 \rangle$ as given and g as unknown and solve the constraint equation for g and the other matrix elements of a_0 in the limit of small but nonzero $\langle 0 | a_0 | 0 \rangle$. Critical coupling as a function of M and N is shown in Fig. 8. We see that the solution seems to be consistent with our earlier results: there is quick convergence as N increases and a logarithmic divergence as M increases.

Thus, we believe that the ansatz introduced in Eq. (4.9) produces misleading results. If one were to extend the ansatz to include more particles, the convergent result for the critical coupling would disappear.

As an example, let us examine some matrix elements of a_0 for the truncation $M = 3$ and $N = 3$. In this case, $S = 20$ and we restrict our attention to states with $P^+ \leq 3(2\pi)/d$ (numerical calculations are performed with all 20 states). Arrange the basis states in the following order:

$$\begin{aligned} P^+ = 0 : & \quad |0, 0, 0\rangle; \\ P^+ = 1 \frac{2\pi}{d} : & \quad |1, 0, 0\rangle; \\ P^+ = 2 \frac{2\pi}{d} : & \quad |0, 1, 0\rangle; \\ & \quad |2, 0, 0\rangle; \\ P^+ = 3 \frac{2\pi}{d} : & \quad |0, 0, 1\rangle; \\ & \quad |1, 1, 0\rangle; \\ & \quad |3, 0, 0\rangle; \\ & \quad \vdots \end{aligned} \quad (5.3)$$

Given $g = 2.8$ above the critical coupling $g_{\text{critical}} = 2.301$, we solve the constraint equation,

$$a_0 = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & -0.287 & & & \\ & & -0.287 & 0 & & & \\ & & & & 0 & -0.281 & 0 \\ & & & & -0.281 & 0 & -0.458 \\ & & & & 0 & -0.458 & 0 \\ & & & & & & \ddots \end{pmatrix}, \quad (5.4)$$

which leads to a Z_2 symmetric Hamiltonian. Now, we lower the coupling to $g = 1.8$. The solution for the unbroken phase is now

$$a_0 = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & -0.318 & & & \\ & & -0.318 & 0 & & & \\ & & & & 0 & -0.315 & 0 \\ & & & & -0.315 & 0 & -0.499 \\ & & & & 0 & -0.499 & 0 \\ & & & & & & \ddots \end{pmatrix}. \quad (5.5)$$

In addition, we have two solutions for the broken phase. There is a solution with $\langle 0|a_0|0 \rangle > 0$,

$$a_0 = \begin{pmatrix} 0.787 & & & & & & \\ & -0.246 & & & & & \\ & & -0.251 & -0.314 & & & \\ & & -0.314 & 0.072 & & & \\ & & & & -0.270 & -0.310 & 0.005 \\ & & & & -0.310 & 0.070 & -0.499 \\ & & & & 0.005 & -0.499 & -0.019 \\ & & & & & & \ddots \end{pmatrix}, \quad (5.6)$$

along with the $\langle 0|a_0|0 \rangle < 0$ solution,

$$a_0 = \begin{pmatrix} -0.787 & & & & & & \\ & 0.246 & & & & & \\ & & 0.251 & -0.314 & & & \\ & & -0.314 & -0.072 & & & \\ & & & & 0.270 & -0.310 & -0.005 \\ & & & & -0.310 & -0.070 & -0.499 \\ & & & & -0.005 & -0.499 & 0.019 \\ & & & & & & \ddots \end{pmatrix}. \quad (5.7)$$

When we substitute the solutions for the broken phase of a_0 into the Hamiltonian (2.10) we get two Hamiltonians H^+ and H^- corresponding to the two signs of $\langle 0|a_0|0 \rangle$ and the two branches of the curve in Fig. 1. This is the new paradigm for spontaneous symmetry breaking: multiple vacua are replaced by multiple Hamiltonians. Picking the Hamiltonian defines the theory in the same sense that picking the vacuum defines the theory in the equal-time paradigm. The two solutions for a_0 are related to each other in a very specific way. Let Π be the unitary operator associated with the Z_2 symmetry of the system; $\Pi a_k \Pi^\dagger = -a_k$, $k \neq 0$. We break up a_0 into an even part $\Pi a_0^E \Pi^\dagger = a_0^E$ and an odd part $\Pi a_0^O \Pi^\dagger = -a_0^O$. The even part a_0^E breaks the Z_2 symmetry of the theory. For $g < g_{\text{critical}}$, the three solutions of the constraint equation are the unbroken phase solution, $a_0^O + a_0^E$ cor-

