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Wormholes in string theory

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We discuss the wormhole effective interactions in string theory, thought of as a sum over two-dimensional field theories on different world sheets. The effective interactions are calculated in the "dilute wormhole approximation," initially by considering the Green's functions on higher-genus Riemann surfaces, and then by calculating the effect of a complete basis of wave functions on scattering amplitudes for a surface with a boundary. The sum over wormholes is equivalent to having a world sheet of trivial topology and summing over different space-time and matter-field backgrounds. To leading order these consist of the massless fluctuations, since the tachyon cancels out when a sum is done over different spin structures going through the wormhole. In this way we recover quantized general relativity as an effective theory, from a sum over field theories on higher-genus Riemann surfaces.

I. INTRODUCTION

It has been widely suggested that manifolds with nontrivial topology might play a role in quantum gravity [1–6]. Most work on this has concentrated on wormholes, which are like handles connecting two large flat regions of space-time. They have been shown to have dramatic effects on the apparent coupling constants of low-energy physics. The ability of wormholes to shift coupling constants has led to a mechanism by which the cosmological constant can naturally be forced to zero [4]. However, such calculations are currenlty plagued with problems. The main problem is that the work done so far is based on a path integral for Euclidean quantum gravity which does not converge, even naively, because the Euclidean action is unbounded from below. This sickness (and others) has led many worker to doubt the validity of the "wormhole calculus." The purpose of this article is to show that one can calculate the effective interactions associated with wormholes in a much simpler theory, also based on a Euclidean path integral over geometries, but which is much better defined than the path integral for Euclidean quantum gravity. This is within the Polyakov approach to string theory [7].

One can regard string-theory wormholes as providing a toy model for wormholes in full quantum gravity. They might reveal new ways of investigating some of the problems with the arguments in the four-dimensional case [8]. However, we shall pursue a different motivation here. We propose that the problems with Euclidean quantum gravity, such as the unboundedness of the action, are so bad that it is likely that Euclidean quantum gravity is just an effective theory which is a low-energy approximation to something more fundamental, such as superstrings. In this case it is important to see how Euclidean quantum gravity emerges, in a low-energy limit, from superstrings. Wormholes in two dimensions can provide an explanation, the first to our knowledge, of how this comes about [9]. The metric and matter fields in four-dimensional space-time would emerge as a set of Coleman α parameters for the two-dimensional wormholes. It is this idea which we shall have in mind in the following development.

Wormhole effects should be present in any quantum theory defined on manifolds of nontrivial topology [9,10]. By a wormhole effect we mean the replacement of handles by bilocal effective interactions, on a manifold with the handles removed. In [10] the wormhole effective interactions are found for bosonic field theory on a torus. The calculations can be done explicitly as the Green's functions are known. However, the connection between effective field-theory interactions and string-theory states was not made explicit. It was not shown, for instance, how the mass-shell condition appears. Also, higher-genus surfaces were not included. This is needed, since string theory is given (in perturbation theory) by a sum of field theories on different Riemann surfaces [7]. It is the sum over wormholes which enables one to exponentiate the bilocal interaction, and turn it into an effective addition to the action in flat space.

In this paper we shall investigate the wormholes which appear in string theory in more detail. In particular, we shall obtain the effective interactions using two approaches.

The first uses the Green's functions, and is an extension of the approach of [10]. We generalize the results obtained there, first to arbitrary n-point functions (Sec. II), and then to arbitrary-genus Riemann surfaces in the Schottky parametrization (Sec. III). We shall see that the mass-shell conditions for the string states appear as poles when we integrate over the moduli parameters of the Riemann surface. We shall also compare the effective interactions for a surface of genus g with the gth power of the bilocal interactions associated with a single wormhole. In this way, the exponentiating of the bilocal interaction can be checked explicitly.

The second approach is closer to that in [1,11,12], except that it has the advantage that it can be done exactly, without resorting to the semiclassical approximation. We start by cutting the Riemann surface across each of the wormholes. This produces a surface with each wormhole replaced by two boundaries. The correlation functions on the original surface can be obtained from those on the cut surface by inserting δ functions (which ensure that the fields take the same values on each boundary) and then integrating over the boundary values [9]. The δ functions can be expanded in a complete basis of wave functions on the boundaries. We shall take a basis which consists of eigenfunctions of the world-sheet Hamiltonian, $L_0+\tilde{L}_0-2$. This is the world-sheet analogue of the Wheeler-DeWitt operator. These basis wave functions can be interpreted as corresponding to wormholes containing different numbers of "particles" in each mode, just as in [1,11,12]. The effect of the wormhole basis states is then studied, in the dilute wormhole approximation, by calculating correlation functions in the region exterior to a disk [13,14]. We obtain in this way an effective interaction for each wormhole basis state. The mass-shell conditions for the wormhole states appear as poles when we integrate over the radius of the wormhole end. This "wave-function approach" to world-sheet wormholes is the subject of Sec. IV. An important point is that the wave functions for wormholes need not satisfy the Hamiltonian constraint; i.e. they need not be "on-shell." This contrasts with the approach of [1,15] in which only solutions of the Wheeler-DeWitt equation are allowed.

In Sec. V we use the wave-function approach to discuss the tachyon effective interactions in superstring theories. We shall use the Ramond-Neveu-Schwarz (RNS) [16,17] formulation of the superstring, in which there are world-sheet fermions present. As in Sec. IV, the wave functions are taken to be eigenfunctions of the world-sheet Hamiltonian, which now includes fermions as well as bosons. For the fermionic modes, one must sum over different spin structures going through the wormhole in order to have invariance under the "large diffeomorphism" on the world sheet (the diffemorphisms which are not connected to the identity). This leads to the tachyonic interactions being projected out [18]. The dominant effective interactions will then be associated with massless space-time particles, and gravitons. In Sec. VI we describe how quantum gravity could emerge as an effective theory, from a "wormhole resummation" [19]. The effective theory is a string theory (at the tree level) interacting with a background of free massless quantum fields, including gravitons. A treatment beyond the dilute wormhole approximation would give an effective interacting background. We cannot answer the question of whether the effective background geometries themselves have wormholes. To do this one would need to go beyond the dilute wormhole approximation. Finally, we summarize our conclusions in Sec. VII, and provide some results on prime forms, Green's functions, and period matrices (in the Schottky parametrization) in an appendix. These results are used in Sec. III.

II. THE SINGLE WORMHOLE

We start with an investigation of the effect of a single wormhole on the n-point functions of a free field theory in two dimensions. We shall first describe the underlying space. In general, oriented Riemann surfaces can be obtained by identifying part of the complex plane under a finitely generated subgroup G of SL(2,$\mathbb{C}$) called the Schottky group. In the single-handle case the Schottky group consists of powers of a single element T which can be written in the form

$$\frac{Tz-z_+}{Tz-z_-}=q\frac{z-z_+}{z-z_-}\ , \tag{2.1}$$

where $|q|<1$ is a complex parameter called the "multiplier," and $z_\pm$ are the two fixed points. Here, the parameters $z_\pm,q$ are related to the standard parameters a,b,c,d (where $ad-bc=1$) of the SL(2,$\mathbb{C}$) transformation

$$z'=\frac{az+b}{cz+d}, \tag{2.2}$$

via

$$\begin{aligned} &a=\Delta(z_+-qz_-),\quad b=\Delta z_-z_+(q-1)\ ,\\ &c=\Delta(1-q),\quad d=\Delta(qz_+-z_-)\ , \end{aligned} \tag{2.3}$$

where $\Delta^{-1}=\sqrt{q}\,(z_+-z_-)$.

Every element of G has the same two fixed points $z_\pm$ and the multiplier of $T^n\in G$ is given by q^n. The fundamental region F of the Schottky group is the region outside two isometric circles. The transformation T identifies one circle with the other (see Fig. 1). No two points in F are related by an element of G, but every point in the complex plane can be obtained from a point in F by an element of G. We shall be interested in the limit of small $|q|$ in which the "ends" of the wormhole are roughly the points $z_\pm$, while the "radius" of the wormhole is roughly $\sqrt{|q|}\,|z_+-z_-|$. The argument of q gives the amount of relative "twist" between the wormhole ends before they are "glued together." This construction can be straightforwardly generalized to the case where there is more than one handle, where there would be a generator T_R for each handle and the fundamental region would consist of the part of the complex plane outside an even number of circles, which are isometric in pairs.

It is well known that the different conformal structures on a torus are parametrized by a single complex parameter τ. Here, however, there seem to be three parameters $z_\pm$ and q. How is this apparent paradox resolved? The answer is that, in the extended Schottky parametrization,

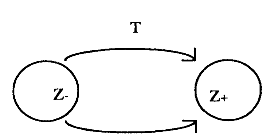

FIG. 1. The Schottky parametrization of the torus.

one still has the remaining freedom to do an overall SL(2,$\mathbb{C}$) transformation on the extended complex plane. Such a transformation does not change the conformal structure, but can be used to move any three points to arbitrary positions. There is therefore a redundancy in the description of moduli space by three complex numbers. In the calculation of string amplitudes the overall SL(2,$\mathbb{C}$) has to be gauge fixed in some way. The points $z_\pm$ can be moved to standard placed on the complex plane (e.g., 0 and 1). Then there is still an additional U(1)$\times$U(1) redundancy. This is usually fixed (in the case of calculating scattering amplitudes) by moving an external vertex operator to a standard place (e.g., the point at infinity). One is left with just one [SL(2,$\mathbb{C}$)-invariant] parameter q, which corresponds to τ, via $q=e^{2\pi i\tau}$. This resolves the apparent discrepancy in the dimension of moduli space. However, in the calculation of Green's functions we shall not do any gauge fixing of SL(2,$\mathbb{C}$). It is not necessary, except in the calculation of scattering amplitudes, to do this. Thus we shall describe the wormhole with all three parameters $z_\pm, q$.

Now we consider the case of small $|q|$. The Green's function for a free massless bosonic field on this space can be obtained to leading order in $|q|$ by solving the Laplace equation with a δ-function source. If the point at infinity is removed then the solution is completely unambiguous, up to the addition of a constant. It can be obtained by the method of images. To leading order, it is sufficient to put image charges at the points $z_\pm$. The Green's function takes the asymptotic form

$$G(z,w)\sim -\ln|z-w|^2 + \frac{1}{2\ln|q|}\ln\left|\frac{z-z_-}{z-z_+}\right|^2 \ln\left|\frac{w-z_-}{w-z_+}\right|^2 . \tag{2.4}$$

This expression is equal to

$$G_0(z,w)+G_1(z,w;z_-,z_+) , \tag{2.5}$$

where

$$G_1(z,w;z_-,z_+)=\frac{1}{2\ln|q|}[G_0(z,z_-)-G_0(z,z_+)] \times [G_0(w,z_-)-G_0(w,z_+)] \tag{2.6}$$

is a kind of interaction term, and $G_0(z,w)=-\ln|z-w|^2$ is the Green's function on the complex plane for the free bosonic theory. Using Wick's theorem. the last term is seen to be equivalent to the operator insertion

$$\frac{1}{4\ln|q|}[\phi(z_-)-\phi(z_+)]^2 , \tag{2.7}$$

on the complex plane, where ϕ is the bosonic field on the wormhole space. Here we are also assuming no self-contractions in the interaction term (i.e., normal order, or just take the creation operator part). The combinatorics of free field theory now tell us that the effect of the wormhole on arbitrary n-point functions can be obtained from the operator insertion

$$\exp\left[\frac{1}{4\ln|q|}[\phi(z_-)-\phi(z_+)]^2\right] \tag{2.8}$$

on the complex plane. Notice that it is *not* replaced by a bilocal interaction on the complex plane. One can of course expand the exponential function as a power series and recover an infinite sum of bilocal interactions but it is not clear that this is necessarily a useful thing to do. However, one can represent it in a useful way as the integral of a bilocal operator over a new variable k, by taking a Fourier transform.

