Flavor symmetry and mass-splitting formulas for the sub-SU(3) Skyrmion in SU(N)

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New mass-splitting formulas for the octet and decuplet baryons are obtained in the sub-SU(3) Skyrme model in SU(N) with $N=3$ to 8 by the perturbative treatment. The higher-order-term contributions to the perturbative expansions are examined. We find that $N=5$ or 6 makes the mass spectrum most compatible with data. They are much better than the pure SU(3) model. Some useful Clebsch-Gordan coefficients of SU(N) with $N=3,4,5,6,7$, and 8 are calculated and listed.

I. INTRODUCTION

In recent years the Skyrme model[1] has aroused extensive interest as a phenomenological model, in which the baryons emerge as topological solitons (i.e., Skyrmions) in the current-algebra chiral Lagrangians (or the effective QCD Lagrangians[2]). Various properties of nucleons and Δ's have been deliberated in detail in the SU(2) Skyrme model by using collective-coordinate quantization of the hedgehog soliton, and they agree with observables reasonably well.[3] In other words, the Skyrme picture is fairly reliable in describing nonstrange baryons. The investigation has also been expanded to apply similar ideas to the strange-baryon case, i.e., to construct an SU(3) Skyrme model.[4–7] In contrast with the SU(2) case, however, there are two novelties. The first is, of course, that the basic flavor symmetry is SU(3) instead of SU(2), and in the second place a perturbative treatment for flavor-symmetry breaking in SU(3) is needed. The results show qualitative agreement with the experimental data, but they are not satisfactory.[5–7] One of the main deficiencies is that the mass-splitting formulas for the octet and decuplet baryons do not work well in either the absolute or relative scales, and in particular they cannot be improved by adjusting the model's parameters nor by adding any higher-derivative terms to the pristine model. In view of this difficulty, some authors[8,9] have proposed new methods or a new formalism to improve it. In these works the defects of the original model are attributed to the perturbative treatment of SU(3)-symmetry breaking, but the basic flavor symmetry, i.e., pure SU(3), of the model is not questioned. In our opinion, the flavor symmetry of the model may also be a fundamental issue, because there is no *a priori* reason to assume the flavor symmetry for the usual baryon octet to be purely SU(3) rather than an SU(3) subgroup in SU(N) with $N>3$ [sub-SU(3) in SU(N)]. Actually, since there exist at least three generations of quarks in the real world, the flavor symmetry for light quarks u, d, and s may well be sub-SU(3) instead of pure SU(3). Hence, we believe the idea of regarding the light-flavor baryons as topological solitons of the sub-SU(3)$\otimes$SU(3) chiral-symmetry sector in the SU(N)$\otimes$SU(N) effective Lagrangians (i.e., sub-SU(3) Skyrme model in SU(N)[10]) is worth exploring. In the present paper we investigate the mass-splitting formulas for baryons with strangeness in this model from $N=3$ to 8. At $N=3$, the results are the original mass-splitting formulas for Guadagnini.[4] For $N=5-6$, the mass spectra for octet and decuplet baryons are most compatible with data. For the $N<5$ and $N>6$ cases, the deviations become large. This result is interesting, because there are five known quark flavors. Including the expected top quark, there will be six. At this stage, we do not know whether it is purely accidental or the sub-SU(3) picture is closer to reality after all.

Since we restrict ourselves to the model's symmetry and the relative magnitude of the mass splitting, the perturbative method developed by Guadagnini,[4] Manohar,[5] Chemtob[6] and Praszalowicz[7] (GMCP method) for SU(3) breaking is employed in this paper. Even though this method is poor for calculating the absolute magnitude of the mass splitting,[6,7] it could still be expected to be an appropriate approximation for reflecting the relative mass splitting and symmetry of the model. The advantage of the GMCP method is that the analytic calculations can be done explicitly and the results can be shown as analytic (not numerical) expressions, which is convenient for discussing the symmetry. In addition, it is possible to estimate the higher-order perturbative corrections to the GMCP method by analytic calculations. We find that, due to the special structure of the perturbative Hamiltonian, the GMCP method seems to be self-consistent and suitable for our purpose. Of course, a similar study on the basis of nonperturbative methods is needed, but, it is out of the scope of this present paper and will be left to further investigations.

The quark-mass term $\mathcal{L}_m(N)$ for the model, which serves to break chiral SU(N) symmetry explicitly and relates to the sub-SU(3) Skyrmion mass splitting, can be obtained in the following considerations. In the ordinary

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chiral SU(3) Lagrangian theory for QCD,[11] the θ dependence in the context of strong interactions can be neglected and the quark mass parameters enter the dynamics through a quark mass term given by

$$\mathcal{L}_m(3)=\frac{1}{8}\frac{m_\pi^2F_\pi^2}{m_u+m_d}\mathrm{Tr}[M_q(U+U^\dagger-2)] , \tag{1}$$

where $U\in\mathrm{SU}(3)$ and the quark mass matrix $M_q=\mathrm{diag}(m_u,m_d,m_s)$. It is standard that $\mathcal{L}_m(3)$ be directly related to PCAC (partial conservation of axial-vector current) and to the breaking of SU(3) symmetry. In order to break chiral symmetry in the sub-SU(3) Skyrme model in SU(N) explicitly, a generalization of $\mathcal{L}_m(3)\rightarrow\mathcal{L}_m(N)$ is needed. The requirements for this generalization are that, as $N=3$, $\mathcal{L}_m(N)$ should go back to $\mathcal{L}_m(3)$, i.e., $\mathcal{L}_m(N=3)=\mathcal{L}_m(3)$, and the sub-SU(3) sector in $\mathcal{L}_m(N)$ should also have the same structure as $\mathcal{L}_m(3)$ given by Eq. (1). The simplest way to realize these is to set $U\in\mathrm{SU}(N)$ and $M_q=\mathrm{diag}(m_u,m_d,m_s,m_c,\dots,m_N)$, then $\mathcal{L}_m(3)$ of Eq. (1) becomes $\mathcal{L}_m(N)$. Thus, under the requirements mentioned above, a reasonable quark mass term in the sub-SU(3) model in SU(N) is obtained. It should be emphasized that, due to the fact that $m_c,m_b,\dots\gg m_u,m_d,m_s$ and F_π, the interpretations of pseudoscalar mesons as Goldstone bosons and of Skyrmions as baryons are only held in the sub-SU(3) configuration space of SU(N), and the heavy-flavor mesons and baryons cannot be treated by this picture.[10]

The contents of this paper are organized as follows. In Sec. II, we recapitulate the sub-SU(3) Skyrme model in SU(N). In Sec. III, new mass-splitting formulas for octet and decuplet Skyrmions in the sub-SU(3) Skyrme model in SU(N) with $N=4,\dots,8$ are presented. Finally, we discuss our results. In the Appendix, we provide the useful Clebsch-Gordan coefficients related to the octet and decuplet baryons states in the representations of SU(N) with $N=3,\dots,8$, which are needed in this paper. To the best of our knowledge, they have not yet been reported in the literature except for the cases of $N=3$ and 4.[12]