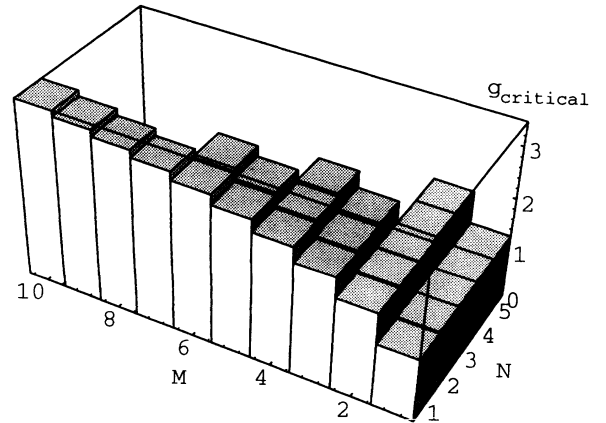


FIG. 8. Critical coupling vs M and N .

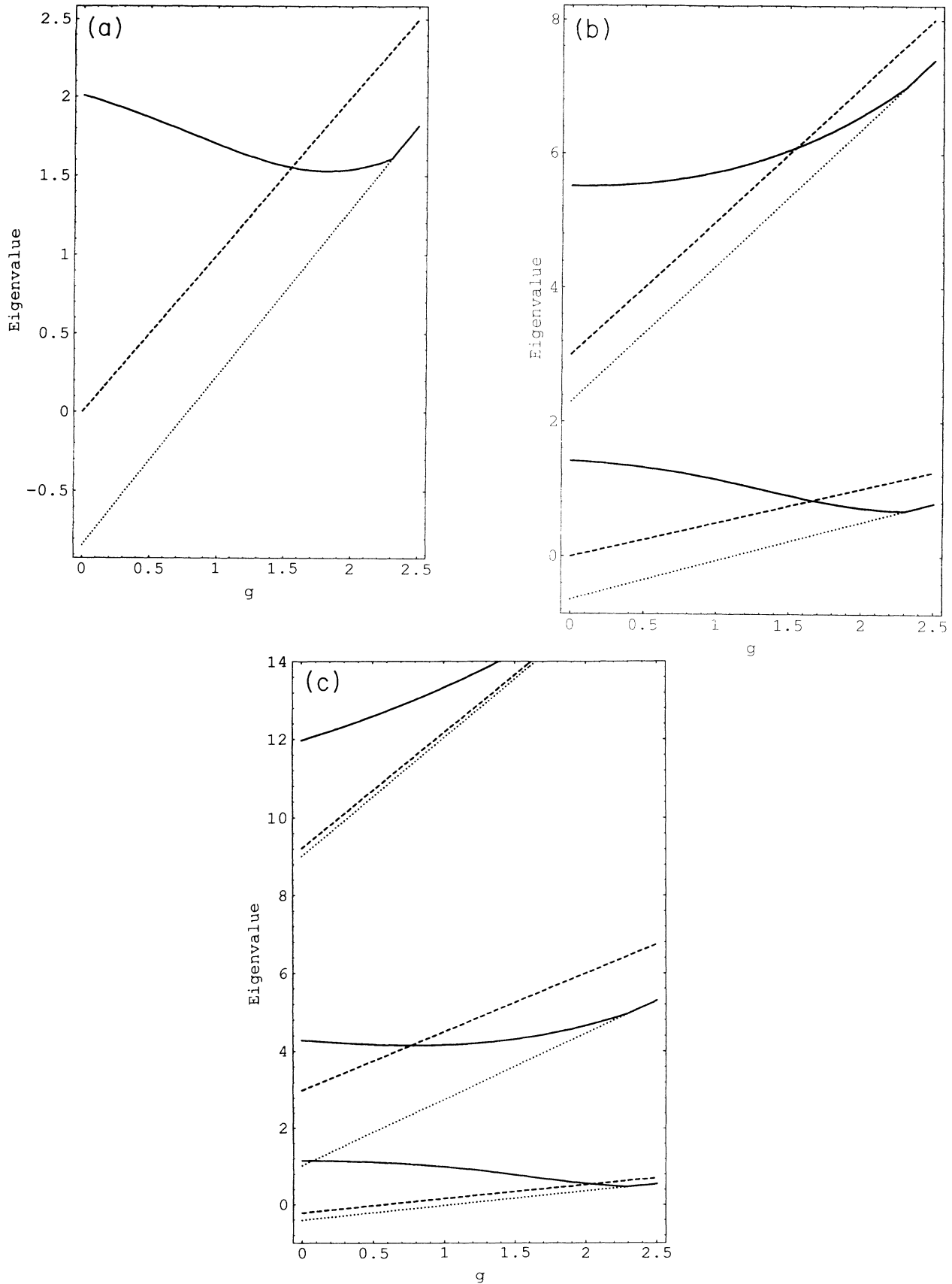


FIG. 9. The spectrum for $M = 3$, $N = 3$ and (a) $P^+ = 1(2\pi)/d$, (b) $P^+ = 2(2\pi)/d$, and (c) $P^+ = 3(2\pi)/d$. The dashed line shows the spectrum with no zero mode. The dotted line is the unbroken phase and the solid line is the broken phase.

responding to the $\langle 0|a_0|0\rangle > 0$ solution, and $a_0^O - a_0^E$ for the $\langle 0|a_0|0\rangle < 0$ solution. Thus, the two Hamiltonians are

$$H^+ = H(a_k, a_0^O + a_0^E) \quad (5.8)$$

and

$$H^- = H(a_k, a_0^O - a_0^E) \quad (5.9)$$

where H has the property

$$H(a_k, a_0) = H(-a_k, -a_0) \quad (5.10)$$

and a_k represents the nonzero modes. Since Π is a unitary operator, if $|\Psi\rangle$ is an eigenvector of H with eigenvalue E then $\Pi|\Psi\rangle$ is an eigenvector of $\Pi H \Pi^\dagger$ with eigenvalue E . Since

$$\begin{aligned} \Pi H^- \Pi^\dagger &= \Pi H(a_k, a_0^O - a_0^E) \Pi^\dagger = H(-a_k, -a_0^O - a_0^E) \\ &= H(a_k, a_0^O + a_0^E) = H^+, \end{aligned} \quad (5.11)$$

H^+ and H^- have the same eigenvalues.

Using the $M = 3, N = 3$ case as an example, let us examine the spectrum of H . Removing the zero mode entirely, the rescaled Hamiltonian, Eq. (2.10), is