Recalling the identity

$$\int dk\, e^{-ak^2/2}e^{ikx}=\left[\frac{2\pi}{a}\right]^{1/2} e^{-x^2/2a} , \tag{2.9}$$

and substituting $x=\phi(z_-)-\phi(z_+)$ and $a=-2\ln|q|$, we see that the operator insertion may be reexpressed as

$$\left[\frac{-\ln|q|}{\pi}\right]^{1/2}\int dk\,|q|^{k^2}e^{ik\phi(z_-)}e^{-ik\phi(z_+)} . \tag{2.10}$$

We now discuss the physical interpretation of such an expression. Firstly, the quantity k has a nice interpretation as the scalar charge carried by the wormhole. It is the conserved quantity corresponding to the translation symmetry $\phi(z)\to\phi(z)+C$ in field space. Now consider the situation in bosonic string theory with a flat Euclidean background space-time of the critical dimension. This is just the same, except there are now 26 free bosonic fields X^μ, corresponding to the coordinates of space-time, instead of just ϕ. The scalar charge k_μ now has an interpretation as the space-time momentum of the state which is going through the wormhole. The state is generated by $e^{ik\cdot X}$ acting on the Fock-space vacuum. In string theory this is a tachyon vertex operator, which is on shell if $k^2=2$. We shall now see how this condition appears in this formalism.

The wormhole parameters (being not observed) have to be integrated over. The measure in the q integration is known for arbitrary genus [20]. We need the integration measure in the region $|q|\to 0$. We are interested in the low-$|q|$ limit because this is what gives the poles on the mass shell. If the external energies are low, there will be significant contributions in the low-$|q|$ limit only from the massless wormhole states, if the tachyon states are projected out by summing over spin structures (as we discuss in Sec. V).

Consider the measure in the case of a single wormhole. This is

$$\frac{d^2q}{|q|^4}(-\ln|q|)^{-13} , \tag{2.11}$$

in the region of low $|q|$. But the operator insertion is

$$\left[\frac{-\ln|q|}{\pi}\right]^{13}\int d^{26}k\,|q|^{k^2}e^{ik\cdot X(z_-)}e^{-ik\cdot X(z_+)} . \tag{2.12}$$

We can interchange the order of integration, doing the integration over q first. Then the factors of $\ln|q|$ cancel and the integral over $|q|$ (near $|q|=0$) yields a pole

$$\frac{1}{k^2-2} \tag{2.13}$$

in the remaining integral over k. This remaining integral over k looks like a space-time one-loop Feynman integral where one has a propagator $1/(k^2-2)$ and a vertex $e^{ik\cdot X}$. The wormhole has been replaced by a propagator with a vertex at one end and the complex-conjugate vertex at the other end. The integral, although dominated by the on-shell tachyon at $k^2=2$, gets contributions from off-shell tachyons as well. Thus a wormhole is like a virtual loop in a Feynman diagram.

Now let us consider the effect, on the Green's function, of the terms which are higher order in $|q|$. These come from image charges which are not quite at the points $z_\pm$. Before, we approximated Tz and Tw as z_+, where z and w are the two points of the Green's function. Now we take the next term, where we approximate T^2z and T^2w as z_+. The next-order term in q in the Green's function can be seen to be

$$G_2(z,w;z_-,z_+)=-2\,\mathrm{Re}\left[q(z_+-z_-)^2\left[\frac{1}{(z-z_+)(w-z_-)}+(z\leftrightarrow w)\right]\right]$$
$$=-2\,\mathrm{Re}\{q(z_+-z_-)^2[\partial G_0(z,z_+)\partial G_0(w,z_-)+(z\leftrightarrow w)]\}\ . \tag{2.14}$$

This is in the approximation where w is close to one wormhole end and z is close to the other. In other words we are assuming that $|q|\ll|z-z_\pm|$, $|w-z_\mp|\ll|z_+-z_-|$, where either the upper signs or the lower signs may be taken. The effective interaction from G_2 is therefore given by

$$q(z_+-z_-)^2\partial X^\mu(z_+)\partial X^\mu(z_-)+\text{c.c.} \tag{2.15}$$

Once again, the combinatorics of free field theory tell us that the effect on arbitrary n-point functions is obtained by exponentiating the effect on two-point functions:

$$\exp[q(z_+-z_-)^2\partial X^\mu(z_+)\partial X^\mu(z_-)+\text{c.c.}]\ . \tag{2.16}$$

This time, however, we have no reason to believe more than the first term in a power-series expansion of the exponential. This is because we have already neglected $O(q^2)$ terms in the Green's function and so we can expect $O(q^2)$ corrections to this formula. We must remember, though, that eventually we shall be integrating over $\arg(q)$. Thus only terms which depend on $|q|$ will contribute to the wormhole. This is just like averaging over the orientation of the wormhole end. All the $O(q^2)$ corrections to this interaction will average to zero under the integration over $\arg(q)$. Thus we can believe the power-series expansion up to (and including) $O(q^2)$ terms, providing we assume that we are doing this averaging. This leads to an interaction term

$$|q|^2|z_+-z_-|^4\partial X^\mu(z_+)\bar\partial X^\nu(z_+)\partial X^\mu(z_-)\bar\partial X^\nu(z_-)\ . \tag{2.17}$$

Remembering that the "radius" of the wormhole is roughly $\sqrt{|q|}\,|z_+-z_-|$ it is pleasing that this gives an interaction of the correct dimension.

Finally, putting together the G_1 and G_2 terms leads to the interactions

$$\exp\left[\frac{1}{4\ln|q|}[X(z_-)-X(z_+)]^2\right][1+|q|^2|z_+-z_-|^4\partial X^\mu(z_+)\bar\partial X^\nu(z_+)\partial X^\mu(z_-)\bar\partial X^\nu(z_-)]\ . \tag{2.18}$$

For the mass-shell condition, the same discussion goes through for the second term in this expression as for the first. One can take a Fourier transform, and do the integration over $|q|$, at least in the small-$|q|$ region. The two extra powers of $|q|$ lead to a shift in the pole from $k^2=2$ to $k^2=0$. The remaining integral over k looks like a space-time one-loop Feynman integral with a propagator $1/k^2$ and vertex $e^{ik\cdot X}\partial X^\mu\bar\partial X^\nu$. Thus we have the interpretation of the second term in terms of virtual massless states. The integral over k gets contributions from off-shell massless states as well as on-shell ones. It should be noticed that, unlike in the case of the space-time wormholes discussed by Giddings and Strominger [5], there is no relation between the "radius" of the wormhole and the momentum (or scalar charge) flowing through. They are both just parameters, to be integrated over.

The series expansion in q contains higher-order terms which correspond to virtual states of positive m^2. However, for a low-energy treatment in which $|q|$ is small, it should be sufficient to stop at the massless states. We shall therefore ignore the states of positive m^2 in this discussion.

It is important to know the space-time tensorial structure of the states which are transmitted by wormholes. This tells us the form of the effective interactions in space-time. The interaction term $\partial X^\mu\bar\partial X^\nu$ can be expressed in a manifestly world-sheet coordinate-invariant way as

$$I^{\mu\nu}=S^{\mu\nu}+A^{\mu\nu}\ , \tag{2.19}$$

where

$$S^{\mu\nu}=\sqrt{g}\,g^{ab}\partial_aX^\mu\partial_bX^\nu\ ,$$
$$A^{\mu\nu}=\sqrt{g}\,\epsilon^{ab}\partial_aX^\mu\partial_bX^\nu\ . \tag{2.20}$$

Here g_{ab} is the metric on the world sheet, with determinant g, and ϵ_{ab} is the Levi-Civita tensor. This two-form exists and is globally well defined, because our world sheet is orientable. $A^{\mu\nu}$ is responsible for the exchange of the antisymmetric tensor states, while $S^{\mu\nu}$ leads to graviton and dilaton states in the wormhole. Now consider the space-time tensor (in D dimensions)

$$G_{\mu\nu\rho\sigma}=\delta_{\mu\nu}\delta_{\rho\sigma}-D\delta_{\mu(\rho}\delta_{\sigma)\nu} . \tag{2.21}$$

This is symmetric and traceless on $\mu\nu$ and $\rho\sigma$. The massless bilocal wormhole interaction has the form

$$I^{\mu\nu}I^{\mu\nu}=A^{\mu\nu}A^{\mu\nu}+\frac{1}{D}(S^2-G_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma}) , \tag{2.22}$$

where $S=S^{\mu\mu}$ and, as is customary in flat Euclidean space, we are sloppy about covariant and contravariant indices. The A^2 term leads to the antisymmetric tensor interaction. The first term in parentheses corresponds to the dilaton interaction. The last term, being symmetric and trace-free on pairs of indices, corresponds to a sum over all possible polarization of gravitons. Longitudinal polarizations are included in the states going through the wormhole. However, polarizations $\zeta^{\mu\nu}$ which do not satisfy the condition $k_\mu\zeta^{\mu\nu}=0$ will decouple from physical processes, ensuring space-time gauge invariance.

We began this section by studying the effect of a single wormhole on n-point functions in bosonic field theory in two dimensions. We considered the "dilute wormhole approximation" in which the parameter $|q|$ is small. By considering the integrals over q, and doing a Fourier transform in space-time, we related the resulting effective interactions to propagators for string states. We found that the states which dominate (apart from the tachyon) are the dilaton, the graviton and the antisymmetric tensor. The mass-shell conditions $L_0+\tilde{L}_0=2$ are not satisfied. However, there is a pole on the mass-shell in the remaining integration over k, the space-time momentum (world-sheet scalar charge) flowing through the wormhole. Thus a world-sheet wormhole is like a propagator of the low-energy field theory in space-time.

III. MULTIWORMHOLE CONFIGURATIONS

It is known that the perturbation series for string scattering amplitudes diverges [21]. This may signify that a new treatment of string theory is needed which includes surfaces of arbitrarily large genus. It is possible that such surfaces are important even at low energies. The view taken throughout this discussion is that, at low energies, the dominant contributions to physical processes come from spaces which look like a large simply connected region, with thin wormholes connecting widely separated points. This assumption means that we can (formally) sum up the effects of multiwormhole configurations by exponentiating the effect of a single wormhole. In this section we shall check whether the multiwormhole configuration gives the correct effective interaction for such an exponentiation to make sense. This will make it clear whether small $|q_R|$ really does correspond to the dilute wormhole approximation.

