Nötzold solution of matter oscillation of solar neutrinos

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(Received 10 May 1989)

An exact analytic solution of the matter oscillation of neutrinos proposed by Nötzold for a hyperbolic-tangent distribution of electron density, which is more realistic for the case of the Sun than other distributions, is improved by an exact treatment of the initial condition at any production point of the neutrino, where it is not on a flat place of $r=-\infty$ as Nötzold took. It must be stressed that the tanh fit can be made precisely even for the central region of the Sun, as well as the exponential tail in the outer region. I then give a compact analytic formula with elementary functions, which is a good approximation to the "exact" solution.

I. INTRODUCTION

Among the many proposals for the solution of the matter oscillation[1] of solar neutrinos, Nötzold's paper[2] has an advantage in the sense that it adopts a hyperbolic-tangent distribution for the electron density, which is more realistic and of wider application than others such as a single exponential distribution,[3] and is, most importantly, capable of an exact analytic solution with elementary functions. But his treatment has a defect in that the neutrino production point was assumed to be a place of flat distribution of $r=-\infty$. This place can obviously provide exact mass eigenstates for the initial amplitudes, but unfortunately it is not the case of the realistic distribution as indicated in a recent calculation of Bahcall and Ulrich[4] for the electrons in the Sun.

In this paper, the Nötzold model is reformulated by treating an exact initial condition on the amplitudes at any neutrino production point for the hyperbolic-tangent distribution. The tanh fit to the electron density distribution in the Sun can be made with a remarkable precision to a root-mean-square accuracy of 3% for $R/R_s<0.31$, where R_s is the sun radius, as well as to the exponential tail of the outer region. Because of the changing density, the initial amplitudes do not in general satisfy an adiabatic condition as pointed out by Barger, Phillips, and Whisnant.[5] But this condition is shown to be satisfied well over a wider range of the mixing-angle parameter $\sin^2 2\theta$ for solar neutrinos. In this connection an approximation for hypergeometric functions is given at the production point where it is actually the center of the Sun with a changing density. As a result the final formula improves the discrepancy involved in the original Nötzold model, which deviates almost 26% larger than the "exact" solution in the values of $\Delta m^2/E$ at least for $\sin^2 2\theta \gtrsim 10^{-4}$ on the high $\Delta m^2/E$ side of the triangle in the $\Delta m^2/E$ vs $\sin^2 2\theta$ plot.

II. FORMULATION

For the level crossing of ν_e and ν_μ, the coupled Schrödinger equations are

$$i\dot{\nu}_e=\left[\frac{m_1^2\cos^2\theta+m_2^2\sin^2\theta}{2E}+\sqrt{2}G_F N_e\right]\nu_e+\frac{\Delta m^2}{2E}\sin\theta\cos\theta\nu_\mu\ , \tag{2.1}$$

$$i\dot{\nu}_\mu=\frac{\Delta m^2}{2E}\sin\theta\cos\theta\nu_e+\frac{m_1^2\sin^2\theta+m_2^2\cos^2\theta}{2E}\nu_\mu\ , \tag{2.2}$$

where m_1,m_2 are masses of the mass eigenstates ν_1,ν_2, respectively, $\Delta m^2=m_2^2-m_1^2$ (hereafter $m_2>m_1$ is assumed), G_F is the Fermi coupling constant, E is the neutrino energy, and N_e is the density of electrons. The mixing angle θ is defined by $\nu_e=\cos\theta\nu_1+\sin\theta\nu_2$, $\nu_\mu=-\sin\theta\nu_1+\cos\theta\nu_2$. ν_e is assumed to be produced at the center of the Sun at $t=0$.

To convert the physical quantities in (2.1) and (2.2) to dimensionless ones, in place of t, $ct=R_S(r-r_0)$ where $r-r_0$ is the distance from the center of the Sun in units of Sun radius R_S ($=6.96\times10^5$ km). The electron density in units of Avogadro's number is given as

$$R_S\sqrt{2}G_F N_e=VU(r)\ , \tag{2.3}$$

where $V=1.718\times10^4$, and

$$U(r)=1-\tanh(r/l)\ , \tag{2.4}$$

where $l=0.155$ and $r_0=-0.105$ as a best fit to the solution of Bahcall and Ulrich. In their solution, the density distribution of electrons is not flat at the center of the Sun ($r=r_0$). Further