II. THE FORMALISM OF THE SUB-SU(3) SKYRME MODEL IN SU(N)

The effective chiral action for the SU(N) case is

$$\Gamma(U)=\int d^4x\,[\mathcal{L}_{\mathrm{sym}}(U,\partial U)+\mathcal{L}_m(N)]+N_c\Gamma_{\mathrm{WZ}}(U) , \tag{2}$$

in which

$$\mathcal{L}_{\mathrm{sym}}(U,\partial U)=\frac{F_\pi^2}{16}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)+\frac{1}{32e^2}\mathrm{Tr}[\partial_\mu UU^\dagger,\partial_\nu UU^\dagger]^2 \tag{3}$$

and

$$\mathcal{L}_m(N)=\frac{1}{8}\frac{m_\pi^2F_\pi^2}{m_u+m_d}\mathrm{Tr}[M_q(U+U^\dagger-2)] , \tag{4}$$

where $U(x)$ is an element of SU(N), which transforms as $U\rightarrow LUR^{-1}$ under an SU(N)$\otimes$SU(N) chiral transformation (L,R), and M_q is a diagonal $N\times N$ current-quark-mass matrix, while

$$\Gamma_{\mathrm{WZ}}(U)=\frac{1}{240\pi^2}\int_Q d\Sigma^{ijklm}\times\mathrm{Tr}(\partial_i UU^\dagger\partial_j UU^\dagger\partial_k UU^\dagger\partial_l UU^\dagger\partial_m UU^\dagger) \tag{5}$$

is the Wess-Zumino term which summarizes the effects of anomalies in current algebra. The Skyrme hedgehog ansatz is

$$U_0(\mathbf{x})=\exp\left[i\sum_{m=1}^{3}\lambda_m\cdot\hat{x}_mF(r)\right] , \tag{6}$$

where λ_m are SU(N) generators. The collective coordinates $A(t)$ can be introduced as

$$U(x)=A(t)U_0(\mathbf{x})A^\dagger(t),\quad A(t)\in\mathrm{SU}(N) . \tag{7}$$

We will treat $A(t)$ as quantum dynamical variables, and construct the baryon states by using the GMCP method. From Eqs. (2), (6), and (7), and ignoring the $\mathcal{L}_m(N)$ momentarily, we get the reduced form of the action in Eq. (2):

$$\Gamma(A)=\int_0^{2\pi}dt\,L(A,\dot{A}) , \tag{8}$$

where the Lagrangian $L(A,\dot{A})$ is

$$L(A,\dot{A})=\dot{\alpha}^ig_{ij}(\alpha)\dot{\alpha}^j+z_i(\alpha)\dot{\alpha}^i,\quad i,j=1,2,\dots,N^2-1 , \tag{9}$$

where α^i are standard coordinates of the SU(N). Denote the vielbein matrix of SU(N) as $C_{ij}(\alpha)$, then $A^\dagger dA=d\alpha^i(t)C_{ij}(\alpha)\lambda_j$ and $g_{ij}(\alpha)=C_{ik}(\alpha)G_{kl}C_{jl}(\alpha)$, where G_{kl} is a diagonal matrix independent of α^i. The elements are (hereafter, we use a subscript c to denote the components corresponding to the Cartan subalgebra generators, i.e., $c=3,8,\dots,N^2-1$)

$$G_{k,k}=-\frac{4\pi}{3}\frac{\Lambda}{e^3F_\pi},\quad k=1,2,3 ,$$
$$\Lambda=\int_0^\infty d\tilde{r}\,\tilde{r}^2\sin^2F\left[1+4\left[F'^2+\frac{\sin^2F}{\tilde{r}^2}\right]\right] ; \tag{10}$$

$$G_{c,c}=0\quad\text{except } c=3 ; \tag{11}$$

$$G_{c+i,c+i}=-\frac{\alpha}{e^3F_\pi},\quad i=1,2,3,4 ,$$
$$\alpha=\pi\int_0^\infty d\tilde{r}\,\tilde{r}^2(1-\cos F)\left[1+F'^2+\frac{2\sin^2F}{\tilde{r}^2}\right] ; \tag{12}$$

$$G_{k,k}=0\quad\text{for the others} ; \tag{13}$$

where $\tilde{r}=eF_\pi r$. After performing the integration in the Wess-Zumino action, we obtain the explicit expression for $z_i(\alpha)$:

$$z_i(\alpha)=iN_c\sum_{k=3}^{N}\left[\frac{2}{k(k-1)}\right]^{1/2}C_{i,k^2-1}(\alpha)\ . \tag{14}$$

$L(A,\dot{A})$ is invariant with respect to $A(t)\rightarrow LA(t)R^{-1}$, where $L\in SU(N)$ and $R\in SU(2)$ embedded in the upper subgroup of SU(N). After the quantization of collective coordinates, the symmetry transformations L and R will give the expected flavor and spin quantum numbers of baryons, respectively. Introducing the conjugate momenta

$$\pi_i=\frac{\partial L}{\partial\dot{\alpha}^i}=2g_{ij}(\alpha)\dot{\alpha}^j+z_i(\alpha)\ , \tag{15}$$

we can now write the Hamiltonian

$$\begin{aligned}&H=\pi_i\dot{\alpha}^i-L=-F_I G_{IJ}^{-1}F_J\ ,\\&I,J=1,2,3,c+1,c+2,c+3,c+4\ ,\\&c=3,8,\ldots,N^2-1\ ,\end{aligned} \tag{16}$$

where

$$F_i=\frac{-i}{2}C_{ij}^{-1}(\alpha)\pi_j,\quad i,j=1,2,\ldots,N^2-1\ . \tag{17}$$

This shows that some of the variables decouple from the dynamics. It arises from the fact that the actual configuration space of the $A(t)$ is not the whole SU(N) manifold, but is the left cosets of SU(N) elements defined up to right multiplication by the subgroup $\mathcal{H}$, i.e.,

$$\mathcal{H}=\exp\left[i\sum_{c=8,15,\ldots}\left[\alpha^c\lambda_c+\sum_{i=5,6,\ldots}\alpha^{c+i}\lambda_{c+i}\right]\right]\ ,$$

because $A(t)$ and $A(t)\mathcal{H}$ give the same $U(x)$ in Eq. (7). Alternatively, we can regard $A(t)$ as the whole SU(N) manifold and treat redundant degrees of freedom of $A(t)$ as constraints on baryon states. Defining $Y_{k^2-1}=\sqrt{2k(k-1)/9}\,F_{k^2-1}$, then $Y_c=1$, ($c=8,15,\ldots,N^2-1$). These are constraints imposed on allowed states. From the above explicit calculations we can see that the essential role of the Wess-Zumino action is selecting the allowed states. Substituting F_i [Eq. (17)] into Eq. (16), we get an explicit expression of the Hamiltonian:

$$H_0=e^3F_\pi\left[\frac{3}{4\pi\Lambda}\mathbf{J}^2+\frac{1}{\alpha}\left[C_2-\mathbf{J}^2-\sum_{k=3}^{N}\frac{9}{2k(k-1)}\right]\right]\ , \tag{18}$$

where $\mathbf{J}^2$ and C_2 are quadratic Casimir operators of SU(2) and SU(N), respectively. After considering the $\mathcal{L}_m(N)$ term, the total Hamiltonian including the classical soliton mass reads

$$H=M_{\rm cl}+H_0+\Delta M \tag{19}$$

in which H_0 is given by Eq. (18), while

$$M_{\rm cl}=\frac{\pi F_\pi}{2e}\int_0^\infty d\tilde{r}\left[\tilde{r}^2F'^2+2\sin^2F+4\sin^2F\left[\frac{\sin^2F}{\tilde{r}^2}+2F'^2\right]+2\mu_N^2\tilde{r}^2(1-\cos F)\right]\ , \tag{20}$$

$$\Delta M=\frac{m_\pi^2\pi}{e^3F_\pi}\frac{1}{m_u+m_d}\int_0^\infty d\tilde{r}\,\tilde{r}^2(1-\cos F)\sum_{l=2,\,k=3}^{N}\left[\frac{2}{k(k-1)}\right]^{1/2}\beta_{l^2-1}D^{(\rm ad)}_{l^2-1,k^2-1}(A)\ , \tag{21}$$

where $D^{(\rm ad)}_{\mu\nu}(A)$ denotes the regular adjoint representation of SU(N) and is characterized by the property $A^\dagger\lambda_\sigma A=D^{(\rm ad)}_{\sigma\rho}(A)\lambda_\rho$, and

$$\mu_N=\frac{m_\pi}{eF_\pi}\left[\frac{2}{N(m_u+m_d)}\right]^{1/2}\left[\sum_{i=1}^{N}m_i\right]^{1/2}\ , \tag{22}$$

$$\beta_{l^2-1}={\rm Tr}(\lambda_{l^2-1}M_q)\ . \tag{23}$$

Here, $m_1=m_u$, $m_2=m_d$, $m_3=m_s$, $m_4=m_c$, and so on. It is difficult to diagonalize H, but it is easy for H_0. The wave functions of the eigenstates of H_0 are $|{}^{\lambda}_{\mu\nu}\rangle=D^{(\lambda)}_{\mu\nu}(A)$, where $D^{(\lambda)}_{\mu\nu}(A)$ are the matrices of the SU(N) representation, $\mu=(I,I_z,Y_8,Y_{15},\ldots)$, and $\nu=(S,-S_z,1,1,\ldots,1)$. $|{}^{\lambda}_{\mu\nu}\rangle$ correspond to light-flavor baryon states with definite flavor and spin quantum numbers. In the sector of the light-flavor Skyrmions, ΔM will be treated perturbatively (i.e., GMCP method), and the higher-order terms' contributions to the perturbative expansions will be discussed in the following section. As to the heavy-flavor baryons, they are incompatible with the sub-SU(3) Skyrme picture. In fact, that the heavy-flavor baryons cannot be regarded as Skyrmions is a natural consequence of the breakdown of chiral symmetry.