The Schottky parametrization easily generalizes from a torus to a Riemann surface of higher genus. For a surface of genus g we take the $3g$ complex parameters $z_{R\pm}$, q_R (where $|q_R|<1$) to describe the surface. The transformations T_R generate the Schottky group $G\subset \mathrm{SL}(2,\mathbb{C})$, where

$$\frac{T_Rz-z_{R+}}{T_Rz-z_{R-}}=q_R\frac{z-z_{R+}}{z-z_{R-}} . \tag{3.1}$$

The complex numbers $z_{R\pm},q_R$ parametrize moduli space up to a single SL(2,$\mathbb{C}$) transformation on the z's. Moduli space therefore has $3g-3$ complex dimensions. The representation of the surface is given by the extended complex plane, quotiented by the action of G. Details of the properties of Riemann surfaces required for this paper are in [20,22] and the Appendix. The Green's function on the surface (minus the point at infinity) specifies the effect of the handles on the n-point functions, just as before. The Green's function is unique up to the addition of a constant, once the point at infinity has been removed. In the limit where all $|q_R|\to 0$ it is obtained by putting image charges at each of the points $z_{R\pm}$. The explicit expression is

$$G(z,w)\sim -\ln|z-w|^2+\frac{1}{8\pi}\sum_{R,S}\left[\ln\left|\frac{z-z_{R+}}{z-z_{R-}}\right|^2\ln\left|\frac{w-z_{S-}}{s-z_{S+}}\right|^2+(z\leftrightarrow w)\right](\mathrm{Im}\Omega)^{-1}_{RS} . \tag{3.2}$$

We shall henceforth use the summation convention on indices R labeling the wormholes. Ω_{RS} is the period matrix of the surface. (See the Appendix for the definition of the period matrix.) The inverse of its imaginary part, $(\mathrm{Im}\Omega)^{-1}_{RS}$, appears as a kind of interaction between wormholes R and S. In the case of the single wormhole, done in Sec. II, this term is the wormhole self-interaction. Now there are, in addition, terms arising from two different wormholes.

To order 1 in the q expansion, the imaginary part of the period matrix is

$$(\mathrm{Im}\Omega)_{RS}\sim\begin{cases}\dfrac{-1}{2\pi}\ln\left|\dfrac{(z_{S+}-z_{R+})(z_{S-}-z_{R-})}{(z_{S+}-z_{R-})(z_{S-}-z_{R+})}\right| & \text{if } R\neq S ,\\[2ex] -\dfrac{1}{2\pi}\ln|q_R| \quad \text{if } R=S .\end{cases} \tag{3.3}$$

Although $|q_R|\to 0$, the off-diagonal terms have to be taken into account as well as the diagonal terms. We are anticipating that the matrix will appear in the exponent. It would then be incorrect, by a multiplicative order-1 factor, to approximate the period matrix as being diagonal.

The Green's function can be expressed in terms of the Green's function on the complex plane as

$$G(z,w)\sim G_0(z,w)+G_1(z,w;z_{R\pm},q_R)\ , \tag{3.4}$$

where

$$G_1(z,w;z_{R\pm},q_R)=\frac{1}{4\pi}[G_0(z,z_{R+})-G_0(z,z_{R-})]\times[G_0(w,z_{S-})-G_0(w,z_{S+})](\mathrm{Im}\Omega)^{-1}_{RS}\ . \tag{3.5}$$

The interaction which duplicates the term G_1 in the Green's function is, therefore,

$$-\frac{1}{8\pi}(\mathrm{Im}\Omega)^{-1}_{RS}[X(z_{R-})-X(z_{R+})]\cdot[X(z_{S-})-X(z_{S+})]\ , \tag{3.6}$$

where we have now replaced the single scalar field by the space-time coordinates X^μ. The effect of the wormholes on n-point functions is given by the exponential of this expression, using Wick's theorem, which relates n-point functions to Green's functions in free field theory.

Just as in Sec. II we can convert this exponential into an integral of local operators inserted at the points $Z_{R\pm}$. This is done by rewriting the interaction in the form

$$(2\det\mathrm{Im}\Omega)^{13}\int\left[\prod_R d^{26}k_R\right]e^{-2\pi k_R\cdot k_S\mathrm{Im}\Omega_{RS}}\times e^{ik_R\cdot[X(z_{R-})-X(z_{R+})]}\ . \tag{3.7}$$

Now consider the integration over $|q_R|$. The region of integration is only known explicitly, in the Schottky parametrization, for low genus. However, we are only interested in the behavior of the integral in the limit of small $|q_R|$ (the so-called "nondividing limit"). This region will give rise to the pole in the amplitude. Near $|q_R|\to 0$ the integration measure contains the factor

$$\prod_R\frac{d^2q_R}{|q_R|^4}(\det\mathrm{Im}\Omega)^{-13}\ . \tag{3.8}$$

This is exactly what is required to cancel out the factors of $\det\mathrm{Im}\Omega$. We note that the only q_R dependence in $e^{-2\pi k_R\cdot k_S\mathrm{Im}\Omega_{RS}}$ comes from the diagonal terms, i.e., $\prod_R|q_R|^{k_R^2}$, with the off-diagonal terms providing an order 1 multiplicative correction. These facts ensure that the integrals over $|q_R|$ lead to a term of the form

$$\prod_R\frac{1}{k_R^2-2}\ . \tag{3.9}$$

This term is a tachyon propagator, which occurs for each wormhole. The final integrations remaining are over k_R. The integrand has a similar form to a tachyon scattering amplitude at the tree level (i.e., on the sphere) where the external tachyons, with momenta $\pm k_R$, are put at the wormhole ends $z_{R\pm}$.

From the form of the interaction we can read off the vertex operator for a wormhole end carrying momentum k. It has space-time dependence $e^{ik\cdot X}$. The normalization factor is $f(k)/\sqrt{k^2-2}$ where $f(k)$ also depends on the positions and momenta of other wormhole ends inserted on the world sheet. The factor $f(k)$ leads to a spoiling of the dilute wormhole approximation, in that the vertex operator depends on the other wormhole ends. However $f(k)$ is analytic, and so the pole structure of the vertex operator is unchanged by it.

The analysis of this section can be repeated to next order in $|q_R|$. One obtains similar results; the tachyon operator is replaced by the graviton, dilaton and antisymmetric tensor vertex operators. In this case the factor $f(k)$ corresponds to the tree-level scattering amplitude of these massless states. In superstring theories, in which the Gliozzi-Scherk-Olive (GSO) projection has been performed, the tachyon is absent [23,24]. The leading-order effects should then come from wormholes carrying massless states. We shall check this explicitly in Sec. V.

IV. WORMHOLE STATES AND WORLD-SHEET PATH INTEGRALS

A. The wormhole basis states

As an alternative approach to calculating the effects of two-dimensional (2D) wormholes one can cut the wormholes around the circles which were originally identified by the Schottky group generators T_R (see Fig. 2). One can describe the wormhole states more easily in this way than by calculating Green's functions directly in a nonsimply connected space [9].

Consider the region of the complex plane exterior to $2g$ discs, where g is the genus of the original surface. The n-point functions on the original surface may be obtained by a functional integral over X on this space, with δ functions to ensure that X takes the same value on disks which were originally identified. Schematically, we have

$$\langle X^{\mu_1}(z_1)\cdots X^{\mu_n}(z_n)\rangle=\int\mathcal{D}X\prod_R\prod_z\delta(X(z)-X(T_Rz))X^{\mu_1}(z_1)\cdots X^{\mu_n}(z_n)e^{-I[X]}\ , \tag{4.1}$$

where $\prod_z$ is a continuous product running around the circle R_-, which encloses z_{R-}. The Schottky generator T_R maps the circle R_- to the circle R_+. They each have the same radius (this can always be arranged by shifting the fundamental region of the Schottky group). The radius is

$$R_R=\frac{\sqrt{|q_R|}}{|1-q_R|}|z_{R-}-z_{R+}|\sim\sqrt{|q_R|}\,|z_{R-}-z_{R+}|\ , \tag{4.2}$$

in the limit as $|q_R|\to 0$.

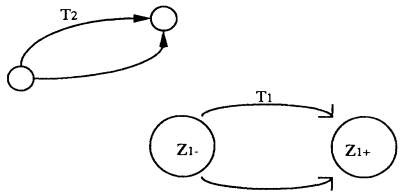

FIG. 2. An arbitrary-genus surface is cut around each of the wormhole ends.

In the small-$|q_R|$ limit we can take $Z_{R\pm}$ to lie at the centers of the circles. The δ function on the circles can then be expanded in a complete set of modes as follows. First the field X on the circles $R_\pm$ is expanded as

$$X(z)=\sum_n X_n^{\pm} e^{in\theta}\,, \tag{4.3}$$

where $\theta=\arg(z-z_{R\pm})/(z_{R\mp}-z_{R\pm})$. Then the δ function becomes a product of δ functions in each of the X_n:

$$\prod_z \delta(X(z)-X(T_R z))=\prod_n \delta(X_n^- - e^{in\phi}X_{-n}^+)\,, \tag{4.4}$$

where $\phi_R=\arg(q_R)$ is the twist angle between the wormhole ends. The relative minus sign between X_n^- and X_{-n}^+ is needed in order to make an orientable surface when the handles are glued together.

We now want to find a complete set of states, described by wave functions, in which we can expand the δ function. These will be interpreted as wormhole basis states. If the $|q_R|$ are very small then, to a first approximation, we can treat each circle separately and consider the region of the complex plane exterior to a single disc. This is the dilute wormhole approximation. The states can be taken to be eigenstates of the string analogue of the Wheeler-DeWitt, or Hamiltonian, operator. This is the Virasoro generator $L_0+\tilde{L}_0-2$. We are considering the two-dimensional analogue of the situation in [15], where space-time wormholes are thought of as solutions to the Wheeler-DeWitt equation. However there is a difference with these wormhole states. Here we are not just considering zero eigenstates of $L_0+\tilde{L}_0-2$. This is because we need to have a complete set of states in which to expand the δ function. If we had only used zero eigenstates, then they would not have formed a complete set. However, we shall show that the dominant contribution comes from the zero eigenstates, when the integration over the size of the wormhole is performed.