$$H = \begin{pmatrix} 0 & & & & & \\ & g & & & & \\ & & \frac{g}{2} & 0 & & \\ & & 0 & 3+2g & & \\ & & & & \frac{g}{3} & 0 & \sqrt{2} \\ & & & & 0 & 3+\frac{3g}{2} & 0 \\ & & & & \sqrt{2} & 0 & 9+3g \\ & & & & & & \ddots \end{pmatrix}. \quad (5.12)$$

For large g the eigenvalues are obviously: $0, g, g/2, 2g, g/3, 3g/2$, and $3g$. However, as we decrease g , one of the last three eigenvalues will be driven negative. This signals the breakdown of the theory near the critical coupling when the zero mode is not included.

Including the zero mode fixes this problem. Figure 9 shows the spectrum for the three lowest nonzero momentum sectors. This spectrum illustrates several characteristics which seem to hold generally (at least for truncations we have examined, $N + M \leq 6$). For the broken phase, the vacuum is the lowest energy state, there are no level crossings as a function of g , and the theory does not break down in the vicinity of the critical point. None of these are true for the spectrum with the zero mode removed or for the unbroken phase below the critical coupling. The lowest eigenvalue has a minimum at or near the critical point and the minimum appears to decrease with the number of modes and particles. A more precise

statement will have to await a fully renormalized treatment.