III. MASS-SPLITTING FORMULAS FOR SUB-SU(3) SKYRMIONS WITH $N=4,\ldots,8$

The chiral-symmetry-breaking term given in Eq. (21) is

$$\Delta M(N)=\Delta M(u,d,s)+\Delta M(c,b,\ldots)\ , \tag{24}$$

in which

$$\begin{aligned}\Delta M(u,d,s)=&-m^*\sum_{k=3}^{N}\left[\frac{6}{k(k-1)}\right]^{1/2}D^{(\rm ad)}_{3,k^2-1}\\&-m\sum_{k=3}^{N}\left[\frac{6}{k(k-1)}\right]D^{(\rm ad)}_{8,k^2-1}\ ,\end{aligned} \tag{25}$$

$$\Delta M(c,b,\ldots)=\tilde{m}\sum_{l=4,\,k=3}^{N}\left[\frac{2}{k(k-1)}\right]^{1/2}\times{\rm Tr}(\lambda_{l^2-1}M_q)D^{(\rm ad)}_{l^2-1,k^2-1}\ , \tag{26}$$

and

$$m^*=\frac{m_d-m_u}{m_u+m_d}\frac{\pi m_\pi^2}{\sqrt{3}e^3F_\pi}\int_0^\infty d\tilde{r}\,\tilde{r}^2(1-\cos F)\ , \tag{27}$$

$$m = \frac{2m_s - m_u - m_d}{m_u + m_d} \frac{\pi m_\pi^2}{3e^3 F_\pi} \int_0^\infty d\tilde{r}\, \tilde{r}^2 (1-\cos F) , \tag{28}$$

and

$$\tilde{m} = \frac{1}{m_u + m_d} \frac{\pi m_\pi^2}{e^3 F_\pi} \int_0^\infty d\tilde{r}\, \tilde{r}^2 (1-\cos F) . \tag{29}$$

Introducing

$$\bar{H}_0 = M_{\rm cl} + H_0 + \Delta M(c,b,\ldots) , \tag{30}$$

then Eq. (19) becomes

$$H = \bar{H}_0 + \Delta M(u,d,s) . \tag{31}$$

For the case $|{}^{\lambda}_{\mu,\nu}\rangle \in$ octet of the sub-SU(3) in SU(N), using the standard formula

$$\langle {}^{\lambda_1}_{\mu_1\nu_1} | D^{(\lambda_2)*}_{\mu_2\nu_2} | {}^{\lambda_3}_{\mu_3\nu_3} \rangle = \sum_{\gamma,i,j} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3^\gamma \\ \mu_1 & \mu_2 & \mu_3^i \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3^\gamma \\ \nu_1 & \nu_2 & \nu_3^j \end{bmatrix} \tag{32}$$

(note that a normalization factor in the RHS of Eq. (32) has been taken to be 1, owing to the fact that we will only calculate diagonal matrix elements below) and the general properties of the SU(N) Clebsch-Gordan coefficients, we find

$$\langle {}^{\lambda}_{\mu\nu} | \bar{H}_0 | {}^{\lambda}_{\mu\nu} \rangle = M_8(N) . \tag{33}$$

We emphasize that $M_8(N)$ is a constant independent of μ and ν. Thus, $\Delta M(c,b,\ldots)$ can be absorbed into $\bar{H}_0$, which serves as the starting point in the perturbative approach. Accordingly, we can use $\Delta M(u,d,s)$ to calculate the mass splitting of the octet baryons perturbatively. For the case $|{}^{\lambda'}_{\mu'\nu'}\rangle \in$ the decuplet baryons of the sub-SU(3) in SU(N), the situation is the same as above; i.e., we have

$$\langle {}^{\lambda'}_{\mu'\nu'} | \bar{H}_0 | {}^{\lambda'}_{\mu'\nu'} \rangle = M_{10}(N) . \tag{34}$$

Neglecting the isospin-breaking part in the $\Delta M(u,d,s)$ and using Eq. (32) and the SU(N) ($N=3,\ldots,8$) Clebsch-Gordan coefficients given in the Appendix, we obtain

$$\begin{aligned} M_N &= M_8(N) - \frac{N}{\Delta(N)} m , \\ M_\Lambda &= M_8(N) - \frac{1}{\Delta(N)} m , \\ M_\Sigma &= M_8(N) + \frac{1}{\Delta(N)} m , \\ M_\Xi &= M_8(N) + \frac{N-1}{\Delta(N)} m , \end{aligned} \tag{35}$$

and

$$\begin{aligned} M_\Delta &= M_{10}(N) - \frac{1}{\Gamma(N)} m , \\ M_{\Sigma^*} &= M_{10}(N) , \\ M_{\Xi^*} &= M_{10}(N) + \frac{1}{\Gamma(N)} m , \\ M_\Omega &= M_{10}(N) + \frac{2}{\Gamma(N)} m , \end{aligned} \tag{36}$$

in which $\Delta(N)$ and $\Gamma(N)$ are N dependent. After a lengthy but direct calculation (see Appendix), we obtain

$$\begin{aligned} &\Delta(3)=10, \quad \Gamma(3)=8 , \\ &\Delta(4)=48, \quad \Gamma(4)=21 , \\ &\Delta(5)=\tfrac{105}{2}, \quad \Gamma(5)=16 , \\ &\Delta(6)=\tfrac{320}{3}, \quad \Gamma(6)=25 , \\ &\Delta(7)=\tfrac{1575}{13}, \quad \Gamma(7)=\tfrac{300}{13} , \\ &\Delta(8)=\tfrac{16800}{89}, \quad \Gamma(8)=\tfrac{2695}{89} . \end{aligned} \tag{37}$$

For the $N=3$ case, the results are the same as those computed in Refs. 4–6. The usual SU(3) Gell-Mann–Okubo relations are held for $N=3,\ldots,8$, i.e.,

$$2(M_N + M_\Xi) = 3M_\Lambda + M_\Sigma , \tag{38}$$

$$M_\Delta - M_{\Sigma^*} = M_{\Sigma^*} - M_{\Xi^*} = M_{\Xi^*} - M_\Omega . \tag{39}$$

In order to see the difference between the sub-SU(3) model and the ordinary or pure SU(3) model, we take M_N and M_Ξ as input, then compute M_Σ, M_Λ, and $(M_\Omega - M_\Delta)/3$ for each case with Eqs. (35)–(37). The results are listed in Table I. In view of the fact that Gell-Mann–Okubo relations in Eq. (39) are satisfied in this model, we do not tabulate the results related to M_{Σ^*} and M_{Ξ^*}.

In Fig. 1 we visually display the relations between the masses M_Σ, M_Λ, and $(M_\Omega - M_\Delta)/3$ and the flavor number N.

The isospin-violating part of $\Delta M(u,d,s)$ is the first term in Eq. (25). Using the symbol $\Delta_{YI} = M_{YI,I_3} - M_{YI,I_3+1}$, we have

$$\Delta_N(N) = \frac{2(N-2)}{\Delta(N)\sqrt{3}} m^* , \tag{40}$$

$$\Delta_\Sigma(N) = \frac{2N-1}{\Delta(N)\sqrt{3}} m^* , \tag{41}$$

$$\Delta_\Xi(N) = \frac{2(N+1)}{\Delta(N)\sqrt{3}} m^* , \tag{42}$$

TABLE I. The prediction of M_Λ, M_Σ, and $(M_\Omega - M_\Delta)/3$ in the sub-SU(3) Skyrme model with $N=3,\ldots,8$.