$$M_0^2=R_S(m_2^2+m_1^2)/(4E)\ , \tag{2.5}$$

$$R_S\Delta m^2/(2E)=x/B\equiv x_0\ , \tag{2.6}$$

with

$$B\equiv E(\mathrm{MeV})/\Delta m^2(\mathrm{eV}^2)\ , \tag{2.7}$$

where $x=1.764\times10^9$, and

$$G=\tfrac{1}{2}x_0\cos2\theta\ , \tag{2.8}$$

$$H=\tfrac{1}{2}x_0\sin2\theta\ . \tag{2.9}$$

If a phase is taken out as

$$\nu_e(r)=v(r)\exp\left[-i\int_{r_0}^{r}[M_0^2+\tfrac{1}{2}VU(r)]dr\right]\ , \tag{2.10}$$

$v(r)$ obeys a second-order differential equation

$$\frac{d^2v}{dr^2}+\{i\tfrac{1}{2}VU(r)'+[\tfrac{1}{2}VU(r)-G]^2+F^2\}v=0 \; . \qquad (2.11)$$

According to Nötzold, with variable transformation,

$$y=1/[1+\exp(-2r/l)] \; , \qquad (2.12)$$

then $U(r)=2(1-y)$, and if $v(r)$ is factorized as

$$v(r)=(1-y)^{\mu}y^{\nu}f(y) \; , \qquad (2.13)$$

then

$$y(1-y)\frac{d^2f}{dy^2}+[c-(a+b+1)y]\frac{df}{dy}-abf=0 \; , \qquad (2.14)$$

with

$$a=\mu+\nu+\lambda \; , \qquad (2.15)$$

$$b=\mu+\nu+1-\lambda \; , \qquad (2.16)$$

$$c=2\nu+1 \; , \qquad (2.17)$$

when

$$\mu=i(l/2)(G^2+H^2)^{1/2} \; , \qquad (2.18)$$

$$\nu=i(l/2)[(V-G)^2+H^2]^{1/2} \; , \qquad (2.19)$$

$$\lambda=i(l/2)V \; . \qquad (2.20)$$

Equation (2.14) is the hypergeometric equation, and it is known that the general solution is given by the hypergeometric functions as[6]

$$f(y)=C_1F(a,b;c;y)_+ + C_2y^{-2\nu}F(a,b;c;y)_- \; , \qquad (2.21)$$

where C_1,C_2 are constants and $F(a,b;c;y)_+ = {}_2F_1(a,b;c;y)$, $F(a,b;c;y)_- = {}_2F_1(a,b;c;y)$ with ν taken as $-\nu$ in a,b,c.

The matter eigenstates $|+\rangle, |-\rangle$ at $y=y_0$, where $r=r_0$, are defined by a mixing angle θ_m as

$$\cos 2\theta_m=\frac{G-V(1-y_0)}{\sqrt{[G-V(1-y_0)]^2+H^2}} \; , \qquad (2.22)$$

where the denominator is half the momentum splitting between the matter eigenstates.

III. INITIAL CONDITION

If the initial point is assumed at a flat place $r_0=-\infty$ where $y_0=0$, as is done by Nötzold, the adiabatic condition is satisfied as

$$|C_1|=\cos^2\theta_m^0 \; , \qquad (3.1)$$

$$|C_2|=\sin^2\theta_m^0 \; , \qquad (3.2)$$

where θ_m^0 is defined by (2.22) with $y_0=0$. But for the initial condition at a place of changing density with any y_0, by boundary conditions $\nu_e(r_0)=1$, $\nu_\mu(r_0)=0$ for (2.1) and (2.2), the constants C_1,C_2 are given as

$$C_1=(1-y_0)^{\mu}y_0^{-\nu}\frac{1}{2}\left[y_0\frac{(\Delta p+\Delta q-V)(\mu+\nu+\lambda-1)}{\Delta q(2\nu-1)}F(a,b;c+1;y_0)_- + \left[1+\frac{G-V}{\Delta q}-\frac{\Delta p+\Delta q-V}{\Delta q}y_0\right]F(a,b;c;y_0)_-\right] \; , \qquad (3.3)$$

$$C_2=(1-y_0)^{\mu}y_0^{\nu}\frac{1}{2}\left[y_0\frac{(\Delta p-\Delta q-V)(\mu-\nu+\lambda-1)}{\Delta q(2\nu+1)}F(a,b;c+1;y_0)_+ + \left[1-\frac{G-V}{\Delta q}+\frac{\Delta p-\Delta q-V}{\Delta q}y_0\right]F(a,b;c;y_0)_+\right] \; , \qquad (3.4)$$