We are therefore led to consider the region of the complex plane outside a disc of radius R. The (Euclidean) action for the field X on this space is

$$I=\frac{1}{8\pi}\int dt\, d\theta\, N\left[\frac{1}{N^2}\dot{X}^2+X'^2\right]\,, \tag{4.5}$$

where $t=-\ln|z|/R$ is a new variable running from $-\infty$ to 0 and $\theta=\arg(z)$ runs from 0 to 2π. We are treating t as the Euclidean "time" variable. Notice that, because of conformal invariance, I has no dependence on R, unlike the case in more than two dimensions. This action is a (partially) gauge-fixed version of a world-sheet differomorphism-invariant action. The variable N is like a Lagrange multiplier (or lapse function) which enforces the Hamiltonian constraint $H\equiv\delta I/\delta N=0$. So far this is all at the classical level. To consider the quantization, we first expand X in Fourier modes as

$$X(t,\theta)=\sum_n X_n(t)e^{in\theta}\,. \tag{4.6}$$

The conjugate momentum to $X_n(t)$ is given by

$$\begin{aligned}P_n&=i\delta I/\delta\dot{X}_n\\&=\frac{i}{2N}\dot{X}_{-n}\,.\end{aligned} \tag{4.7}$$

The Hamiltonian constraint takes the form

$$P_0^2+\sum_{n>0}(2P_n\cdot P_{-n}+\tfrac{1}{2}n^2X_n\cdot X_{-n})=0\,. \tag{4.8}$$

Canonical quantization can now proceed by letting $P_n\to -i\partial/\partial X_n$, where the derivative acts on wave functions $\psi(X_n)$. The Hamiltonian constraint is imposed on the wave function as an operator constraint. We can diagonalize the term in parentheses by introducing new variables (for $n>0$):

$$\begin{aligned}Y_n^{(1)}&=(X_n+X_{-n})/2\,,\\Y_n^{(2)}&=(X_n-X_{-n})/2i\,.\end{aligned} \tag{4.9}$$

In these variables the Hamiltonian constraint becomes

$$H\psi(X_0,Y_n^{(i)})=0\,, \tag{4.10}$$

where

$$H=-\frac{\partial^2}{\partial X_0^2}+\frac{1}{2}\sum_{n>0,i}\left[-\frac{\partial^2}{\partial (Y_n^{(i)})^2}+n^2(Y_n^{(i)})^2\right]\,. \tag{4.11}$$

This is the string analogue of the Wheeler-DeWitt equation. H is the sum of the Hamiltonian of a free particle in the coordinate X_0 and an infinite sum of harmonic oscillators in the coordinates $Y_n^{(i)}$.

The coordinates $X_0, Y_n^{(i)}$, which are real in the classical theory, become Hermitian in the quantum theory, provided the standard inner product $\int\bar{\phi}\psi$ is taken, with the contour of integration along the real axis. One then obtains a complete set of wave functions by taking products of

$$\psi_k(X_0)=e^{ik\cdot X_0} \tag{4.12}$$

with harmonic-oscillator wave functions in the $Y_n^{(i)}$:

$$\psi_{nm}=\left[\frac{n}{\pi 2^{2m}(m!)^2}\right]^{1/4} e^{-n(Y_n^{(i)})^2/2}H_m(\sqrt{n}\;Y_n^{(i)})\,. \tag{4.13}$$

We could have expanded the δ functions in any complete basis, but the effective interactions will have a particularly simple form if we choose a basis of eigenstates of H. We regard the expansion in terms of these eigenfunctions as a sum of products of wormhole states. Each wormhole state has a wave function that is the product of $e^{ik\cdot X_0}$ and harmonic-oscillator wave functions in the $Y_n^{(i)}$.

The non-negative integers $m_n^{\mu(i)}$ give the occupation number of the mode n,μ,i. The eigenfunction for the $n=0$ mode is labeled by k. The eigenvalue of the wormhole state is

$$h=k^2+\sum_{n>0}\sum_{\mu,i} nm_n^{\mu(i)} , \tag{4.14}$$

if the zero-point energies are subtracted.

B. Wave functions and effective interactions

We shall now calculate the effective interactions associated with the wormhole wave functions of the previous section. This will be done in the dilute wormhole approximation. This means that we shall calculate scattering amplitudes outside a single disk on the complex plane. The wave functions are defined on the boundary of this disk. We shall, for simplicity, consider tachyon amplitudes only. As in the analogous calculation of [1], this proceeds in two steps.

First, we calculate the path integral

$$\mathcal{D}(X_0,X_n;p_j)=\int [dX]e^{-I[X]}\prod_j V_j . \tag{4.15}$$

Here, $V_j=\kappa\int d^2z_j{:}e^{ip_j\cdot X(z_j)}{:}$ are M on-shell tachyon vertex operators ($p_j^2=2$). They are inserted in the region exterior to the disk with center z_0 and radius r. The fields X are required to satisfy Dirichlet boundary conditions

$$X|_{\partial M}=\sum_n X_n e^{in\theta} , \tag{4.16}$$

on the boundary of the disk, where $\theta=\arg(z-z_0)$. There is an integration over z_0 and r, as well as the positions of the vertex operators. The measure for these integrations can be found by the requirement of SL(2, $\mathbb{C}$) invariance. This invariance is present because $\mathcal{D}(X_0,X_n;p_j)$ arises from a diffeomorphism- and Weyl-invariant path integral, which is then partially gauge fixed. The SL(2, $\mathbb{C}$) invariance of the integrand leads to a divergence proportional to Vol[SL(2, $\mathbb{C}$)], just as for amplitudes on the sphere. This has to be removed by the Faddeev-Popov gauge-fixing procedure. On the sphere it is usually done by fixing the positions of three of the vertex operators. In this case one can fix any three out of r,z_0,z_j, where z_j, $j=1,\dots,M$ are the positions of the vertex operators. [Fixing r leads to a residual U(1) invariance, with finite volume equal to 2π.]

The second step in the calculation is to compute the effect of the wormhole wave function ψ on tachyon amplitudes. This is given by

$$\mathcal{D}(\psi;p_j)=\int dX_0\left[\prod_{n>0} dX_n dX_{-n}\right] \times\psi(X_0,Y_n^{(i)})\mathcal{D}(X_0,X_n;p_j) . \tag{4.17}$$

The basis wave functions are plane waves in X_0, multiplied by harmonic oscillator wave functions in the $Y_n^{(i)}$. We are particularly interested in the limit in which $r\to 0$. If the radius is held fixed then this is the limit where all the vertex operators are very far from the boundary. The advantage of using the basis wave functions is that they are an orthogonal set. This means that if the function $\mathcal{D}(X_0,X_n;p_j)$ is expanded as a power series in X_n,X_{-n}, then terms which are of lower order than the level of the harmonic-oscillator wave function give a vanishing contribution. This is very similar to the situation in [1]. It turns out that this expansion is also a power series in the radius r, so that it is useful in the limit $r\to 0$.

We shall now calculate $\mathcal{D}(X_0,X_n;p_j)$ with the gauge-fixing conditions $r=1$ and $z_0=0$. One vertex operator z_1 is fixed at infinity. The measure takes the simplest form if r,z_0 and one vertex operator are fixed in this manner. We shall then transform integration variables, so as to examine the limit of "small" r, with z_0 and two vertex operators fixed. First, it is convenient to shift X by the harmonic function which satisfies the required boundary conditions. We write

$$X=\bar{X}+x , \tag{4.18}$$

where

$$\bar{X}=X_0+\sum_{n>0}(X_{-n}z^{-n}+X_n\bar{z}^{-n}) \tag{4.19}$$

is regular outside the wormhole. The path integral is now over x, with $x=0$ on the boundary. We find

$$\mathcal{D}(X_0,X_n;p_j)=\kappa^M\exp\left[iX_0\cdot\sum_1^M p_j-\frac{1}{2}\sum_{n>0}nX_n\cdot X_{-n}\right]\int\left[\sum_2^M d^2z_j\right]\exp\left[i\sum_{n>0}\sum_{j=1}^M p_j\cdot(X_{-n}z_j^{-n}+X_n\bar{z}_j^{-n})\right]\mathcal{D}(0;p_j;z_j) , \tag{4.20}$$

where

$$\mathcal{D}();p_j;z_j)=\int [dx]e^{-I[x]}\prod_1^M {:}e^{ip_j\cdot x(z_j)}{:} \tag{4.21}$$

is performed with vanishing Dirichlet boundary conditions. $\mathcal{D}(0;p_j;z_j)$ can be computed exactly, using the Dirichlet Green's function outside the unit disk. This is

$$G(z,w)=-\ln\left|\frac{z-w}{1-z\bar{w}}\right|^2 . \tag{4.22}$$

To define $G(z,z)$, one must subtract the corresponding Green's function on the complex plane [14] $G(z,w)_{\mathbb{C}}=-\ln|z-w|^2$, and then take the limit $w\to z$.

This gives

$$G(z,z)_{\rm reg}=\ln(1-|z|^2)^2 \ . \qquad (4.23)$$

The normal-ordering symbols on the tachyon vertex operators instruct us to use the regularized Green's functions $G(z,z)_{\rm reg}$ for contractions at the same point. With this subtraction the measure in the integration over z_j ($j=2,\dots,M$) is trivial. We obtain the result

$$\mathcal{D}(0;p_j;z_j)=\left[\prod_2^M(1-|z_j|^2)^2\right]\prod_{1\le i\neq j\le M}\left|\frac{z_i-z_j}{1-z_i\bar z_j}\right|^{p_i\cdot p_j} . \qquad (4.24)$$

We have fixed a six parameter gauge group [SL(2,$\mathbb{C}$)] with the five (real) conditions $z_0=0$, $z_1=\infty$, $r=1$. The extra parameter corresponds to rotations of the disk about the origin. It can be fixed by rotating one of the z_j, $j=2,\dots,M$, say z_2, to the real axis. However, this is not necessary, as the additional integration over $\arg(z_2)$ just corresponds to an additional factor of 2π.

We now consider the boundary of moduli space in which the rim of the disk corresponds to a small wormhole end. With this choice of variables this is the region in which all the vertex operators go out towards infinity. It is easier to see the small wormhole end limit by rescaling our coordinates, so that the radius of the disk is effectively one of the integration variables. The integrals over z_j ($j=2,\dots,M$) can be transformed to integrals over ζ_j ($j=3,\dots,M$) outside the circle $|\zeta_j|=r$, together with an integral over r, with an appropriate measure. First we rotate z_2 to the real axis [fixing the U(1) invariance]. Then we change from the integration variables $|z_2|$, z_j to the variables $r=1/|z_2|$, $\zeta_j=z_j/|z_2|$, where $j=3,\dots,M$ We define $\zeta_0=0$, $\zeta_1=\infty$, $\zeta_2=1$. With this change of variables we obtain

$$\begin{aligned}\mathcal{D}(X_0,X_n;p_j)=2\pi\kappa^M\int_0^1\frac{dr}{(1-r^2)^2}r^{-3+(\sum_1^M p_j)^2}\\ \times\int_{|\zeta_j|>r}\left[\prod_3^M\frac{d^2\zeta_j}{(|\zeta_j|^2-r^2)^2}\right]\prod_{1\le k\neq l\le M}\left|\frac{\zeta_k-\zeta_l}{\zeta_k\bar\zeta_l-r^2}\right|^{p_k\cdot p_l}\\ \times\exp\left[iX_0\cdot\textstyle\sum_1^M p_j\right]\prod_{n>0}\exp\left[-\frac{1}{2}nX_n\cdot X_{-n}+i\textstyle\sum_1^M p_j\cdot(X_{-n}\zeta_j^{-n}+X_n\bar\zeta_j^{-n})r^n\right] .\end{aligned} \qquad (4.25)$$

We have calculated the path integral with Dirichlet boundary conditions. The next step is to weight this with the wormhole wave function on the boundary, and integrate over boundary values. This gives us the effect of the wave function on scattering amplitudes outside the wormhole. We shall now do this calculation for a few examples of wave functions ψ.