Expt. (MeV)	Pure SU(3) (Ref. 4)	With $N=4$	With $N=5$	Sub-SU(3) With $N=6$	With $N=7$	With $N=8$
$M_N=939$	939 (input)	939 (input)	939 (input)	Input	Input	Input
$M_\Xi=1318$	1318 (input)	1318 (input)	1318 (input)	Input	Input	Input
$M_\Lambda=1115.6$	1090.6	1101.5	1107.5	1111.2	1113.9	1115.8
$M_\Sigma=1193.05$	1242.2	1209.7	1191.7	1180.2	1172.3	1166.4
$(M_\Omega - M_\Delta)/3=146.7$	94.75	123.8	138.2	147	153.1	157.4

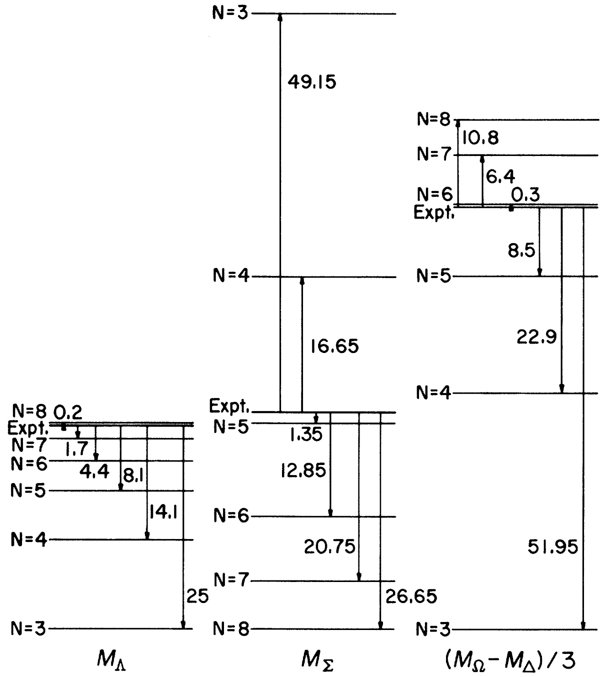


FIG. 1. The relative mass differences between the predictions of the sub-SU(3) Skyrmions with $N=3,4,\ldots,8$ and the experimental data for masses of Λ, Σ, and $(M_\Omega - M_\Delta)/3$. M_N and M_Ξ are as input and the unit is MeV.

$$\Delta_\Delta(N)=\Delta_{\Sigma^*}(N)=\Delta_{\Xi^*}(N)=\frac{2}{\Gamma(N)\sqrt{3}}m^* , \tag{43}$$

in which $\Delta(N)$ and $\Gamma(N)$ are given by Eq. (37). Equations (40)–(43) show $\Delta_N < \Delta_\Sigma < \Delta_\Xi$, which is compatible with experimental data qualitatively. Given the above relations, we again get the usual SU(3) Coleman-Glashow relations for $N=3,\ldots,8$;

$$M_n - M_p + M_{\Xi^-} - M_{\Xi^0} = M_{\Sigma^-} - M_{\Sigma^+} . \tag{44}$$

At this point a natural question is raised: which N will make the sub-SU(3) Skyrme model the closest to the physical picture? It is essential to make clear which N makes the mass spectrum for octet and decuplet baryons the best compared to experiments before evaluating other static properties of them in detail. Comparing our results obtained above with the experimental data, which are visually displayed in Fig. 1, we get the answer: $N=5$ or 6. This is the main result of our investigations of this paper, which seems to be rather interesting and compatible with the fact that there exist three known generations of quarks in the real world.

The mass spectrum of the skyrmions comes from $M_{\rm cl}+H_0+\Delta M$ [Eq. (19)]. In the above, the problem has been investigated by the GMCP method, in which H_0 serves as the unperturbed Hamiltonian, ΔM as the perturbative part and only the first-order term of the perturbative expansion were considered. In order to ensure that this method is legitimate mathematically, we should examine the contributions of the higher-order perturbative terms. For definiteness, we restrict ourselves to the $N=3$ case, and, as usual, we will only concern the mass splitting between the s flavor and u,d flavors. Then, $\Delta M(u,d,s)$, up to a coefficient independent of A, is $D^{(8)}_{8,8}(A)$. In Refs. 4–7 the first-order perturbation of the mass splitting has been calculated. Its second-order perturbation is

$$E_k^{(2)} = \sum_{n\neq k}{}' \frac{|\langle n|\Delta M|k\rangle|^2}{E_k^{(0)}-E_n^{(0)}} . \tag{45}$$

It is related to the nondiagonal matrix elements of $D^{(8)}_{8,8}(A)$ between the baryon states. For the baryon octet, the basic wave functions are $D^{(8)}_{\mu,\nu}(A)$, where

$$\mu=\begin{bmatrix} I\ Y \\ I_z \end{bmatrix}$$

and

$$\nu=\begin{bmatrix} \frac{1}{2}\ 1 \\ -S_z \end{bmatrix} .$$

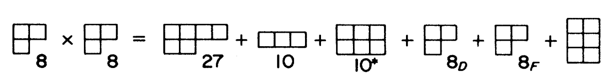
The nondiagonal matrix elements are

$$\langle D^{(8)}_{\mu\nu}|D^{(8)}_{8,8}|D^{(\lambda')}_{\mu'\nu'}\rangle = c\begin{bmatrix} 8 & 8 & \lambda'_i \\ \begin{bmatrix} I\ Y \\ I_z \end{bmatrix} & \begin{bmatrix} 0\ 0 \\ 0 \end{bmatrix} & \mu' \end{bmatrix}\begin{bmatrix} 8 & 8 & \lambda'_i \\ \begin{bmatrix} \frac{1}{2}\ 1 \\ -S_z \end{bmatrix} & \begin{bmatrix} 0\ 0 \\ 0 \end{bmatrix} & \nu' \end{bmatrix} . \tag{46}$$

where c is a constant related to normalizations. We find

$$\langle D^{(8)}_{\mu\nu}|D^{8}_{8,8}|D^{(\lambda')}_{\mu'\nu'}\rangle = 0, \quad \text{unless } \mu'=\begin{bmatrix} I\ Y \\ I_z \end{bmatrix}, \ \nu'=\begin{bmatrix} \frac{1}{2}\ 1 \\ -S_z \end{bmatrix} . \tag{47}$$

If one notices the multiplets on the right side of



one will find that

$$\nu'=\begin{bmatrix} \frac{1}{2}\ 1 \\ -S_z \end{bmatrix}$$

is impossible, because six-quark states (27-plet and 10*-plet) cannot have spin of $\frac{1}{2}$, and the decuplet baryon's spin is $\frac{3}{2}$. Consequently, the second-order perturbation vanishes, and the higher-order perturbations also have no contributions. For the baryon decuplet, the situation is the same as the octet case. This result means that once the whole Hamiltonian [Eq. (19)] is divided into the basic part (H_0) and the perturbative part ($H'=\Delta M$), the GMCP method is legitimate. So we employ this method to calculate the relative mass splitting and to discuss the model's symmetry.

IV. SUMMARY AND DISCUSSION

By employing the sub-SU(3) Skyrme model in SU(N), we have obtained new mass-splitting formulas for octet and decuplet baryons for $N=4,\ldots,8$. We find that $N=5$–6 makes the mass spectrum of them most compatible with data. This implies that in the Skyrmion picture the basic flavor symmetry for light-quark (u, d, and s) baryons could be the SU(3) subgroup in SU(5) or SU(6) instead of pure SU(3).

The starting point of our investigation is based upon the symmetry considerations of the Skyrme model and the dynamics of the heavy flavor quarks are not considered. So, the heavy-flavor mass parameters (e.g., $m_c, m_b, \ldots$) emerged in the above do not possess the real mass meaning in the ordinary heavy-flavor quark dynamics. They are only the model's parameters and serve to reflect the influence of the heavy flavors on the usual Skyrmions. As we fit experimental data, they are absorbed by the input masses. It is an open question to explore the meaning of these parameters in heavy-quark dynamics.