with

$$\Delta p=(G^2+H^2)^{1/2} \; , \qquad (3.5)$$

$$\Delta q=[(V-G)^2+H^2]^{1/2} \; . \qquad (3.6)$$

In the above calculations, the following Wronskian was used:

$$u_2'u_1-u_1'u_2=(1-c)y^{-c}(1-y)^{c-a-b-1} \; , \qquad (3.7)$$

where

$$u_1(y)=F(a,b;c;y)_+ \; , \qquad (3.8)$$

$$u_2(y)=y^{-2\nu}F(a,b;c;y)_- \; . \qquad (3.9)$$

It should be noted that the factors $(1-y_0)^{\mu}y_0^{-\nu}$ and $(1-y_0)^{\mu}y_0^{\nu}$ in C_1 and C_2 are all pure phases. For $y_0\to 0$, C_1 and C_2 become (3.1) and (3.2), respectively, except for phases.

IV. SURVIVAL PROBABILITY

The function of ν_e produced at r_0 is given at r by (2.10), (2.13), and (2.21). When a detection is made at $r=\infty$, where $y=1$ and ${}_2F_1(a,b;c;1-y)=1$, the hypergeometric functions in (2.21) may be transformed as

$$ {}_2F_1(a,b;c;y)=C(-\mu,\nu)[{}_2F_1(a,b;c;1-y)]_{(\mu,\nu)\to(\nu,\mu)}+C(\mu,\nu)(1-y)^{-2\mu}[{}_2F_1(a,b;c;1-y)]_{(\mu,\nu)\to(\nu,-\mu)} \tag{4.1}$$

with the same notation as in Nötzold's paper. The survival probability smeared over wavelengths is then given as

$$\langle P_{\nu_e}\rangle=|C_1|^2\left[\frac{\cos\theta}{\cos\theta_m^0}\right]^2+|C_2|^2\left[\frac{\sin\theta}{\sin\theta_m^0}\right]^2 -\cos2\theta\left[\frac{|C_1|^2}{\cos^2\theta_m^0}-\frac{|C_2|^2}{\sin^2\theta_m^0}\right]P , \tag{4.2}$$

where

$$P=\frac{\cosh(\pi l V)-\cosh[\pi l(\Delta p-\Delta q)]}{\cosh[\pi l(\Delta p+\Delta q)]-\cosh[\pi l(\Delta p-\Delta q)]} . \tag{4.3}$$

This formula of survival probability is the most general expression for neutrino mixing in matter with hyperbolic-tangent distribution of electrons, when ν_e is produced at any point r_0 and detected at $r=\infty$. If (3.1) and (3.2) are adopted, then the probability coincides with that of Nötzold's model.

V. APPROXIMATION AND RESULT

In general it is difficult to get explicit formulas of C_1 and C_2 with elementary functions. But if a condition for adiabatic approximation[4]

$$\left|\frac{dN_e}{dr}\right| \ll \frac{[(2\sqrt{2}G_F N_e-\Delta m^2\cos2\theta)^2+(\Delta m^2\sin2\theta)^2]^{3/2}}{4\sqrt{2}G_F E^2\Delta m^2\sin2\theta} , \tag{5.1}$$

is satisfied at $r=r_0$, it is possible to set

$$|C_1F(a,b;c;y_0)_+|=\cos^2\theta_m , \tag{5.2}$$

$$|C_2F(a,b;c;y_0)_-|=\sin^2\theta_m . \tag{5.3}$$

The condition (5.1) is expressed in terms of dimensionless quantities as

$$\frac{4V}{l}y(1-y) \ll \frac{\{[2V(1-y)-x_0\cos2\theta]^2+(x_0\sin2\theta)^2\}^{3/2}}{x_0\sin2\theta} . \tag{5.4}$$

It would be sufficient to have $\sin^2 2\theta \gtrsim 10^{-4}$ for the initial state ($y=y_0$) to be sufficiently adiabatic from this condition with the parameters of solar neutrinos. Even if $\sin^2 2\theta < 10^{-4}$, this condition is satisfied except for a small range of B around a value at which a neutrino encounters resonance at the production point, i.e., $B=(5.0\text{–}6.5)\times10^4$.