VI. SPECTRUM OF THE FIELD OPERATOR

So far we have examined the zero mode and its effect on the spectrum of the Hamiltonian. How does the zero mode affect the field itself? Since ϕ is a Hermitian operator it is an observable of the system and one can measure ϕ for a given state $|\alpha\rangle$. The result is given by simple quantum mechanics. Let us define the eigenvalues $\tilde{\phi}_i$ and eigenvectors $|\chi_i\rangle$ of $\sqrt{4\pi}\phi$:

$$\sqrt{4\pi}\phi|\chi_i\rangle = \tilde{\phi}_i|\chi_i\rangle, \quad \langle\chi_i|\chi_j\rangle = \delta_{i,j}. \quad (6.1)$$

The probability of obtaining $\tilde{\phi}_i$ as the result of a measurement of $\sqrt{4\pi}\phi$ for the state $|\alpha\rangle$ is $|\langle\chi_i|\alpha\rangle|^2$.

In the limit of large N , the probability distribution becomes continuous. If we ignore the zero mode, the probability of obtaining $\tilde{\phi}$ as the result of a measurement of $\sqrt{4\pi}\phi$ for the vacuum state is

$$P(\tilde{\phi}) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{\tilde{\phi}^2}{2\tau}\right) d\tilde{\phi}, \quad (6.2)$$

where $\tau = \sum_{k=1}^M 1/k$. The probability distribution comes from the ground state wave function of the harmonic oscillator where we identify ϕ with the position operator. This is just the Gaussian fluctuation of a free field. Note that the width of the Gaussian diverges logarithmically in M . When N is finite, the distribution becomes discrete as shown in Fig. 10.

In general, there are $N + 1$ eigenvalues such that $\langle\chi_i|0\rangle \neq 0$, independent of M . Thus if we want to examine the spectrum of the field operator for the vacuum state, it is better to choose Fock space truncations where N is large. With this in mind, we examine the $N = 50$ and $M = 1$ case as a function of g in Fig. 11. Note that near the critical point, Fig. 11(a), the distribution is approximately equal to the free field case shown in Fig. 10. There is no symmetry breaking and the field is symmetric about zero. As we move away from the critical point, Figs. 11(b)–11(d), the distribution becomes increasingly

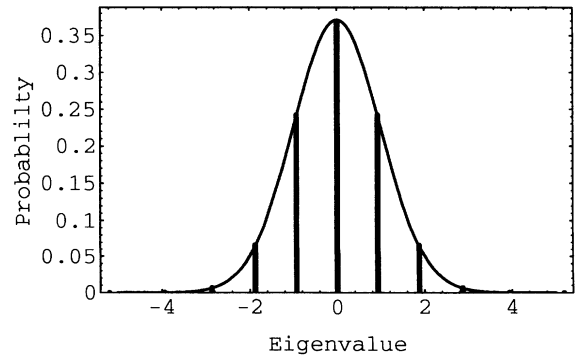


FIG. 10. Probability distribution of eigenvalues of $\sqrt{4\pi}\phi$ for the vacuum with $M = 1, N = 10$, and no zero mode. Also shown is the infinite N limit from Eq. (6.2).

narrow with a peak located at the VEV of what would be the minimum of the symmetric double well potential in the equal-time paradigm. In addition, there is a small peak corresponding to minus the VEV. In the language

of the equal-time paradigm, there is tunneling between the two minima of the potential.

VII. DISCUSSION

In the context of $(\phi^4)_{1+1}$ on the light front, the vacuum of the full theory is always the perturbative vacuum. The zero mode, which satisfies an operator-valued constraint equation, produces the long range physics of the theory including spontaneous breaking of the Z_2 symmetry. We have found that the constraint equation can be solved using a Tamm-Dancoff truncation of the Fock space. Even for the one-mode truncation, we find a critical coupling consistent with the best equal-time calculations. Increasing the Tamm-Dancoff truncation, we find rapid convergence with the total number of particles N . This is to be contrasted with the equal-time approach where an infinite number of particles are required to produce a critical point. In the weak coupling limit, the zero mode ensures that light-front perturbation theory agrees with equal-time perturbation theory. Above the critical coupling the zero mode develops a contribution that breaks the Z_2 symmetry of the theory. There are two such solutions to the constraint equation: one with $\langle 0|\phi|0\rangle > 0$ and one with $\langle 0|\phi|0\rangle < 0$. This, in turn, gives rise to two Hamiltonians which have the same spectrum. The spectrum has the expected behavior: the Fock vacuum is the state of lowest energy and the lowest eigenvalue has a minimum at the critical coupling. Closer inspection of the field shows that tunneling occurs between positive and negative eigenvalues in the broken phase.