In the present paper, we concentrated on the mass-splitting pattern of the baryon octet and decuplet. Other static properties such as the magnetic moments and the F/D ratio will likely serve as a crucial test. We will study them next to further check the validity of the picture given here.

We note that the group-theoretical calculations such as the F/D ratios, $g_{\pi NN}/g_{\pi N\Delta}$, and the $\bar{s}s$ content in the nucleon have been considered as a function of N_c and the large-N_c limit.[5,13] It would be interesting to extend the present framework of embedding the flavor SU(3) to a larger unitary group to include the large N_c analysis as in Refs. 5 and 13.

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APPENDIX: SU(N) ($N=3,\ldots,8$) CLEBSCH-GORDAN COEFFICIENTS

In this Appendix we provide some Clebsch-Gordan (CG) coefficients of SU(N) ($N=3,\ldots,8$) which are useful in this paper. First, we give an outline of the computational method for calculating the coefficients. Then we list some CG coefficients in tables. The first problem for computing CG coefficients is the choice of bases for representations. In particle physics, the Gel'fand basis has been used extensively. The Gel'fand basis[14] of SU(N) is an irreducible basis adapted to the canonical subgroup chain

$$\begin{array}{ccccc} \mathrm{SU}(n) \supset & \mathrm{SU}(n-1)\supset & \cdots & \mathrm{SU}(2)\supset & \mathrm{SU}(1) \\ |[m_{i,n}], & [m_{i,n-1}], & ,\ldots, & [m_{i,2}], & [m_{i,1}]\rangle\ , \end{array}$$

where

$$[m_{i,k}]=[m_{1k},m_{2k},\ldots,m_{kk}]$$

is the partition specifying that the Gel'fand basis vector belongs to the irrep$[m_{ik}]$ of SU(k). We usually use the Gel'fand symbol

$$\begin{vmatrix} [\nu] \\ (m) \end{vmatrix} \equiv \begin{vmatrix} m_{1n}m_{2n}m_{3n}\cdots m_{n-2\,n}m_{n-1\,n}m_{nn} \\ m_{1\,n-1}\cdots m_{n-1\,n-1} \\ \cdots \\ m_{12}m_{22} \\ m_{11} \end{vmatrix}$$

to label a Gel'fand basis vector. We can also use the Weyl tableau to label it. The one to one correspondence between the Gel'fand symbol and the Weyl tableau is realized in the following way:

$$\begin{pmatrix} [\nu] \\ (m) \end{pmatrix} = \begin{array}{|c|c|c|c|} \hline f_{11}\ \ 1\text{'s} & f_{12}\ \ 2\text{'s} & \cdots & f_{1n}\ \ n\text{'s} \\ \hline f_{22}\ \ 2\text{'s} & f_{23}\ \ 3\text{'s} & \cdots & f_{2n}\ \ n\text{'s} \\ \hline \multicolumn{2}{|c|}{\cdots\cdots} \\ \cline{1-2} f_{nn}\ \ n\text{'s} \\ \cline{1-1} \end{array}$$

where $f_{1k}=m_{1k}-m_{1\,k-1}$, $f_{2k}=m_{2k}-m_{2\,k-1},\ldots$, $f_{k-1\,k}=m_{k-1\,k}-m_{k-1\,k-1}$, $f_{kk}=m_{kk}$, and $1,2,\ldots,n$ in the boxes of the above Young diagram is a set of ordered single-particle states. The Gel'fand basis is a multiplicity-free and orthonormal basis; namely, it is complete enough to uniquely identify each state in an irrep. Above all, the most remarkable feature is that there are algebraic expressions of matrix elements of the SU(n) generators between Gel'fand bases. In fact, Baird and Biedenharn[15] have given those expressions. We duplicate their results as follows:

$$E_{n,n-1}\begin{vmatrix} m_{1n}m_{2n}m_{3n}\cdots m_{n-2\,n}m_{n-1\,n}m_{nn} \\ m_{1\,n-1}\cdots m_{i\,n-1}\cdots m_{n-1\,n-1} \\ \cdots \\ m_{12}m_{22} \\ m_{11} \end{vmatrix}$$

$$=\sum_{i=1}^{n-1}\left|\frac{\prod_{j=1}^{n-2}(m_{j,n-2}-m_{i,n-1}-j+i)}{\prod_{j=1,j\neq i}^{n-1}(m_{j,n-1}-m_{i,n-1}-j+i)}\right|^{1/2}$$

$$\times\left[\frac{(-1)\prod_{j=1}^{n}(m_{j,n}-m_{i,n-1}-j+i+1)}{\prod_{j=1,j\neq i}^{n-1}(m_{j,n-1}-m_{i,n-1}-j+i+1)}\right]^{1/2}$$

$$\times\left(\begin{matrix} m_{1n}m_{2n}m_{3n}\cdots m_{n-2\,n}m_{n-1\,n}m_{nn} \\ m_{1\,n-1}\cdots m_{i\,n-1}-1\cdots m_{n-1\,n-1} \\ \cdots \\ m_{12}m_{22} \\ m_{11} \end{matrix}\right), \quad (48)$$

where $E_{n,n-1}$ is one of the SU(n) generators. An SU(n) has n^2-1 generators E_{ij}, which in the fundamental representation are

$$(E_{ij})_{\alpha\beta}=\delta_{i\alpha}\delta_{j\beta}-\frac{1}{n}\delta_{ij}\delta_{\alpha\beta}\ ,$$

The commutation relations of these generators E_{ij} are

$$[E_{ij},E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj}\ .$$

The formula of Eq. (48) is very useful in practical calculation. It also embodies the sign convention for SU(N) generators: namely, the matrix elements of $E_{n,n-1}$ are defined to be non-negative. The full SU(N) CG coefficient can be constructed with SU(N) isoscalar factors and SU($N-1$) CG coefficients

$$C_{\text{CG}}[\text{SU}(N)]=\left[\begin{matrix}\nu_1 & \nu_2 \\ \mu_1 Z_1 & \mu_2 Z_2\end{matrix}\middle|\begin{matrix}\nu \\ \mu Z\end{matrix}\right]C_{\text{CG}}[\text{SU}(N-1)]\ .$$

In the following, we only give isoscalar factors for SU(N) ($N=2,\ldots,7$). Using Tables II and III and familiar SU(2) CG coefficients, we can easily obtain the complete SU(N) ($N=3,\ldots,8$) CG coefficients we need. Now, we make some necessary explanation for using the tables. We only discuss the $N=4$ case, and the same consideration can be carried out for $N>4$. Each state in an SU(4) irrep is usually labeled by a set of quantum numbers, the isospin I and the z component I_z, the hypercharge Y, and another quantum number Z, which are related to quark numbers of u, d, s, and c:

$$I_z=(n_u-n_d)/2,\quad Y=(n_u+n_d-2n_s)/3\ ,$$

$$Z=(n_u+n_d+n_s-3n_c)/4\ .$$

In an SU(4) irrep we define a lexical ordering for the basis vectors by considering the vectors

$$m=(Z,Y,I,I_z)\ .$$

If the first nonvanishing component of $m-m'$ is positive, then the basis vector $\binom{[\nu]}{m}$ is higher than $\binom{[\nu]}{m'}$. Between the SU(4) irreps, the higher is defined to be the one that contains the SU(3) submultiplet with the highest Z. We only tabulate the coefficients for $[\nu_1]\geq[\nu_2]$. As for $[\nu_1]<[\nu_2]$ cases, we take the phase convention

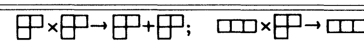
$$\left[\begin{matrix}\nu_2 & \nu_1 \\ \mu_2 Z_2 & \mu_1 Z_1\end{matrix}\middle|\begin{matrix}\nu \\ \mu Z\end{matrix}\right]=\left[\begin{matrix}\nu_1 & \nu_2 \\ \mu_1 Z_1 & \mu_2 Z_2\end{matrix}\middle|\begin{matrix}\nu \\ \mu Z\end{matrix}\right].$$

In the following, we tabulate the values of C_2 (Casimir operators) and the SU(N)-isoscalar factors for $N=3,\ldots,8$.