C. Examples

The effect of the basis wormhole wave functions $\psi(X_0,Y_n^{(i)})$ on scattering amplitudes $\mathcal{D}(\psi;p_j)$ can now be calculated using (4.17), where $\mathcal{D}(X_0,X_n;p_j)$ is given by the expression (4.25). In this section we perform the integrations over the zero modes X_0, and the higher modes X_n,X_{-n} (or equivalently over the $Y_n^{(i)}$). We are interested in the $r\to 0$ limit. We shall exhibit the results for the lowest few wormhole states and then write down the general rule for replacing each factor in the wave function with an effective interaction.

The easiest of these integrations is the one over X_0. The $e^{ik\cdot X_0}$ factor in ψ ensures that we obtain a momentum-conserving δ function of the form

$$(2\pi)^D\delta\left[k+\sum_{j=1}^M p_j\right] , \qquad (4.26)$$

from the X_0 integral, where D is the dimension of space-time.

The integrals over each of the $Y_n^{(i)}$ can also be done exactly, for an arbitrary wormhole basis state. We start by rewriting each factor

$$\exp\left[-\frac{1}{2}nX_n\cdot X_{-n}+i\sum_1^M p_j\cdot(X_{-n}\zeta_j^{-n}+X_n\bar\zeta_j^{-n})r^n\right] \qquad (4.27)$$

in (4.25), as

$$\exp\left[\sum_i[-n(Y_n^{(i)})^2/2+ir^nk_n^{(i)}\cdot Y_n^{(i)}]\right] , \qquad (4.28)$$

where

$$k_n=2\sum_{j=1}^M p_j\zeta_j^{-n} , \qquad (4.29)$$

and $k_n^{(1)},k_n^{(2)}$ denote the real and imaginary parts of k_n, respectively. It is now clear that the integral over $Y_n^{(i)}$ is the Fourier transform, with respect to $r^nk_n^{(i)}$, of

$$e^{-n(Y_n^{(i)})^2/2}\psi_m(\sqrt{n}\,Y_n^{(i)}) , \qquad (4.30)$$

where $m=m_n^{\mu(i)}$ is the occupation number of the mode n,μ,i. Using the standard Fourier integral [25]

$$\int dy\, e^{iky-y^2}H_m(y)=\sqrt{\pi}(ik)^m e^{-k^2/4} , \qquad (4.31)$$

we see that each integral over $Y_n^{(i)}$ contributes a factor proportional to

$$\frac{r^{nm}(ik_n^{(i)})^m}{\sqrt{m}!(2n)^{m/2}}e^{-r^2(k_n^{(i)})^2/4n}\,, \tag{4.32}$$

up to a (n-dependent) multiplicative constant.

The infinite product over modes has to be regularized in some way, by including an n-dependent factor for each mode. If this were not done there would be an infinite factor in $\mathcal{D}(\psi;p_j)$, even in the simplest case where the wormhole is in the ground state in each of the oscillators. We have chosen this factor in such a way that the effective interaction from each mode which is in its ground state is unity (in the $r\to 0$ limit).

The Gaussian exponential factor in (4.32) contributes higher-order interactions (as $r\to 0$) which we shall therefore ignore from now on. The important factor in (4.32) is $r^{nm}(ik_n^{(i)})^m$ which leads to the dominant effective interactions. We shall now discuss this in more detail for the lowest few wormhole states.

For the wave function which is the ground state in each of the $Y_n^{(i)}$ oscillators, multiplied by $e^{ik\cdot X_0}$, $\mathcal{D}(\psi;p_j)$ takes the form

$$\mathcal{D}(\psi;p_j)\propto\delta\left[\sum_0^M p_j\right]\int_0^1 dr\, r^{-3+k^2}\lim_{\zeta_1\to\infty}|\zeta_1|^4\int\left[\prod_3^M d^2\zeta_j\right]\prod_{0\le k\ne l\le M}|\zeta_k-\zeta_l|^{p_k\cdot p_l}f(r;\zeta_j;p_j)\,, \tag{4.33}$$

where $f(r;\zeta_j;p_j)\to 1$ as $r\to 0$ and , for simplicity of notation, we have defined $p_0=k$.

There are two important features of this expression. The first is that it is clear that the integration over r, near $r=0$, yields a pole at $k^2=2$, the on-shell value of the tachyon momentum squared. This is just like in Sec. II where this pole appeared in the integral over $|q|$. The second important feature is that the quantity multiplying this pole is proportional to the scattering amplitude for $(M+1)$ on-shell tachyons. The amplitude is automatically SL(2, $\mathbb{C}$) gauge fixed. The extra tachyon is at $\zeta_0=0$, the center of the wormhole disk. Two other tachyons are fixed at $\zeta_1=\infty$, $\zeta_2=1$. This example shows that the leading effect of the wormhole wave function $e^{ik\cdot X_0}$, defined on a circle, is the same as the insertion of a tachyon vertex operator, $:e^{ik\cdot X(0)}:$, at the center of the circle. This is a result that has already been obtained by other methods, but it is instructive to see it emerge in this formalism.

The next example is more interesting since it corresponds to massless particle interactions. These will be the leading interactions in superstring theories, where one sums over spin structures. The case where just one of the harmonic oscillators $Y_1^{\mu(i)}$ is in its first excited energy level leads to a trivial result, zero, because of rotational invariance of the integrals over ζ. The first nontrivial result comes from the wave function $\psi_{\mu\nu}^{(i)(j)}$, which has occupation numbers $m_1^{\mu(i)}=m_1^{\nu(j)}=1$. In this wave function we cannot have both $\mu=\nu$ and $i=j$ simultaneously. At this order of magnitude in r we must also consider the effect of the wave function $\psi_\mu^{(i)}$, which has $m_1^{\mu(i)}=2$. The other occupation numbers are taken to be zero in each case.

Keeping only the $O(r^{nm_n^{\mu(i)}})$ term in each factor, (4.32), and inserting the resulting expression into (4.25), we obtain the following behavior (up to a multiplicative constant) as $r\to 0$:

$$\mathcal{D}(\psi_{\mu\nu}^{(i)(j)};p_j)\sim\delta^D\left[\sum_0^M p_j\right]\int_0 dr\, r^{-1+k^2}\sum_{k,l=1}^M p_k^\mu p_l^\nu\begin{cases}\mathcal{S}_{[kl]} & \text{if } i\ne j,\\ \mathcal{S}_{(kl)} & \text{if } i=j,\end{cases}$$

$$\mathcal{D}(\psi_\mu^{(i)};p_j)\sim\delta^D\left[\sum_0^M p_j\right]\int_0 dr\, r^{-1+k^2}\sum_{k,l=1}^M p_k^\mu p_l^\mu\mathcal{S}_{(kl)}\quad\text{no sum on }\mu\,, \tag{4.34}$$

where

$$\mathcal{S}_{kl}=\lim_{\zeta_1\to\infty}|\zeta_1|^4\int\left[\prod_3^M d^2\zeta_j\right]\frac{1}{\zeta_k\bar\zeta_l}\times\prod_{0\le r\ne s\le M}|\zeta_r-\zeta_s|^{p_r\cdot p_s} \tag{4.35}$$

is the basic quantity appearing in tree-level scattering amplitudes for M on-shell tachyons and one massless particle. The position of the massless particle vertex operator is fixed at $\zeta_0=0$. The leading interaction from $\psi_{\mu\nu}^{(i)(j)}$ is thus the same as an antisymmetric tensor (if $i\ne j$) or a graviton vertex operator (if $i=j$). The effect of the sum of wave functions $\psi^{(i)}=\sum_\mu\psi_\mu^{(i)}$ is the same as a dilaton vertex operator. More precisely, the wave functions $\psi^{(i)}$ correspond to the vertex operators $\pm\frac{1}{2}(\partial X\pm\bar\partial X)^2$, where the upper (lower) signs correspond to $i=1$ (2). However there is an integration over $\arg(\zeta)$ which projects onto the $\partial X\bar\partial X$ part of these vertex operators. Thus the two wave functions both correspond to the same effective interaction. This happens also for the wave functions $\psi_{\mu\nu}^{(1)(1)}$ and $\psi_{\mu\nu}^{(2)(2)}$ corresponding to graviton vertex operators. The integral over $\arg(\zeta)$ ensures that only wave functions obeying the conditon $(L_0-\tilde L_0)\psi=0$ give a nonvanishing effective interaction.

The integral over r in (4.34), near $r=0$, yields a pole in k^2, which shows that the states are massless. This $1/k^2$ factor is like a space-time propagator for the state. The path integral $\mathcal{D}(\psi)$ provides an off-shell (i.e., $k^2\ne 0$) continuation of the amplitudes. However, as $k^2\to 0$, the effective interaction factorizes into the scattering amplitude $S^{\mu\nu}=\sum_{k,l}p_k^\mu p_l^\nu\mathcal{S}_{kl}$, and the propagator $1/k^2$.

Note that the path integral which we used to define the quantity $\mathcal{D}(X_0,X_n;p_j)$ depended on the use of the angular coordinate to parametrize the circle. This was also used in the definition of ψ. We used a mode expansion of X which is not invariant under reparametrizations. It is not clear whether it is desirable for off-shell quantities, such as $\mathcal{D}(\psi;p_j)$, to depend on the parametrization of the circle used [26], or whether they should be independent of this choice [14]. This is really the question of whether one should integrate over all reparametrizations, or just those which do not reparametrize the boundary, in the path integral over metrics. Further discussion of these points is in [14,26,27].

We conclude this section by exhibiting the general rule for the higher states. The general form of (4.32) as $r \to 0$ shows that the general state will yield a singularity in the r integration, as $r \to 0$, of the form

$$\int_0 dr\, r^{-3+h} , \tag{4.36}$$

where $h = k^2 + \sum_{n>0} \sum_{\mu,i} n m_n^{\mu(i)}$ is the eigenvalue of the wormhole state. This produces a pole of the form

$$\frac{1}{h-2} \tag{4.37}$$

in the effective interaction. This pole is at exactly the place where the states are on-shell, and obey $(L_0 + \tilde{L}_0 - 2)\psi = 0$. It can be thought of as a propagator for the state. The other crucial factor in (4.32) is the quantity $[\sum p_j^\mu \mathrm{Re}(\zeta_j^{-n})]^m$, or $[\sum p_j^\mu \mathrm{Im}(\zeta_j^{-n})]^m$, which occur for $i=1,2$. This can be thought of as arising from the vacuum expectation values

$$\langle :[\mathrm{Re}\partial^n X^\mu(0)]^m: :\exp\left[i \sum p_j \cdot X(\zeta_j) \right]: \rangle$$
$$\langle :[\mathrm{Im}\partial^n X^\mu(0)]^m: :\exp\left[i \sum p_j \cdot X(\zeta_j) \right]: \rangle , \tag{4.38}$$

on the complex plane. They thus lead to the effective interactions $:[\mathrm{Re}\partial^n X^\mu(0)]^m:$, or $:[\mathrm{Im}\partial^n X^\mu(0)]^m:$, respectively. Each reciprocal power of ζ corresponds to a derivative of the Green's function on the complex plane $G(\zeta,0)_{\mathrm{C}} = -\ln|\zeta|^2$. There is also a factor of $:e^{ik\cdot X}:$ in the effective interaction, which is present for all the basis states, and arises from the $e^{ik\cdot X_0}$ factor in the wave function. Note that the integration over $\arg(\zeta)$ will ensure that only the rotationally invariant interactions survive. This is equivalent to projecting onto states satisfying the condition $(L_0 - \tilde{L}_0)\psi = 0$.

