RAPID COMMUNICATIONS

PHYSICAL REVIEW D VOLUME 41, NUMBER 2 15 JANUARY 1990

Generating functional for the energy-momentum tensor in two-dimensional conformal field theory

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(Received 14 June 1989)

We discuss a power-series solution for the generating functional $Z[h]$ of the two-dimensional (Euclidean) energy-momentum tensor resulting from Belavin-Polyakov-Zamolodchikov Ward identities. We derive another formula for $Z[h]$ in terms of a quasiconformal transformation determined by the Beltrami differential h.

The conformal Ward identities of Belavin, Polyakov, and Zamolodchikov[1] (BPZ) determine the n-point correlation functions of the two-dimensional energy-momentum tensor. Hence, by summation we can get the generating functional $Z[h]$ for the correlation functions. The generating functional is related to the exponential of an effective action arising from the interaction of quantum matter fields with an external gravitational field h.[2,3] In this Rapid Communication we derive a formula for the generating functional of the energy-momentum tensor. First, we get a power-series expansion for $Z[h]$ as a solution of the BPZ Ward identities. Then, using the (Euclidean) functional integral we obtain another formula for $Z[h]$ expressed by the Schwartzian $S(h;z)$ for a conformal map determined by the Beltrami differential h.

The holomorphic energy-momentum tensor in a conformal two-dimensional field theory satisfies the Ward identities[1] ($\partial=\partial_1-i\partial_2$)

$$\langle T(z)T(z_1)\cdots T(z_n)\rangle=\sum_{j=1}^{n}\left[2(z-z_j)^{-2}+(z-z_j)^{-1}\frac{\partial}{\partial z_j}\right]\langle T(z_1)\cdots T(z_n)\rangle +C(z-z_j)^{-4}\langle T(z_1)\cdots \hat{T}(z_j)\cdots T(z_n)\rangle, \tag{1}$$

where $\hat{T}(z_j)$ indicates absence of the factor $T(z_j)$. Denote

$$T(h)=(2\pi)^{-1}\int T(z)h(z).$$

Then

$$\langle T(h)^{n+1}\rangle=n\langle T(h)^{n-1}T(h*h)\rangle+nChKh\langle T(h)^{n-1}\rangle, \tag{2}$$

where

$$hKh=(2\pi)^{-2}\int h(z)(z-z')^{-4}h(z')=-(12\pi)^{-1}h\,\partial^3\bar{\partial}^{-1}h, \tag{3}$$

with

$$\bar{\partial}^{-1}(z,z')=(2\pi)^{-1}(z-z')^{-1}$$

and

$$h*h(z)=(2\pi)^{-1}h(z)\int(z-z')^{-2}h(z')+(2\pi)^{-1}\partial h(z)\int(z-z')^{-1}h(z')\equiv h\,\partial\bar{\partial}^{-1}h(z)+\partial h\,\bar{\partial}^{-1}h(z). \tag{4}$$

Let

$$Z[h]=\sum(n!)^{-1}\langle T(h)^n\rangle; \tag{5}$$

then, multiplying Eq. (2) by $(n!)^{-1}t^n$ we get an equation for $Z[h]$:

$$\partial_t Z[h]=\int h*h(z)\frac{\delta}{\delta h(z)}Z[th]+CthKhZ[th]. \tag{6}$$

If $Z[th]=\exp(W[th])$, then we have for W

$$\partial_t W[th]=AW[th]+Cthkh, \tag{7}$$

where

$$A=\int h*h(z)\frac{\delta}{\delta h(z)}. \tag{8}$$

The solution of Eq. (7) is (we set $t=1$)

$$W[h]\equiv\ln Z[h]=C\int_0^1 ds\exp[(1-s)A]shKh. \tag{9}$$

The power-series expansion in h has the form

$$\ln Z[h]=(C/2)hKh+(C/6)h*h\frac{\delta}{\delta h}hKh+(C/24)h*h\frac{\delta}{\delta h}h*h\frac{\delta}{\delta h}hKh+\cdots. \tag{10}$$

Let us write down the series (10) more explicitly to the third order:

$$W_3[h] = -(24\pi)^{-1} h\,\partial^3\bar\partial^{-1}h - (36\pi)^{-1}C\left(\int [(\partial h)\bar\partial^{-1}h](\partial^3\bar\partial^{-1}h) - \int (h\,\partial\bar\partial^{-1}h)(\partial^3\bar\partial^{-1}h)\right). \quad (11)$$

We can see from the formula (9) that $Z[h]$ depends on the model only through the "anomalous charge" C. So, it is sufficient to compute $Z[h]$ in a free theory.