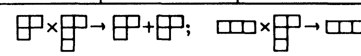
TABLE II. Values of quadratic Casimir operator C_2 in SU(N) ($N=3,4,5,6,7,8$) irreps.

C_2	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$	$SU(7)$	$SU(8)$
	3	39/8	33/5	33/4	69/7	183/16
	6	63/8	48/5	45/4	90/7	231/16

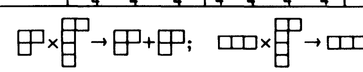
TABLE III. SU($N-1$) content of SU(N) ($N=3,\ldots,8$) irreps.

N=3 

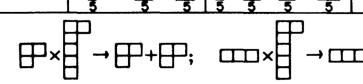
Irrep of $SU(3)$	8	10
$SU(2)$ submultiplets	2 3+1 2	4 3 2 1
Y	1 0 −1	1 0 −1 −2

N=4 

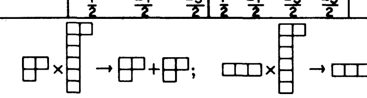
Irrep of $SU(4)$	20	20	15
$SU(3)$ submultiplets	8 6+3 3	10 6 3 1	3 8+1 3*
Z	$\frac{3}{4}$ $\frac{-1}{4}$ $\frac{-5}{4}$	$\frac{3}{4}$ $\frac{-1}{4}$ $\frac{-5}{4}$ $\frac{-9}{4}$	1 0 −1

N=5 

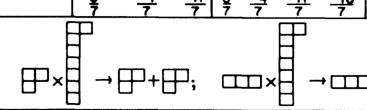
Irrep of $SU(5)$	40	35	24
$SU(4)$ submultiplets	20 10+6 4	20 10 4 1	4 15+1 4*
$Z^{(1)}$	$\frac{3}{5}$ $\frac{-2}{5}$ $\frac{-7}{5}$	$\frac{3}{5}$ $\frac{-2}{5}$ $\frac{-7}{5}$ $\frac{-12}{5}$	1 0 −1

N=6 

Irrep of $SU(6)$	70	56	35
$SU(5)$ submultiplets	40 15+10 5	35 15 5 1	5 24+1 5*
$Z^{(2)}$	$\frac{1}{2}$ $\frac{-1}{2}$ $\frac{-3}{2}$	$\frac{1}{2}$ $\frac{-1}{2}$ $\frac{-3}{2}$ $\frac{-5}{2}$	1 0 −1

N=7 

Irrep of $SU(7)$	112	84	48
$SU(6)$ submultiplets	70 21 + 15 6	35 15 5 1	6 35+1 6*
$Z^{(3)}$	$\frac{3}{7}$ $\frac{-4}{7}$ $\frac{-11}{7}$	$\frac{3}{7}$ $\frac{-4}{7}$ $\frac{-11}{7}$ $\frac{-18}{7}$	1 0 −1

N=8 

Irrep of $SU(8)$	168	120	63
$SU(7)$ submultiplets	112 28+21 7	84 28 7 1	7 48+1 7*
$Z^{(4)}$	$\frac{3}{8}$ $\frac{-5}{8}$ $\frac{-13}{8}$	$\frac{3}{8}$ $\frac{-5}{8}$ $\frac{-13}{8}$ $\frac{-21}{8}$	1 0 −1

TABLE IV. SU(N) ($N=3,\ldots,8$) isoscalar factors.

$N=3.$	(a):	(1) $8\times 8\to 8_D+8_F$;	(2) $10\times 8\to 10$.
$N=4.$	(b):	(1) $15\times 20\to 20_D+20_F$;	(2) $15\times 20'\to 20'$.
$N=5.$	(c):	(1) $24\times 40\to 40_D+40_F$;	(2) $24\times 35\to 35$.
$N=6.$	(d):	(1) $35\times 70\to 70_D+70_F$;	(2) $35\times 56\to 56$.
$N=7.$	(e):	(1) $48\times 112\to 112_D+112_F$;	(2) $48\times 84\to 84$.
$N=8.$	(f):	(1) $63\times 168\to 168_D+168_F$;	(2) $63\times 120\to 120$.

(a)

$$(1)\quad \left[\begin{array}{cc|c} 8 & 8 & 8^i \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y \end{array}\right]$$

$(\lambda Y)=(2,1)$

$\lambda_1 Y_1$;	$\lambda_2 Y_2$	8_D	8_F
2,1;	1,0	$-1/\sqrt{20}$	1/2
2,1;	3,0	$3/\sqrt{20}$	1/2
1,0;	2,1	$-1/\sqrt{20}$	$-1/2$
3,0;	2,1	$3/\sqrt{20}$	$-1/2$

$(\lambda Y)=(3,0)$

$\lambda_1 Y_1$;	$\lambda_2 Y_2$	8_D	8_F
2,1;	2, −1	$-\sqrt{3/10}$	$1/\sqrt{6}$
1,0;	3,0	$1/\sqrt{5}$	0
3,0;	1,0	$1/\sqrt{5}$	0
3,0;	3,0	0	$-\sqrt{2/3}$
2, −1;	2,1	$-\sqrt{3/10}$	$-\sqrt{1/6}$

$(\lambda Y)=(1,0)$

$\lambda_1 Y_1$;	$\lambda_2 Y_2$	8_D	8_F
2,1;	2, −1	$-1/\sqrt{10}$	$-1/\sqrt{2}$
1,0;	1,0	$-1/\sqrt{5}$	0
3,0;	3,0	$-\sqrt{3/5}$	0
2, −1;	2,1	$1/\sqrt{10}$	$-1/\sqrt{2}$

$(\lambda Y)=(2,-1)$

$\lambda_1 Y_1$;	$\lambda_2 Y_2$	8_D	8_F
1,0;	2, −1	$-1/\sqrt{20}$	1/2
3,0;	2, −1	$-3/\sqrt{20}$	$-1/2$
2, −1;	1,0	$-1/\sqrt{20}$	$-1/2$
2, −1;	3,0	$-3/\sqrt{20}$	1/2

$$(2)\quad \left[\begin{array}{cc|c} 10 & 8 & 10 \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y \end{array}\right]$$

$(\lambda Y)=(4,1)$

$\lambda_1 Y_1$;	$\lambda_2 Y_2$	10
4,1;	1,0	$-1/\sqrt{8}$
4,1;	3,0	$\sqrt{5/8}$
3,0;	2,1	1/2

$(\lambda Y)=(3,0)$

$\lambda_1 Y_1$;	$\lambda_2 Y_2$	10
4,1;	2, −1	$1/\sqrt{3}$
3,0;	1,0	0
3,0;	3,0	$1/\sqrt{3}$
2, −1;	2,1	$1/\sqrt{3}$

$(\lambda Y)=(2,-1)$

$\lambda_1 Y_1$;	$\lambda_2 Y_2$	10
3,0;	2, −1	$1/\sqrt{2}$
2, −1;	1,0	$1/(2\sqrt{2})$
2, −1;	3,0	$-1/(2\sqrt{2})$
1, −2;	2,1	1/2

$(\lambda Y)=(1,-2)$

$\lambda_1 Y_1$;	$\lambda_2 Y_2$	10
2, −1;	2, −1	$1/\sqrt{2}$
1, −2;	1,0	$1/\sqrt{2}$

(b)

$$(1)\quad \left[\begin{array}{ccc} 15 & 20 & 20_i \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{array}\right]$$

$(\mu Z)=(8,3/4)$

$\mu_1 Z_1$;	$\mu_2 Z_2$	20_D	20_F
3,1;	3*, −1/4	$\sqrt{2/13}$	$\frac{-3}{4}\sqrt{3/26}$
3,1	6, −1/4	$\sqrt{2/13}$	$17/\sqrt{1248}$
1,0;	8,3/4	$-\sqrt{1/13}$	$-1/(2\sqrt{39})$
(8,0;	8,3/4)$_D$	0	$\frac{-13}{4}\sqrt{5/78}$
(8,0;	8,3/4)$_F$	$-2\sqrt{2/13}$	$\frac{5}{4}\sqrt{1/78}$

$$(2)\quad \left[\begin{array}{ccc} 15 & 20' & 20' \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{array}\right]$$

$(\mu Z)=(10,3/4)$

$\mu_1 Z_1$;	$\mu_2 Z_2$	$20'$
3,1;	6, −1/4	$2/\sqrt{21}$
1,0;	10,3/4	$-1/\sqrt{21}$
8,0;	10,3/4	$-4/\sqrt{21}$

TABLE IV. (*Continued*).