In this work, we apply a simple mass renormalization to the symmetric ordered Hamiltonian. This prescription removes tadpoles from ordinary interaction terms and would properly renormalize the theory if the zero mode were removed. However, inclusion of the zero mode produces logarithmic divergences in the constraint equation and in the resulting Hamiltonian. This can be seen from Figs. 6 and 8 where g_{critical} grows with the number of modes. The number of modes is equivalent to a large P^+ cutoff in a standard discretized light-front quantization calculation. We therefore believe that this behavior is a renormalization effect related to the fact that the new interactions are not normal ordered. To a very good accuracy,

$$g_{\text{critical}} \approx (3 - \sqrt{3}) \sum_{m=1}^M \frac{1}{m}. \quad (7.1)$$

This type of logarithmic growth indicates that the system needs further renormalization. The most direct approach would be to find some nonperturbative method of normal ordering the resulting Hamiltonian. Another approach is to add a constant to g and multiply λ by a constant so that logarithmic divergences are removed. In the one particle case, this will work. However, it is unclear if we can successfully extend it to the more general case $N > 1$. Another possibility is to add more operators to the Hamiltonian. For instance, if we add a term of the

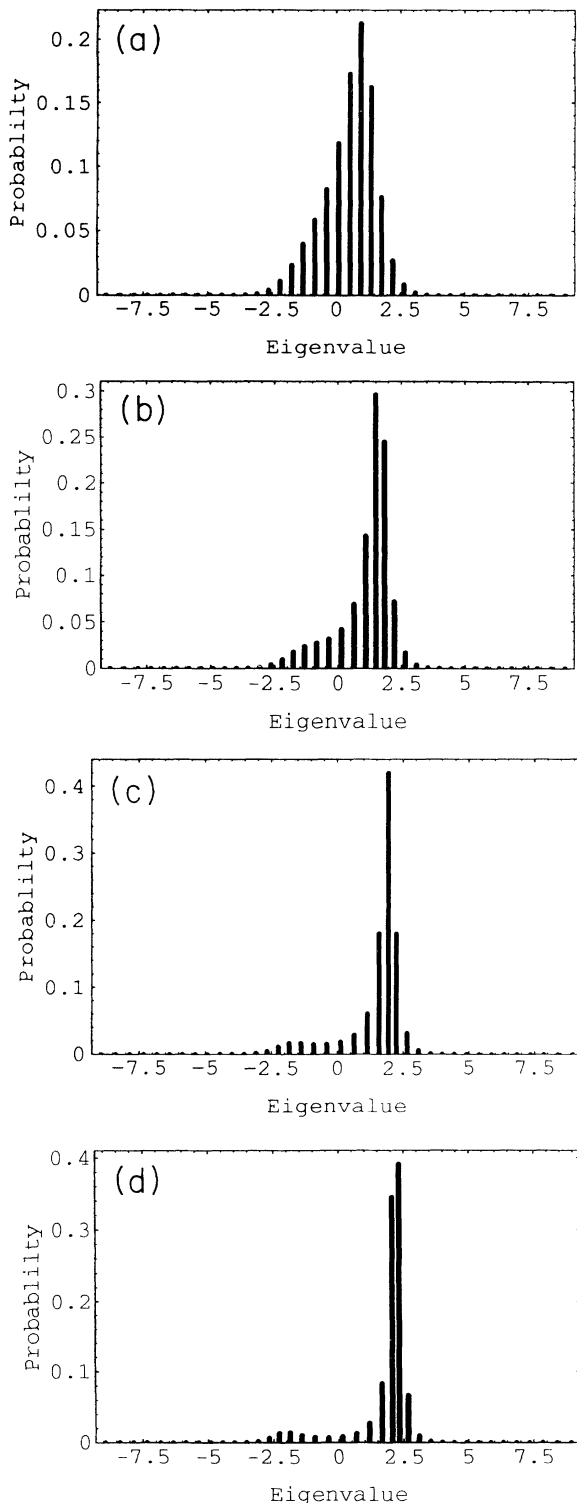


FIG. 11. Probability distribution of eigenvalues of $\sqrt{4\pi}\phi$ for the vacuum with couplings (a) $g = 1$, (b) $g = 0$, (c) $g = -1$, and (d) $g = -2$. $M = 1$, $N = 50$, and the positive VEV solution to the constraint equation is used.