Let us consider first the free real scalar field ϕ. Define (for $|h| < 1$)

$$\begin{aligned} Z[h,\bar h] &= \int d\phi \exp\left(-(2\pi)^{-1}\int (1-h\bar h)^{-1}|\bar\partial\phi - h\,\partial\phi|^2\right) \\ &= \int d\phi \exp\left(-(2\pi)^{-1}\int [\bar\partial\phi\,\partial\phi - h(1-h\bar h)^{-1}\partial\phi\,\partial\phi]\right) \\ &\qquad \times\exp\left((2\pi)^{-1}\int [\bar h(1-h\bar h)^{-1}\bar\partial\phi\,\bar\partial\phi - 2h\bar h(1-h\bar h)^{-1}\partial\phi\,\bar\partial\phi]\right). \end{aligned} \quad (12)$$

Hence

$$\frac{\delta Z[h,\bar h]}{\delta h(z)} = \int d\phi \exp\left(-(2\pi)^{-1}\int (1-h\bar h)^{-1}|\bar\partial\phi - h\,\partial\phi|^2\right)\left(\frac{1}{\pi}T(z) + \bar h\text{-dependent terms}\right), \quad (13)$$

where

$$T(z) = \tfrac{1}{2}:\partial\phi\,\partial\phi:. \quad (14)$$

We assume that h has its support in the upper half plane U (we denote the lower half plane by L). It can be seen that the Lagrangian in the functional integral (12) can be transformed by a change of coordinates into the free Lagrangian L_0 (in the isothermal coordinates). In fact, let $\zeta(z)$ be the quasiconformal transformation determined by the Beltrami differential h:

$$\bar\partial\zeta = h\,\partial\zeta. \quad (15)$$

Then, the formula (12) takes the form

$$Z[h,\bar h] = \int d\phi \exp\left(-(2\pi)^{-1}\int_{U(h)}\left|\frac{\partial\phi}{\partial\bar\zeta}\right|^2 - (2\pi)^{-1}\int_L\left|\frac{\partial\phi}{\partial\bar z}\right|^2\right), \quad (16)$$

where $U(h)$ is the image of the upper half plane by the quasiconformal transformation ζ [Eq. (15)].

The correlation functions of the energy-momentum tensor T are determined by derivatives over h: i.e.,

$$\langle T(z_1)\cdots T(z_n)\rangle = \frac{\delta}{\delta h(z_1)}\cdots\frac{\delta}{\delta h(z_n)}Z[h,\bar h]\Bigg|_{h=0}. \quad (17)$$

So, only the dependence on h is relevant. For the computation of the generating functional of T we may set $\bar h = 0$ (we denote $Z[h,0]\equiv Z[h]$). Now, Eq. (13) can be expressed as

$$\frac{\delta}{\delta h(z)}\ln Z[h] = \frac{1}{\pi}\langle T(z)\rangle_h, \quad (18)$$

where

$$\langle F\rangle_h = Z[h]^{-1}\int d\phi \exp\left(-\int_{U(h)}L_0\right)F \quad (19)$$

is the expectation value in the free scalar theory on $U(h)$.

Let G_h be the Green's function of $U(h)$. Then,

$$\begin{aligned} \langle T(z)\rangle &= \lim_{w\to z}\tfrac{1}{2}\langle\partial\phi(z)\,\partial\phi(w) - \partial_z\partial_w\ln(z-w)\rangle_h \\ &= \lim_{w\to z}\tfrac{1}{2}[\partial_z\partial_w G_h(z,w) - \partial_z\partial_w\ln(z-w)] \\ &= \tfrac{1}{12}S(h;z), \end{aligned} \quad (20)$$

where

$$S(h;z) = \partial^2\ln\partial\zeta - \tfrac{1}{2}(\partial\ln\zeta)^2 \quad (21)$$

is the Schwartzian of the map (15).

The limit on the right-hand side of Eq. (20) is shown in Ref. 4. Let us briefly describe the argument. The quasiconformal map ζ of U extends to an analytic function in the lower half plane L (because $h=0$ in L). This analytic function conformally maps L onto $U(h)$. In such a case,

$$G_h(z,w) = G(\zeta(z),\zeta(w)) = -\ln[\zeta(z)-\zeta(w)]. \quad (22)$$

Equation (20) follows from Eq. (22) by standard calculations.

Inserting Eq. (20) onto Eq. (18), we get the equation

$$\frac{\delta Z[h]}{\delta h(z)} = (12\pi)^{-1}S(h;z)Z[h] \quad (23)$$

with the solution

$$Z[h] = \exp\left(C(12\pi)^{-1}\int_0^1 dt\int d^2z\,S(th;z)h(z)\right). \quad (24)$$

$C=1$ for the free scalar field discussed above, but as mentioned before, if the formula is correct in this particular case, then it holds true in general conformal field theory (1) (with $C\neq 1$).