Thus we have the general correspondence

$$\begin{aligned} &e^{ik\cdot X_0} \leftrightarrow r^{k^2} :e^{ik\cdot X(0)}: ,\\ &\psi_m(Y_n^{\mu(1)}) \leftrightarrow r^{nm} :[\mathrm{Re}\partial^n X^\mu(0)]^m: ,\\ &\psi_m(Y_n^{\mu(2)}) \leftrightarrow r^{nm} :[\mathrm{Im}\partial^n X^\mu(0)]^m: , \end{aligned} \tag{4.39}$$

between wormhole wave functions and effective interactions.

V. SUMMING OVER SPIN STRUCTURES: THE SUPERSTRING

Up to now the only string theory which we have considered has been purely bosonic. However, there are several reasons why bosonic strings are not regarded as realistic. The main ones are the appearance of a tachyon and the absence of space-time fermions. The tachyon is a constant source of embarrassment, having a spacelike momentum in Lorentzian space-time. Attempts at solving the tachyon problem by, for instance, shifting to another vacuum have achieved little success. However, even if the tachyon problem can be solved in bosonic string theory, the lack of fermions is clearly in contradiction with our everyday experience.

It has been claimed in earlier sections that for superstrings, there are no tachyon effective interactions coming from wormholes. One expects this to follow from factorization of multihandle amplitudes [28]. It should be possible to replace the handles by a complete set of vertex operators of the theory. Since the theory does not contain a tachyon, one expects it to be absent in the effective interactions. We shall see that this is indeed the case. The easiest way in which to see this is in the RNS formulation of superstrings [16,17], in which there are fermions on the world-sheet.

The inclusion of spinors on the world-sheet leads to a new kind of structure, a "spin structure." This tells us how a spinor changes when it is parallel transported around a closed loop. A vector (in an orthonormal frame) rotates by an SO(2) matrix (for Euclidean signature). When a spinor is parallel transported it (loosely speaking) changes by the square root of this matrix. There is a sign ambiguity in taking the square root. For a contractible loop there is no ambiguity because the sign must be continuous as the loop is smoothly contracted down to a point. Since the spinor only changes infinitesimally, if it is parallel transported around an infinitesimal loop, this shows that the square root taken must tend to the identity matrix as the loop shrinks to a point. However, for noncontractible loops there is no longer any reason for preferring one choice of square root over any other. Such a choice of signs, for each noncontractible loop, is called a spin structure.

The Riemann surfaces corresponding to world-sheet wormholes have handles. Each handle adds two more noncontractible closed loops. One of them threads itself through the wormhole, while the other goes round one of the ends. The number of spin structures on a surface of genus g is given by 2^{2g}. In a string theory which includes world-sheet spinors it is natural to sum over all these spin structures, just as one sums over topologies. In fact, this is necessary for the consistency of the theory. One can fix the relative weights of the terms almost uniquely, by the requirements of modular invariance and unitarity [18]. It turns out [18] that the sum over spin structures is equivalent to the GSO projection [23,24], which removes the tachyon.

We shall show in this section, using the wave function formalism that we have developed, that the tachyonic effective interaction vanishes, when the spin structures are summed over, as in [18].

A. World-sheet spinors

The obvious way in which to include spinors on the world sheet, which is manifestly covariant under reparametrizations and local SO(2) [or SO(1,1)] transformations, is to add the following (minimal) term to the world-sheet action:

$$I_\psi = \int d^2\sigma\, e\tilde{\psi}\rho^\alpha \nabla_\alpha \psi \ . \tag{5.1}$$

The notation here is as follows. ρ^α denote the two-dimensional gamma matrices, in the coordinate basis α. $e=\sqrt{g}$ is the determinant of the zweibein. ∇_α is the spinor covariant derivative, obtained from the (standard) spin connection in which the zweibein is covariantly constant. A tilde denotes Dirac conjugation ($\tilde{\psi}=\psi^\dagger\rho^{\hat{0}}$) and indices with carets denote components in a local orthonormal frame. if the two-component world-sheet spinor fields ψ also transform as space-time vector fields ψ^μ then one can incorporate world-sheet supersymmetry between ψ^μ and X^μ. (Space-time indices μ will be suppressed in this discussion.) We shall also restrict the components of ψ to be real in the basis for which ρ^α are both real matrices. This is possible since in this basis the Dirac equation does not mix the real and imaginary parts of ψ. Spinors satisfying this reality condition (which we shall assume from now on) are called Majorana. All spinors will also be taken to be anticommuting. It is easy to show that for anti-commuting Majorana spinors in two dimensions the term in I_ψ involving the spin connection vanishes, so that we can use the simpler action

$$I_\psi = \int d^2\sigma\, e\tilde{\psi}\rho^\alpha \partial_\alpha \psi \ . \tag{5.2}$$

For the massless Dirac equation we can also split ψ into its chiral components, which obey independent equations. Defining $\bar{\rho}=i\rho^{\hat{0}}\rho^{\hat{1}}$ (if the signature is Euclidean) or $\rho^{\hat{0}}\rho^{\hat{1}}$ (if it is Lorentzian), we see that $\bar{\rho}$ has eigenvalues ± 1. We can write $\psi=\psi_+ + \psi_-$, where $\psi_\pm$ has eigenvalue ± 1. Then the Majorana condition becomes $\psi_- = \tilde{\psi}_+$ for Euclidean signature, while it is $\mathrm{Im}\psi_\pm = 0$ in the Lorentzian case. We can see that, while for Lorentzian world sheets the two chiral components can be thought of as completely independent, in the Euclidean case they are related by the Majorana condition. This leads to a problem with deriving the correct Hamiltonian for Majorana spinors on the world sheet, starting from a purely Euclidean action. We shall therefore, temporarily, work with a Lorentzian signature world sheet. We use the basis in which $\rho^{\hat{0}}, i\rho^{\hat{1}}$ are the first two Pauli matrices. The action then becomes

$$I_\psi = -i\int d\theta\, dt [\psi_+(\dot{\psi}_+ + \psi_+') + \psi_-(\dot{\psi}_- - \psi_-')] \ . \tag{5.3}$$

Here we have specialized to a flat world sheet with coordinates t,θ, and a dot (prime) denotes a derivative with respect to t (θ). However, a similar action can be obtained if the world sheet is curved.

Now we shall discuss the possible boundary conditions on spinors, and their physical properties. If we consider a propagating closed string, the world sheet is a cylinder. There is a single noncontractible closed loop, which goes around the string at a single moment of time (with $0 \le \theta < 2\pi$). If we take a flat metric on the cylinder, then worldsheet vectors are unchanged as they are parallel transported around this loop. Spinors must therefore change by ± 1. Thus there are two possible boundary conditions on spinors. They must be either periodic or antiperiodic around the loop. The two cases lead to different field theories on the world sheet. The periodic (antiperiodic) case is known as Ramond (Neveu-Schwarz) boundary conditions. One can expand $\psi_\pm$ in modes on the cylinder, as

$$\psi_\pm = \sum_n \psi_n^\pm(t) e^{in\theta} \ , \tag{5.4}$$

where n runs over values in $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ in the Ramond or Neveu-Schwarz case, respectively. The equations for ψ_+ and ψ_- decouple, so one can specify the boundary conditions on each independently. (This would have caused problems for a Euclidean signature world sheet, since the Majorana condition would then relate ψ_+ and ψ_-.) One thus obtains four possibilities for boundary conditions: (R-R), (R-NS), (NS-R), (NS-NS). Substituting the mode expansions back into the action we obtain a fermionic quantum mechanical system consisting of an infinite set of fermionic oscillators. The Hamiltonian can be obtained by introducing a lapse function, as in Sec. IV. One changes the metric on the cylinder to

$$ds^2 = -N^2(t)dt^2 + d\theta^2 \ , \tag{5.5}$$

and varies the action with respect to $N(t)$. The Hamiltonian turns out to be

$$H_\psi = \sum_n n\,\psi_{-n}^+ \psi_n^+ + \sum_m m\,\psi_m^- \psi_{-m}^- \ , \tag{5.6}$$

in the gauge where $N=1$. Here n and m can take integral or half-integral values, depending on which of the four boundary conditions are being used.

We can quantize this system of fermionic oscillators in exactly the same way as in [11]. The only nontrivial anticommutation relations are

$$\{\psi_{-n}^\pm, \psi_n^\pm\} = 1 \ . \tag{5.7}$$

$\psi_{-n}^\pm, \psi_n^\pm$ can be interpreted as fermion annihilation and creation operators since the Majorana condition now implies that $\psi_{-n}^\pm$ is the adjoint operator to $\psi_n^\pm$. There is a holomorphic representation in which annihilation operators are represented by derivatives with respect to creation operators. States are represented by holomorphic functions of the (anticommuting) creation operators. The annihilation operators are ψ_n^+, ψ_{-n}^- and the creation operators are ψ_{-n}^+, ψ_n^-, where $n>0$. However, for Ramond boundary conditions there is a zero-mode spinor ψ_0. The algebra of the ψ_0^μ is the same (up to a constant factor) as the Clifford algebra of Dirac gamma matrices. Further, each ψ_0^μ commutes with the Hamiltonian. Thus, in the Ramond sector, the ground state must transform like a space-time spinor, rather than a scalar. However, we are here considering the coupling of the tachyon, which is the lowest energy space-time scalar state and resides in the (NS-NS) sector. The question is, does this cancel out when we sum over spin structures going through the wormhole? Thus we shall not need the physics of the Ramond sector in this discussion.

B. Spin structures and the GSO projection

Thus far we have considered the boundary conditions on the spinors going around a closed string. Such a choice of spin structure has to be made for each external closed string state. However, a (now Euclidean signature) world sheet contributing to the Polyakov path integral may have other noncontractible loops. These are associated with internal handles. The simplest example where an internal handle is present is the case where the world sheet is a torus. We take ξ^α as coordinates on the simple torus with metric $(d\xi^0)^2+(d\xi^1)^2$. There are identifications, $\xi^0\rightarrow\xi^0+1$ and $\xi^1\rightarrow\xi^1+1$, say, which make the topology that of a torus instead of flat Euclidean space. ξ^0 is treated like Euclidean time and ξ^1 is the "space" coordinate. Both the loop with constant ξ^0 and the loop with constant ξ^1 are noncontractible. A one-component spinor, ψ_- say, thus has four different spin structures, $(--)$, $(-+)$, $(+-)$, $(++)$, on the torus. But it is not consistent, in the Polyakov path integral, to restrict attention just to one of the spin structures. This is because there are (so-called "large") diffeomorphisms, such as $(\xi^0,\xi^1)\rightarrow(\xi^0,\xi^0+\xi^1)$, which exchange spin structures [in this case $(--)$ and $(+-)$]. The path integral must be invariant under *all* diffeomorphisms, including large ones.