(c)

(1) $\left[\begin{array}{ccc} 24 & 40 & 40_i \\ \mu_1^{(1)}\mathbf{Z}_1^{(1)} & \mu_2^{(1)}\mathbf{Z}_2^{(1)} & \mu^{(1)}\mathbf{Z}^{(1)} \end{array}\right]$

$(\mu^{(1)}\mathbf{Z}^{(1)})=(20,3/5)$

$\mu_1^{(1)}\mathbf{Z}_1^{(1)}$;	$\mu_2^{(1)}\mathbf{Z}_2^{(1)}$	40_D	40_F
4,1;	$6,-2/5$	$-\sqrt{1/10}$	$4/\sqrt{210}$
4,1;	$10,-2/5$	$\sqrt{1/10}$	$6/\sqrt{210}$
1,0;	20,3/5	0	$-1/\sqrt{28}$
(15,0;	$20,3/5)_D$	$2/\sqrt{65}$	$-61/\sqrt{5460}$
(15,0;	$20,3/5)_F$	$-4\sqrt{3/65}$	$-4/\sqrt{455}$

(2) $\left[\begin{array}{ccc} 24 & 35 & 35 \\ \mu_1^{(1)} & \mu_2^{(1)}\mathbf{Z}_2^{(1)} & \mu^{(1)}\mathbf{Z}^{(1)} \end{array}\right]$

$(\mu^{(1)}\mathbf{Z}^{(1)})=(20',3/5)$

$\mu_1^{(1)}\mathbf{Z}_1^{(1)}$;	$\mu_2^{(1)}\mathbf{Z}_2^{(1)}$	35
4,1;	$10,-2/5$	$\sqrt{5/32}$
1,0;	20,3/5	$-\sqrt{3/128}$
15,0;	20,3/5	$-\sqrt{105/128}$

(d)

(1) $\left[\begin{array}{cc|c} 35 & 70 & 70i \\ \mu_1^{(2)}\mathbf{Z}_1^{(2)} & \mu_2^{(2)}\mathbf{Z}_2^{(2)} & \mu^{(2)}\mathbf{Z}^{(2)} \end{array}\right]$

$(\mu^{(2)}\mathbf{Z}^{(2)})=(40,1/2)$

$\mu_1^{(2)}\mathbf{Z}_1^{(2)}$;	$\mu_2^{(2)}\mathbf{Z}_2^{(2)}$	70_D	70_F
5,1;	$10,-1/2$	$-1/\sqrt{12}$	$5/(8\sqrt{6})$
5,1;	$15,-1/2$	$1/\sqrt{12}$	$7/(8\sqrt{6})$
1,0;	40,1/2	0	$-3/\sqrt{480}$
(24,0;	$40,1/2)_D$	$-\sqrt{5/6}$	$1/\sqrt{960}$
(24,0;	$40,1/2)_F$	0	$-3\sqrt{7/80}$

(2) $\left[\begin{array}{cc|c} 35 & 56 & 56 \\ \mu_1^{(2)}\mathbf{Z}_1^{(2)} & \mu_2^{(2)}\mathbf{Z}_2^{(2)} & \mu^{(2)}\mathbf{Z}^{(2)} \end{array}\right]$

$(\mu^{(2)}\mathbf{Z}^{(2)})=(35,1/2)$

$\mu_1^{(2)}\mathbf{Z}_1^{(2)}$;	$\mu_2^{(2)}\mathbf{Z}_2^{(2)}$	56
5,1;	$15,-1/2$	$2/\sqrt{30}$
1,0;	35,1/2	$-1/(5\sqrt{3})$
24,0;	35,1/2	$-8/(5\sqrt{3})$

(e)

(1) $\left[\begin{array}{cc|c} 48 & 112 & 112i \\ \mu_1^{(3)}\mathbf{Z}_1^{(3)} & \mu_2^{(3)}\mathbf{Z}_2^{(3)} & \mu^{(3)}\mathbf{Z}^{(3)} \end{array}\right]$

$(\mu^{(3)}\mathbf{Z}^{(3)})=(70,3/7)$

$\mu_1^{(3)}\mathbf{Z}_1^{(3)}$;	$\mu_2^{(3)}\mathbf{Z}_2^{(3)}$	112_D	112_F
6,1;	$15,-4/7$	$-1/\sqrt{14}$	$\sqrt{2/35}$
6,1;	$21,-4/7$	$1/\sqrt{14}$	$\frac{4}{3}\sqrt{2/35}$
1,0;	70,3/7	0	$-1/(3\sqrt{10})$
(35,0;	$70,3/7)_D$	$-\sqrt{6/7}$	$1/(3\sqrt{210})$
(35,0;	$70,3/7)_F$	0	$-4\sqrt{7/135}$

(2) $\left[\begin{array}{cc|c} 48 & 84 & 84 \\ \mu_1^{(3)}\mathbf{Z}_1^{(3)} & \mu_2^{(3)}\mathbf{Z}_2^{(3)} & \mu^{(3)}\mathbf{Z}^{(3)} \end{array}\right]$

$(\mu^{(3)}\mathbf{Z}^{(3)})=(56,3/7)$

$\mu_1^{(3)}\mathbf{Z}_1^{(3)}$,	$\mu_2^{(3)}\mathbf{Z}_2^{(3)}$	84
6,1;	$21,-4/7$	$\frac{1}{2}\sqrt{7/15}$
1,0;	56,3/7	$-1/(2\sqrt{30})$
35,0;	56,3/7	$-\sqrt{7/8}$

(f)

(1) $\left[\begin{array}{cc|c} 63 & 168 & 168_i \\ \mu_1^{(4)}\mathbf{Z}_1^{(4)} & \mu_2^{(4)}\mathbf{Z}_2^{(4)} & \mu^{(4)}\mathbf{Z}^{(4)} \end{array}\right]$

$(\mu^{(4)}\mathbf{Z}^{(4)})=(112,3/8)$

$\mu_1^{(4)}\mathbf{Z}_1^{(4)}$;	$\mu_2^{(4)}\mathbf{Z}_2^{(4)}$	168_D	168_F
7,1	$21,-5/8$	$-1/4$	$\frac{7}{24}\sqrt{3/5}$
7,1;	$28,-5/8$	1/4	$\frac{3}{8}\sqrt{3/5}$
1,0;	112,3/8	0	$-\frac{1}{2}\sqrt{1/35}$
(48,0;	$112,3/8)_D$	$-\sqrt{7/8}$	$\frac{1}{24}\sqrt{6/35}$
(48,0;	$112,3/8)_F$	0	$-\sqrt{6/7}$

(2) $\left[\begin{array}{cc|c} 63 & 120 & 120 \\ \mu_1^{(4)}\mathbf{Z}_1^{(4)} & \mu_2^{(4)}\mathbf{Z}_2^{(4)} & \mu^{(4)}\mathbf{Z}^{(4)} \end{array}\right]$

$(\mu^{(4)}\mathbf{Z}^{(4)})=(84,3/8)$

$\mu_1^{(4)}\mathbf{Z}_1^{(4)}$;	$\mu_2^{(4)}\mathbf{Z}_2^{(4)}$	120
7,1;	$28,-5/8$	$2\sqrt{2/77}$
1,0;	84,3/8	$\frac{-1}{7}\sqrt{3/11}$
48,0;	84,3/8	$\frac{-4}{7}\sqrt{30/11}$