It appears instructive to repeat the argument leading to Eq. (24) for anticommuting fields. Consider the ghost system (b,c) (spin 1 and 0) with the Lagrangian

$$L_0 = (2\pi)^{-1}\int b\,\bar\partial c. \quad (25)$$

The generating functional is (h has its support in the upper half plane U)

$$Z[h]=\int db\,dc\exp\left[-(2\pi)^{-1}\int b(\bar\partial-h\,\partial)c\right]. \qquad (26)$$

Under the quasiconformal transformation ζ [Eq. (15)],

$$\bar\partial-h\partial=\frac{\bar\partial\zeta}{\partial z}(1-h\bar h)\frac{\partial}{\partial\bar\zeta},\quad d^2z=d^2\zeta(1-h\bar h)^{-1}\left|\frac{\partial\zeta}{\partial z}\right|^{-2}.$$

The change of variables $b(z)\to b'(z)(\partial\zeta/\partial z)\dot b(\zeta(z))$, $c(z)\to c'(z)=c(\zeta(z))$ transforms $Z[h]$ into a partition function of the free (b,c) system defined on the image $U(h)$ of U. In such a case Eqs. (20) and (24) follow.

The formula for an effective action of the two-dimensional gravity discussed in Refs. 2, 3, and 5–7 is different from Eq. (24). However, these authors work in a pseudo-Riemannian space with the light-cone gauge. The light-cone gauge does not exist in the Riemannian space. There is some similarity between Beltrami parametrization (applied here) and the light-cone gauge. The distinction of these two notions and the corresponding coordinates ζ may be a reason for a difference between the Polyakov effective action (Ref. 2) and the action (24). Nevertheless, we can show that Polyakov's equations of motion $\bar\partial^3 h=0$ as well as the SL$(2,R)$ current algebra follow from the action (24).

The equations of motion resulting from the effective action (24) are

$$S(h;z)=0. \qquad (27)$$

It is well known that the general solution of Eq. (27) is

$$\zeta=Gz=(az+\beta)(\gamma z+\delta)^{-1}, \qquad (28)$$

where

$$G=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in \mathrm{SL}(2,R).$$

If $\zeta\in U$, then G may depend on $\bar z$. Then, the Beltrami differential is

$$h=(\bar\partial\zeta)(\partial\zeta)^{-1}\equiv -J_+z^2+2J_1z+J_-, \qquad (29)$$

where

$$\begin{pmatrix}J_1 & J_-\\ J_+ & -J_1\end{pmatrix}=G^{-1}\bar\partial G. \qquad (30)$$

The action (24) is invariant under SL$(2,R)$ transformations $\zeta\to(a\zeta+b)(c\zeta+d)^{-1}$. Under such transformations J [Eq. (30)] transforms as an SL$(2,R)$ current (this current has been derived also in Ref. 7 in the formalism of the light-cone gauge).

For string theory (in noncritical dimensions) the quantization of $W(h)$ is important. For this purpose a study of the functional integral $dh\,|\exp[-W(h)]|^2$ is required. This problem would lead us beyond the scope of this note.

We wish to derive a perturbative expansion resulting from the action (24) and compare it to Eq. (9). We can write Eq. (15) in the form

$$\partial\zeta=1+\partial\bar\partial^{-1}(h\,\partial\zeta). \qquad (31)$$

Hence,

$$\partial\zeta=1+\partial\bar\partial^{-1}h+\partial\bar\partial^{-1}(h\,\partial\bar\partial^{-1}h)+\cdots,$$

$$\ln\partial\zeta=\partial\bar\partial^{-1}h+\partial\bar\partial^{-1}(h\partial\bar\partial^{-1}h)-\tfrac{1}{2}(\partial\bar\partial^{-1}h)^2+\cdots.$$

So, to the third order in h,

$$C(2\pi)^{-1}\int_0^1 dt\int d^2z\,h(z)S(th;z)=C(24\pi)^{-1}h\,\partial^3\bar\partial^{-1}h$$
$$-C(36\pi)^{-1}\left[\int\partial^3h\,\bar\partial^{-1}(h\,\partial\bar\partial^{-1}h)+\tfrac{1}{2}\int h\,\partial^2(\partial\bar\partial^{-1}h)^2+\tfrac{1}{2}\int h(\partial^2\bar\partial^{-1}h)^2\right]. \qquad (32)$$

Showing that the expression (32) is equal to (11) involves some integration by parts [it becomes easier if we set $h=\partial f$, then the term in the large parentheses in Eq. (32) is equal to $f\,\partial\bar\partial f\,\partial^3 f-\bar\partial f\,\partial f\,\partial^3 f$, evidently coinciding with (11)]. We are not able to show that Eq. (9) coincides with Eq. (24) at any order of h. However, we have derived both formulas from basic principles of conformal field theory.

This work was supported by Sonderforschungsbereich Grant No. SFB 237 Bochum-Essen-Düsseldorf.

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