form $\int dx^- dy^- \phi(x)\phi(y)$, we can remove the logarithmic divergence in the constraint equation.

For $(\phi^4)_{1+1}$ we believe that a vanishing of the mass gap is associated with the critical coupling [9]. In Fig. 7 we see that the gap between the vacuum and the lowest energy excited state is minimized at the critical coupling. One could imagine that, in the limit of large M , the gap between the vacuum and the first excited state goes to zero at the critical coupling. However, the first excited state is the one with the largest longitudinal momentum and is always at the “edge” of our Fock space truncation in M . Thus, our Fock space truncation may not be particularly well suited for an investigation of the vanishing of the mass gap.

We chose $(\phi^4)_{1+1}$ as a model for its simplicity. One problem with this choice is that the scalar field ϕ is dimensionless. Power counting arguments suggest that arbitrary powers of ϕ are allowed in the Hamiltonian. Consequently, the zero mode can be a very complicated object for this model. In contrast, for ϕ^4 theory in 3+1 dimensions the only allowed local operators are ϕ^n , $n \leq 4$. For physically interesting theories like QCD, the opera-

tors that are allowed by power counting arguments are much more restricted [22]. However, they still allow a very large set of operators and are therefore in some sense similar to the $(\phi^4)_{1+1}$ example. Solution of the zero mode problem for the model we chose may be conceptually more difficult.

In theories like QCD we expect to have, in addition, zero modes that are dynamical degrees of freedom. A recent study of pure glue QCD in 1+1 dimensions shows that the zero mode of A^+ has this property [13, 23]. Thus, in QCD, we expect to have dynamical as well as nondynamical zero modes making the problem quite complicated.