Now every diffeomorphism can be obtained by composing a diffeomorphism which is connected to the identity (and does not change the spin structures) with a respresentative large diffeomorphism given by

$$(\xi^0,\xi^1)\rightarrow(k\xi^0+l\xi^1,m\xi^0+n\xi^1)\ , \tag{5.8}$$

where k,l,m,n are integers satisfying $kn-lm=1$. It is invariance under this discrete group [in this case $SL(2,\mathbb{Z})$] which is ensured by summing over spin structures. The large diffeomorphisms mix up the spin structures $(--)$, $(-+)$, $(+-)$, so if we include any one of them in the sum over surfaces, then we must include them all. By considering fermion determinants with different spin structures, on the torus, Seiberg and Witten [18] have found the weights with which each spin structure contributes, in order to have full invariance under the large diffeomorphisms. It turns out that the sum over $(--)$ and $(+-)$ spin structures in the NS sector is equivalent to the GSO projection [23,24] on states which have an odd number of excited world-sheet fermion modes. The GSO projection arises since the path integral on the torus with spin structure $(--)$ computes tr $\exp(i\tau H)$ while the $(+-)$ spin structure gives $-$tr $\exp(i\tau H)(-1)^F$, where F is the number of excited fermion modes. Thus the sum over the two types of NS spin structures computes tr $\exp(i\tau H)[1-(-1)^F]$, which is a trace over just those states for which F is odd. A similar projection occurs in the Ramond sector [although this is harder to see since the spin structure $(++)$ is invariant all by itself].

C. Wormhole wave functions

In the (NS-NS) sector, the total Hilbert space can be represented in terms of bosonic (discussed in Sec. IV) and fermionic harmonic oscillator wave functions. We describe the fermionic states in the holomorphic representation, as in [11]. The states are represented by holomorphic functions of fermion creation operators ψ^+_{-r}, ψ^-_r, while the annihilation operators ψ^+_r, ψ^-_{-r} act on the wave functions as derivatives. (The dual Hilbert-space states are represented by antiholomorphic functions of annihilation operators.) Here r is positive and $2r$ is an odd integer throughout. We can compute correlation functions on a genus g surface by considering the path integral on the region of the complex plane outside $2g$ discs. This is just like the description of Sec. IV, only for fermions. The path integral on the cut surface produces a function which is holomorphic in the ψ^+_{-r}, ψ^-_r on g of the circles, and antiholomorphic in the ψ^+_r, ψ^-_{-r} on the other g circles. We then need to insert some kind of delta function in each fermionic mode, just as we did for each bosonic mode. This is easily done using the fact that the inner product in the holomorphic representation is

$$\langle\Phi|\Psi\rangle=\int d\theta\, d\bar\theta\, e^{-\theta\bar\theta}\Phi(\bar\theta)\Psi(\theta)\ , \tag{5.9}$$

where the integrals are interpreted as Berezin integration over anticommuting variables (i.e., $\int d\theta\,\theta=1$ and $\int d\theta=0$).

Thus the delta functions that we need (for the $+$ and - modes) are

$$\begin{aligned}&\exp\left[-\sum_r \psi^+_r\bar\psi^+_{-r}\right]\ ,\\ &\exp\left[-\sum_r \psi^-_{-r}\bar\psi^-_r\right]\ ,\end{aligned} \tag{5.10}$$

for each of the g pairs of circles which are being "sewn together." Just as a conventional delta function can be expanded in a complete set of oscillator states, as $\delta(x-y)=\sum\psi_n(x)\psi_n(y)$, so this δ function has a particularly simple expansion as

$$\exp\left[-\sum_r \psi^+_r\cdot\bar\psi^+_{-r}\right]=1-\sum_r \psi^+_r\cdot\bar\psi^+_{-r}+\cdots\ , \tag{5.11}$$

and similarly for the "$-$" modes. This is an expansion in terms of the number of positive fermions which go through the wormhole. The first corresponds to $F=0$, the second to $F=1$ and so on. It seems that there is no restriction to states of odd F, and the tachyon (with $F=0$) would be included. However we have not so far taken account of the two different spin structures which can go through each handle. We cannot just have $e^{-\theta\bar\theta}$ but must also include $e^{+\theta\bar\theta}$, where we have changed the relative sign of the fermions at each end, and θ is a generic fermion variable. The relative phase with which each spin structure contributes can only be obtained by a careful evaluation of fermion determinants (carried out in [18]). It turns out that we need a relative minus sign between $e^{-\theta\bar\theta}$ and $e^{+\theta\bar\theta}$. Thus the sum over the spin structures going through the wormhole leads to the insertion of the product of the two functions:

$$\exp\left[-\sum_r \psi_r^+\cdot\bar{\psi}_{-r}^+\right]-\exp\left[+\sum_r \psi_r^+\cdot\bar{\psi}_{-r}^+\right],$$
$$\exp\left[-\sum_r \psi_{-r}^-\cdot\bar{\psi}_r^-\right]-\exp\left[+\sum_r \psi_{-r}^-\cdot\bar{\psi}_r^-\right] \quad (5.12)$$

in the path integral over fermion boundary values. This quantity is included for each pair of circles to be "sewn together."

Whereas the function $e^{-\theta\bar{\theta}}$ corresponds to a complete sum of states, the sum $e^{-\theta\bar{\theta}}-e^{+\theta\bar{\theta}}$ projects onto the $(-1)^F=-1$ states. Thus the tachyonic state, which has $F=0$ for both the $+$ and $-$ sectors, is not included in the sum over states contributing to each wormhole. The first term which is included has $F=1$ for each of the $\pm$ sectors. This is a state of zero space-time mass, and two space-time indices $\mu\nu$. It corresponds to the graviton, dilaton and antisymmetric tensor states.

VI. LINEARIZED QUANTUM GRAVITY FROM WORLD-SHEET INTERACTIONS

Summarizing the results of the previous sections, we have shown that in string theory the leading effect of long thin handles on the world sheet is the same as having a sum of bilocal interactions. In superstrings (for which the tachyon is absent) these are associated with the massless states of the theory. The massive states are suppressed, provided the modular parameter integration $|q|$ is dominated by handles which are very thin, $|q|\to 0$. This is expected to be the case in certain low-energy (but high-genus) situations. In such situations it is possible to formally "sum up" the topological expansion. Although the full topological expansion diverges [21], the sum over wormholes is just an exponential series, and so converges. It can be thought of as being like a sum over an infinite class of Feynman diagrams in field theory, which often has relevance even if the complete perturbation series diverges.

The effect of one wormhole is given by the bilocal operator

$$\frac{1}{2}\sum_{i,j}\Delta_{ij}\int d^2z\int d^2w\; V_i(z,\bar{z})V_j(w,\bar{w}) , \quad (6.1)$$

where the wormhole "propagator matrix" is the part of the operator which is the propagator for each state going through the wormhole (e.g., $\Delta_{ij}\sim(1/k^2)\delta^D(k+k')\delta_{\mu\rho}\delta_{\nu\sigma}$ for the massless states), and the V_i are the vertex operators for these states (e.g., $e^{ik\cdot X}\partial X^\mu\bar{\partial}X^\nu$). The sum over i consists of the integral over the momentum k, as well as a sum over polarizations. The effect of n wormholes (taking into account a factor of $n!$ from the possibility of permuting the wormholes) is given by

$$\frac{1}{2^n n!}\left[\sum_{i,j}\Delta_{ij}\int d^2z\int d^2w\; V_i(z,\bar{z})V_j(w,\bar{w})\right]^n . \quad (6.2)$$

Thus the sum over wormholes has the effect

$$\exp\left[\frac{1}{2}\sum_{i,j}\Delta_{ij}\int d^2z\int d^2w V_i(z,\bar{z})V_j(w,\bar{w})\right] . \quad (6.3)$$

The exponentiating of vertex operators was checked in Sec. III. It works (formally), provided we are purely interested in the behavior of V_i close to the pole at $k^2=0$. It gives a bilocal addition to the action for strings on a flat background. One can turn this into a local addition to the action, by introducing α parameters for each vertex operator in the manner first suggested in [29]. We rewrite the exponential as

$$\int\left[\prod_i d\alpha_i\right]\times\exp\left[-\frac{1}{2}\sum\alpha_i(\Delta^{-1})_{ij}\alpha_j-\sum\alpha_i\int d^2z\; V_i(z,\bar{z})\right] \quad (6.4)$$

showing that the effect of the wormholes is to add to the action the term

$$\sum\alpha_i\int d^2z V_i(z,\bar{z}) , \quad (6.5)$$

which is then averaged over α_i with a Gaussian weight. Each of the terms $\alpha_i V_i(z,\bar{z})$ shifts the coupling constant associated with V_i by an amount α_i. The effect of wormholes is thus to renormalize the coupling constants. However, there is an integral over α_i. This effectively divides the theory into noninteracting superselection sectors, labeled by α_i. But the α_i can also be thought of as space-time fields, as we shall now show.

Consider, for example, the case of wormholes containing gravitons. In Sec. II we found that the wormhole vertex operator $V_g^{\mu\nu}$ (for bosonic strings) is the trace-free part of

$$V^{\mu\nu}=e^{ik\cdot X}\partial X^{(\mu}\bar{\partial}X^{\nu)} . \quad (6.6)$$

Thus the term added to the action takes the form

$$\int d^Dk\,\alpha^g_{\mu\nu}(k)\int d^2z V_g^{\mu\nu}(k;z,\bar{z})=\int d^2z h_{\mu\nu}(X)\partial X^\mu\bar{\partial}X^\nu , \quad (6.7)$$

where $h_{\mu\nu}$ is the shift of the background space-time metric. We see from (6.7) that $\alpha^g_{\mu\nu}(k)$ is the Fourier transform of the linearized gravitational fluctuations about a flat space-time background:

$$\alpha_{\mu\nu}(k)=\tilde{h}_{\mu\nu}(k) . \quad (6.8)$$

The wormholes have been replaced by a sum of string theories on different curved backgrounds. Each background is weighted by $e^{-I_{\rm lin}}$, where $I_{\rm lin}$ is the (gauge-fixed) action for linearized Einstein gravity:

$$I_{\rm lin}=-\tfrac{1}{2}\int d^DX h_{\mu\nu}\Box h_{\mu\nu} . \quad (6.9)$$

Thus string theory on the sphere, coupled to linearized Einstein gravity, has appeared as an effective theory. But there is still a functional integral over $h_{\mu\nu}(X)$, since there is an integral over all the α_i in (6.4). Thus it is *quantum* gravity which emerges, rather than the classical theory. The action $I_{\rm lin}$ is manifestly positive definite, because it is written in terms of the trace-free parts of the metric fluctuations. These are the physical degrees of freedom of (linearized) gravity (except the transverse condition is absent). The notorious divergence of the conformal factor is not present. This only arises if gauge degrees of freedom are included in the action, such as the trace of the metric fluctuations.