Finally, we present several examples to show how to use the Tables III and IV to calculate the elements of the Skyrmion mass-splitting matrix [Eqs. (35)–(37)]. For the SU(3) (i.e., $N=3$) case, according to Eq. (25) the s-flavor-breaking Hamiltonian reads

$$\Delta M(u,d,s)=-mD_{8,8}^{(8)} .$$

Its expectation value between the proton states is

$$
\begin{aligned}
\langle p\uparrow|\Delta M(u,d,s)|p\uparrow\rangle &= -m\,\langle D^{(8)}_{\mu_p\nu_p}|D^{(8)}_{8,8}|D^{(8)}_{\mu_p\nu_p}\rangle \\
&= -m\sum_{\gamma}\left[\begin{array}{ccc} 8 & 8 & 8_\gamma \\ \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{\frac12}\end{bmatrix} & \begin{bmatrix}0 & 0\\ \multicolumn{2}{c}{0}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{\frac12}\end{bmatrix}\end{array}\right]\left[\begin{array}{ccc} 8 & 8 & 8_\gamma \\ \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{-\frac12}\end{bmatrix} & \begin{bmatrix}0 & 0\\ \multicolumn{2}{c}{0}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{-\frac12}\end{bmatrix}\end{array}\right] \\
&= -m\sum_{\gamma}\left[\begin{array}{cc|c} 8 & 8 & 8_\gamma \\ (\frac12\ 1) & (0\ 0) & (\frac12\ 1)\end{array}\right]C^{1/2\,0\,1/2}_{1/2\,0\,1/2}\left[\begin{array}{cc|c} 8 & 8 & 8_\gamma \\ (\frac12\ 1) & (0\ 0) & (\frac12\ 1)\end{array}\right]C^{1/2\,0\,1/2}_{-1/2\,0\,-1/2}
\end{aligned}
$$

in which Eq. (32) has been used and $C^{l_1\ l_2\ l}_{m_1\ m_2\ m}$ is the usual SU(2) CG coefficient. From Table IV(a)(1)

$$
\left[\begin{array}{cc|c} 8 & 8 & 8_D \\ (\frac12\ 1) & (0\ 0) & (\frac12\ 1)\end{array}\right]=\frac{-1}{\sqrt{20}},\quad \left[\begin{array}{cc|c} 8 & 8 & 8_F \\ (\frac12\ 1) & (0\ 0) & (\frac12\ 1)\end{array}\right]=\tfrac12 .
$$

Then

$$
\begin{aligned}
\langle p\uparrow|\Delta M(N=3)|p\uparrow\rangle &= -m\left[\frac{-1}{\sqrt{20}}\times1\times\frac{-1}{\sqrt{20}}\times1+\frac12\times1\times\frac12\times1\right] \\
&= -m\times\frac{3}{10} .
\end{aligned}
$$

Comparing with Eq. (35), we have

$$\Delta(3)=10 .$$

This is the first one in Eq. (37).

Similarly, for the SU(4) case,

$$\Delta M(u,d,s)=-mD^{(15)}_{8,8}-\frac{1}{\sqrt2}mD^{(15)}_{8,15} .$$

We need to compute $\langle D^{(15)}_{8,8}\rangle$ and $\langle D^{(15)}_{8,15}\rangle$. First,

$$
\begin{aligned}
\langle p\uparrow|D^{(15)}_{8,8}|p\uparrow\rangle &= \sum_{\gamma ij}\left[\begin{array}{ccc} 15 & 20 & 20_\gamma \\ \begin{bmatrix}0 & 0\\ \multicolumn{2}{c}{0}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{\frac12}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{\frac12}\end{bmatrix} \\ \mu=8,\ Z=0 & \mu=8,\ Z=\frac34 & \mu=8_i,\ Z=\frac34\end{array}\right]\left[\begin{array}{ccc} 15 & 20 & 20_\gamma \\ \begin{bmatrix}0 & 0\\ \multicolumn{2}{c}{0}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{-\frac12}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{-\frac12}\end{bmatrix} \\ \mu=8,\ Z=0 & \mu=8,\ Z=\frac34 & \mu=8_j,\ Z=\frac34\end{array}\right] \\
&= \sum_{\gamma ij}\left[\begin{array}{cc|c} 15 & 20 & 20_\gamma \\ \mu=8,\ Z=0 & \mu=8,\ Z=\frac34 & \mu=8_i,\ Z=\frac34\end{array}\right]\left[\begin{array}{ccc} 8 & 8 & 8_i \\ \begin{bmatrix}0 & 0\\ \multicolumn{2}{c}{0}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{\frac12}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{\frac12}\end{bmatrix}\end{array}\right] \\
&\quad\times\left[\begin{array}{cc|c} 15 & 20 & 20_\gamma \\ \mu=8,\ Z=0 & \mu=8,\ Z=\frac34 & \mu=8_j,\ Z=\frac34\end{array}\right]\left[\begin{array}{ccc} 8 & 8 & 8_j \\ \begin{bmatrix}0 & 0\\ \multicolumn{2}{c}{0}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{-\frac12}\end{bmatrix} & \begin{bmatrix}\frac12 & 1\\ \multicolumn{2}{c}{-\frac12}\end{bmatrix}\end{array}\right] .
\end{aligned}
$$

From Table IV(b)(1) we have

$$
\begin{aligned}
&\left[\begin{array}{cc|c} 15 & 20 & 20_D \\ \mu=8,\ Z=0 & \mu=8,\ Z=\frac34 & \mu=8_D,\ Z=\frac34\end{array}\right]=0 , \\
&\left[\begin{array}{cc|c} 15 & 20 & 20_D \\ \mu=8,\ Z=0 & \mu=8,\ Z=\frac34 & \mu=8_F,\ Z=\frac34\end{array}\right]=-2\sqrt{2/13} , \\
&\left[\begin{array}{cc|c} 15 & 20 & 20_F \\ \mu=8,\ Z=0 & \mu=8,\ Z=\frac34 & \mu=8_D,\ Z=\frac34\end{array}\right]=-\tfrac{13}{4}\sqrt{5/78} , \\
&\left[\begin{array}{cc|c} 15 & 20 & 20_F \\ \mu=8,\ Z=0 & \mu=8,\ Z=\frac34 & \mu=8_F,\ Z=\frac34\end{array}\right]=\tfrac54\sqrt{1/78} .
\end{aligned}
$$

Using Table IV(a)(1) again, we have

$$\begin{bmatrix} 8 & 8 & 8_D \\ \begin{bmatrix} 0\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}\ 1 \\ \pm\frac{1}{2} \end{bmatrix} & \begin{bmatrix} \frac{1}{2}\ 1 \\ \pm\frac{1}{2} \end{bmatrix} \end{bmatrix} = -\sqrt{1/20}, \quad \begin{bmatrix} 8 & 8 & 8_F \\ \begin{bmatrix} 0\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}\ 1 \\ \pm\frac{1}{2} \end{bmatrix} & \begin{bmatrix} \frac{1}{2}\ 1 \\ \pm\frac{1}{2} \end{bmatrix} \end{bmatrix} = -\tfrac{1}{2} .$$

Then

$$\begin{aligned} \langle p\uparrow|D^{(15)}_{8,8}|p\uparrow\rangle &= (-\sqrt{1/20} \quad (-\tfrac{1}{2}) \begin{bmatrix} 0 & -\frac{13}{4}\sqrt{5/78} \\ -2\sqrt{2/13} & \frac{5}{4}\sqrt{1/78} \end{bmatrix} \begin{bmatrix} 0 & -2\sqrt{2/13} \\ -\frac{13}{4}\sqrt{5/78} & \frac{5}{4}\sqrt{1/78} \end{bmatrix} \begin{bmatrix} -\sqrt{1/20} \\ -\frac{1}{2} \end{bmatrix} \\ &= \tfrac{2}{13} + \tfrac{1}{78} \\ &= \tfrac{1}{6} . \end{aligned}$$

Next, through same procedures, it can be found that

$$\langle p\uparrow|D^{(15)}_{8,15}|p\uparrow\rangle = -\frac{\sqrt{2}}{12} .$$

Then

$$\begin{aligned} \langle p\uparrow|\Delta M(u,d,s)|p\uparrow\rangle &= -m\langle p\uparrow|D^{(15)}_{8,8}|p\uparrow\rangle - \frac{1}{\sqrt{2}} m\langle p\uparrow|D^{(15)}_{8,15}|p\uparrow\rangle \\ &= -\frac{1}{12}m . \end{aligned}$$

Comparing with Eq. (35), we get

$$\Delta(4) = 48 .$$

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