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- [1] P. A. M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949).
 - [2] S. J. Brodsky, G. McCartor, H. C. Pauli, and S. S. Pinsky, *Particle World* **3**, 109 (1993); S. J. Brodsky and H. C. Pauli, in *Proceedings of the 30th Schlading Winter School*, 1991 (to be published).
 - [3] I. Tamm, *J. Phys. (USSR)* **9**, 449 (1945); S. M. Dancoff, *Phys. Rev.* **78**, 382 (1950).
 - [4] T. Maskawa and K. Yamawaki, *Prog. Theor. Phys.* **56**, 270 (1976).
 - [5] G. McCartor, *Z. Phys. C* **41**, 271 (1988); **52**, 611 (1991); G. McCartor and D. G. Robertson, *ibid.* **53**, 679 (1992).
 - [6] F. Lenz, in *Proceedings of the NATO Advanced Summer Institute on Nonperturbative Quantum Field Theory*, Cargese, France, 1989, edited by D. Vautherin, F. Lenz, and J. W. Negele (Plenum, New York, 1990); F. Lenz, M. Thies, S. Levit, and K. Yazaki, *Ann. Phys. (N.Y.)* **208**, 1 (1991).
 - [7] K. Hornbostel, *Phys. Rev. D* **45**, 3781 (1992).
 - [8] T. Heinzl, S. Krusche, S. Simburger, and E. Werner, *Z. Phys. C* **56**, 415 (1992); *Phys. Lett. B* **272**, 54 (1991); **275**, 410 (1992); *Nucl. Phys. A* **532**, 4290 (1991).
 - [9] A. Harindranath and J. P. Vary, *Phys. Rev. D* **37**, 1076 (1988); **36**, 1141 (1987).
 - [10] D. Robertson, *Phys. Rev. D* **47**, 2549 (1993).
 - [11] M. Burkardt, *Phys. Rev. D* **47**, 4628 (1993).
 - [12] S. Huang and N. Lin, “On the Zero Mode Problem of the Light-Cone Quantization,” University of Washington Report, 1992 (unpublished).
 - [13] H. C. Pauli, Heidelberg Report No. MPIH-V21-1991 (unpublished); A. C. Kalloniatis and H. C. Pauli, Heidelberg Report No. MPIH-V6-93 (unpublished).
 - [14] C. M. Bender, S. S. Pinsky, and B. van de Sande, *Phys. Rev. D* **48**, 816 (1993).
 - [15] S. J. Chang, R. G. Root, and T. M. Yan, *Phys. Rev. D* **7**, 1133 (1973); S. J. Chang and T. M. Yan, *ibid.* **7**, 1147 (1973).
 - [16] M. Maeno, Osaka U. Report No. OU-HET-175 (unpublished).
 - [17] R. S. Wittman, in *Nuclear and Particle Physics on the Light Cone*, Proceedings of the Workshop, Los Alamos, New Mexico, 1988, edited by M. B. Johnson and L. S. Kisslinger (World Scientific, Singapore, 1989).
 - [18] C. M. Bender, L. R. Mead, and S. S. Pinsky, *Phys. Rev. Lett.* **56**, 2445 (1986).
 - [19] S. J. Chang, *Phys. Rev. D* **13**, 2778 (1976); J. Abad, J. G. Esteve, and A. F. Pacheco, *ibid.* **32**, 2729 (1985); M. Funke, V. Kaulfuss, and H. Kummel, *ibid.* **D 35**, 621 (1987); H. Kroger, R. Girard, and G. Dufour, *ibid.* **35**, 3944 (1987).
 - [20] C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., *Phys. Rev. Lett.* **58**, 2615 (1987); C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., *Phys. Rev. D* **37**, 1472 (1988).
 - [21] A MATHEMATICA package written by one of the authors was used to perform many of the calculations discussed in this paper. Persons interested in obtaining a copy of this package should contact the authors via E-mail at vandesande@ohstpy.mps.ohio-state.edu.
 - [22] S. Glazek, A. Harindranath, S. S. Pinsky, J. Shigemitsu, and K. Wilson, *Phys. Rev. D* **47**, 1599 (1993); D. Mustaki, S. S. Pinsky, J. Shigemitsu, and K. Wilson, *ibid.* **43**, 3411 (1991).
 - [23] A. C. Kalloniatis, H. C. Pauli, and S. S. Pinsky (unpublished).

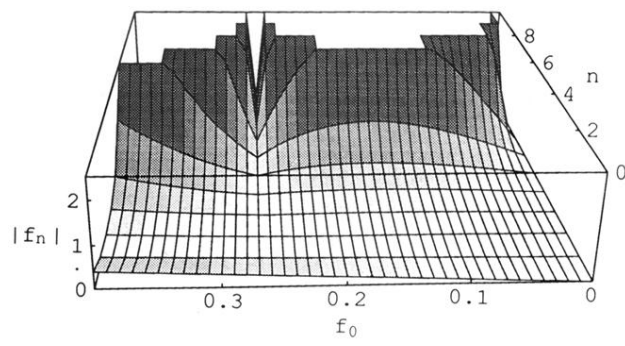


FIG. 2. $|f_n|$ vs n and f_0 for $g = 1.2$ using Eq. (3.3).

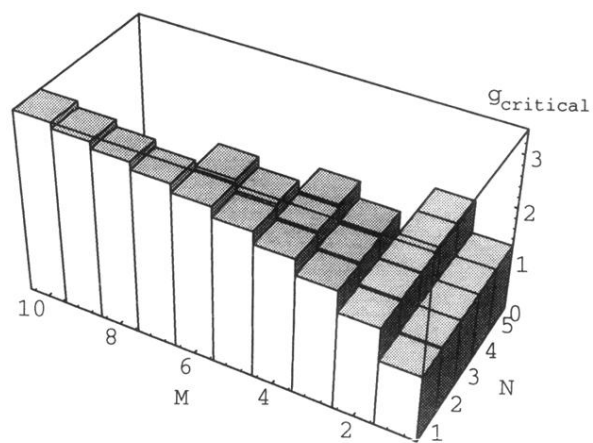


FIG. 8. Critical coupling vs M and N .