The α_i for other vertex operators can be associated with space-time fields in a similar manner. This leads to an effective theory of strings coupled to quantized massless background fields. One can regard the α_i as the component fields of a single "third quantized" field α on superspace (in the sense of Wheeler, rather than the supersymmetry meaning of the word)

$$\alpha(X_0,X_n)=\sum \alpha_i f_i(X_0,X_n) \ . \tag{6.10}$$

Here, f_i are a complete basis of functions on superspace. In Sec. IV we took the basis consisting of plane waves in X_0, and harmonic oscillator wave functions in X_n for $n\neq 0$. This choice leads to the possibility of reducing to a finite number of massless space-time fields $\alpha_{\mu\nu}(X_0)$, in the thin wormhole approximation. In effect, a reduction has been made from superspace, to a "minisuperspace" just consisting of the space-time coordinates X_0 [9]. A different choice of basis would be possible, but it would not lead to such a simple reduction.

VII. SUMMARY

We have investigated the effects of wormholes on the world sheet of string theory. In the dilute wormhole approximation they can be replaced by a sum of effective interactions. We used two main techniques for calculating the interactions associated with wormholes.

(1) We calculated n-point functions directly (for bosonic fields), on an arbitrary Riemann surface in the Schottky parametrization. The dilute wormhole approximation becomes an expansion in the modular parameters $|q_R|$ of the wormholes. By integrating over the $|q|$ we were able to show that the interactions are dominated by on-shell (i.e., $L_0+\tilde{L}_0-2=0$) vertex operators, although off-shell ones contribute as well. The integral over $\arg(q_R)$ gave the condition that the vertex operators are U(1) invariant (i.e., $L_0-\tilde{L}_0=0$). For thin handles, the dominant vertex operators are those of the tachyon and massless fields.

(2) We used the Riemann surface (with boundaries) obtained by cutting across each wormhole to describe the interactions in terms of basis wormhole wave functions. Then we found the effective interactions associated with the basis wave functions, in the dilute wormhole approximation, by calculating scattering amplitudes in the region outside the disk. To leading order the effective interactions are due to tachyons and massless particles. The basis states are off shell, but the integral over r near $r=0$ produces a pole on the mass shell, like a propagator.

The wave function formalism enabled us to see why there is no tachyon effective interaction in superstring theories with world-sheet fermions. The sum over spin structures going through the wormhole eliminates the tachyon, as it projects onto states with odd fermion number in each of the $\pm$ movers. We used the holomorphic representation of the states, as in [11].

We concluded with a discussion of the interpretation of the α-parameters for wormholes in string theory. They can be thought of as an infinite set of quantum fields in space-time, including the graviton. One can also view them as components of a single quantum field on superspace. This is the "third-quantized" viewpoint advocated by Giddings and Strominger [6] and, more recently, by Hawking [9]. We see how quantum gravity could emerge as an effective theory, from a fundamental theory of strings moving in flat space-time with arbitrary genus world sheets. Further work (beyond the dilute wormhole approximation) may shed light on issues in quantum gravity, such as whether the effective space-times themselves can have wormholes. This will have to await future investigation.

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APPENDIX

This appendix summarizes the results on (oriented) Riemann surfaces (in the Schottky parametrization) which are required for the calculations of Sec. III. A more complete account of Riemann surfaces and their applications to string theory is in [20,22]. The results here were first obtained in the Schottky parametrization by Mandelstam [20].

We recall that a Riemann surface of genus g can be represented as the quotient of the extended complex plane $\mathbb{C}^\infty=\mathbb{C}\cup\{\infty\}$ by a finitely generated group G, where $G\subset \mathrm{SL}(2,\mathbb{C})$. G is the Schottky group, with generators T_R. The generators are often represented as

$$\frac{T_R z - z_{R+}}{T_R z - z_{R-}} = q_R \frac{z-z_{R+}}{z-z_{R-}} \ , \tag{A1}$$

where $z_{R\pm}$ are the fixed points of T_R, and q_R ($|q_R|<1$) is called the multiplier of T_R. The fundamental domain F of the Schottky group is given by the region exterior to $2g$ circles, which are isometric in pairs. The Rth pair of circles have radii

$$R_R=\frac{\sqrt{|q_R|}}{|1-q_R|}\,|z_{R+}-z_{R-}| \tag{A2}$$

and centers

$$J=\frac{z_R - q_R z_{R+}}{1-q_R} \ ,$$
$$J'=\frac{z_{R+}-q_R z_{R-}}{1-q_R} \ . \tag{A3}$$

The complex parameters $q_R, z_{R\pm}$ are chosen such that none of the circles are intersecting. With this definition, every point on $\mathbb{C}^\infty$ can be obtained from a point of F by an element of G, and no two points of F are related by an element of G. Thus it makes sense to take $\mathbb{C}^\infty/G$ as a representation of the surface. Such a representation exists for an arbitrary genus Riemann surface.

A Riemann surface of genus g has exactly g holomorphic differentials. This is because there are $2g$ independent noncontractible loops, and each one can be thought of as a dual to a harmonic one-form on the surface. The inner product between the loops and the one-forms is given by the contour integral of the one-form along the loop. A basis for the one-forms can be taken which consists of holomorphic and antiholomorphic one-forms (or differentials). Exactly half of them (i.e., g) are holomorphic. In the Schottky parametrization there is a natural choice of generators for the noncontractible loops. For the Rth pair of circles, one loop goes round one of the circles and the other goes from one circle to the other. The holomorphic differentials which are dual to this choice of basis are

$$\omega_R=\frac{1}{2\pi i}\sum_\alpha{}' \left[\frac{1}{z-T_\alpha z_{R+}}-\frac{1}{z-T_\alpha z_{R-}}\right]dz \ , \tag{A4}$$

where the sum goes over all the elements of G which do not have T_R^m as their rightmost factor, This is constructed so that it is G invariant. The sum converges to a well-defined differential on F. The integral of ω_R around z_{S+} gives δ_{RS}, using Cauchy's theorem. The integral of ω_R from a point on the $S-$ circle to its G image on the $S+$ circle is a quantity called the period matrix:

$$\Omega_{RS}=\int_{S-}^{S+}\omega_R \ . \tag{A5}$$

This has the property that it is symmetric, and has a positive imaginary part:

$$\begin{aligned}&\Omega_{RS}=\Omega_{SR} \ ,\\ &(\mathrm{Im}\Omega)_{RS}>0 \ .\end{aligned} \tag{A6}$$

In the Schottky parametrization, the period matrix takes the form [20]

$$\Omega_{\mathrm{RS}}=\frac{1}{2\pi i}\left[\delta_{\mathrm{RS}}\ln q_R-{}^{(R)}\sum_\alpha{}'^{(S)}\ln\frac{(z_{S+}-T_\alpha z_{R-})(z_{S-}-T_\alpha z_{R+})}{(z_{S+}-T_\alpha z_{R+})(z_{S-}-T_\alpha z_{R-})}\right] , \tag{A7}$$

where $T_\alpha \neq T_S^k(\ )T_R^m$, and for $R=S$ the sum does not include the unit element.

The holomorphic differentials and the period matrix are the basic ingredients from which analytic functions on the Riemann surface are formed. One first forms the combination

$$E(z,w)=(z-w)\prod_{\alpha\neq I}{}'\frac{(z-T_\alpha w)(w-T_\alpha z)}{(z-T_\alpha z)(w-T_\alpha w)} \ , \tag{A8}$$

where $\prod'_{\alpha\neq I}$ indicates that the product is restricted to one of each pair $(T_\alpha, T_\alpha^{-1})$, and the unit element is not included. $E(z,w)$ is called the prime form. Its main properties are that it is antisymmetric in (z,w) and is pseudoperiodic upon going around one of the basis loops. It has a simple zero at the point $z=w$ and no other zeros in F. The periodicity properties (around the loop going from $R-$ to $R+$) are expressed by

$$E(T_R z,w)=\frac{i}{C_R z+D_R}\times\exp\left[2\pi i\int_w^z\omega_R-\frac{1}{2}\pi i\Omega_{RR}\right]E(z,w) \ , \tag{A9}$$

while it is single-valued upon going around a loop encircling $R-$ or $R+$. Here, the quantities C_R and D_R are given by

$$T_R z=\frac{A_R z+B_R}{C_R z+D_R} \ . \tag{A10}$$

The quantity

$$F(z,w;z_1,z_2)=\frac{E(z,w)E(z_1,z_2)}{E(z,z_2)E(z_1,w)} \tag{A11}$$

has much simpler periodicity properties as z or w go from the $R-$ circle to the equivalent point on the $R+$ circle. It transforms as

$$F(T_R z,w;z_1,z_2)=\exp\left[-2\pi i\int_w^{z_2}\omega_R\right]F(z,w;z_1,z_2) \ , \tag{A12}$$

and is symmetric under the interchange $(z\leftrightarrow w)$, $(z_1\leftrightarrow z_2)$. We define the Green's function on the surface (with the point at ∞ removed) to be the unique (up to the addition of a constant) function $G(z,w)$ which is harmonic and symmetric in both variables and behaves like $-\ln|z-w|^2$ as $z\rightarrow w$, and like $-\ln|z|^2$ or $-\ln|w|^2$ as z or $w\rightarrow\infty$. It can be built out of the quantity $-\ln|F(z,w;z_1z_2)|^2$ [30], in the limit $z_1,z_2\rightarrow\infty$. (The arbitrariness in the way the limit is taken is just a constant addition to the Green's function.) In effect the extra image charge which is needed on a compact surface has been moved to ∞. There is an ambiguity with Green's functions on all compact surfaces which can be thought of as being due to the zero mode of the scalar Laplacian. It disappears when a point is removed from the surface.

We cannot just use $-\ln|F|^2$ as the Green's function, as this is not periodic by itself. A term must be added to make the Green's function periodic. However, this term is of the form holomorphic + antiholomorphic, so the Green's function remains harmonic everywhere on $F\times F$ (except at $z=w$). The Green's function is therefore given by

$$G(z,w) = -\ln\left|(z-w)\prod_{\alpha\neq I}{}' \frac{(z-T_\alpha w)(w-T_\alpha z)}{(z-T_\alpha\infty)(w-T_\alpha\infty)}\right|^2 + 4\pi\sum_{RS}\left[\mathrm{Im}\int_\infty^z \omega_R\right](\mathrm{Im}\Omega)^{-1}_{RS}\left[\mathrm{Im}\int_\infty^w \omega_S\right] . \tag{A13}$$

The expressions used in (3.3) and (3.2) for the period matrix and the Green's function are obtained from (A7) and (A13) as the leading behavior in the limit $|q_R|\to 0$ while z and w tend to any two of the $2g$ limit points $z_{R\pm}$. All the limit points $z_{R\pm}$ are kept fixed while this limit is taken.

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