To get C_1 and C_2, it can be shown that the following approximation works very well:

$$|F(a,b;c;y_0)_+|=\cos\theta_m/\cos\theta_m^0 , \tag{5.5}$$

$$|F(a,b;c;y_0)_-|=\sin\theta_m/\sin\theta_m^0 , \tag{5.6}$$

which instead of (3.1) and (3.2) results in

$$|C_1|=\cos\theta_m\cos\theta_m^0 , \tag{5.7}$$

$$|C_2|=\sin\theta_m\sin\theta_m^0 . \tag{5.8}$$

For the validity of the proposed approximations (5.5) and (5.6), both sides of (5.5) are shown to become $1/(1-y_0)$, and those of (5.6) become 1 when $x_0\to0$. On the other hand, both sides of (5.5) and (5.6) become 1 when $x_0\to\infty$. Also expanding by y_0 under $(l\Delta q)^2 \gg l(\Delta p-\Delta q+V) \gg 1$, both sides of (5.5) can be shown to become

$$1+[V/(2\Delta q)][1-(G-V)/\Delta q]y_0 , \tag{5.9}$$

and those of (5.6),

$$1-[V/(2\Delta q)][1+(G-V)/\Delta q]y_0 , \tag{5.10}$$

up to y_0. Finally, the survival probability is given as

$$\langle P_{\nu_e}\rangle=\tfrac{1}{2}[1+\cos2\theta\,\cos2\theta_m(1-2P)] , \tag{5.11}$$

where θ_m and P are defined by (2.22) and (4.3), respectively.

VI. CONCLUSION

The survival probability (4.2) with (3.3), (3.4), and (4.3) is the most general and exact expression of the matter oscillation of neutrinos produced at any place in the tanh distribution of electrons. And the final result (5.11) is an approximate formula for the probability with elementary functions, which in contrast with Nötzold's model can be applied for solar neutrinos produced at a place with changing density.

The validity of the approximation in (5.11) is due to the adiabatic condition (5.4), which is well satisfied for $\sin^2 2\theta \gtrsim 10^{-4}$. This can actually be confirmed in an "exact" solution of the differential equations (2.1) and (2.2). In Fig. 1 the triangle of $\Delta m^2/E$ vs $\sin^2 2\theta$ plot in comparison with the Nötzold model is presented for the case of survival probability 0.3. It is seen that a discrepancy of almost 26% in the values of $\Delta m^2/E$ is improved on the

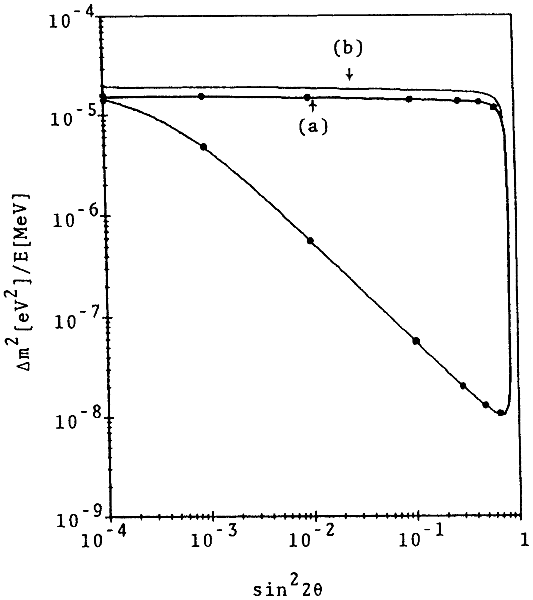

FIG. 1. $\Delta m^2/E$ vs $\sin^2 2\theta$ for solar neutrinos with $\langle P_{\nu_e} \rangle = 0.3$. The bold line (a) denotes the probability of neutrinos produced at $r_0 = -0.105$, while the thin line (b) denotes the result of the Nötzold model for $r_0 = -\infty$. The solid circle denotes the "exact" solution of Eq. (2.1) and (2.2) with a hyperbolic-tangent potential fitted to the electron density calculated by Bahcall and Ulrich.

high $\Delta m^2/E$ side of the triangle at least for $\sin^2 2\theta \gtrsim 10^{-4}$.

Finally a few remarks should be added. The tanh function fit (2.4) reproduces well the electron density calculated by Bahcall and Ulrich,[4] to a root-mean-square accuracy of 3% for $r - r_0 < 0.31$. Even at the nearest point to the center $r - r_0 = 0.001\,953$ it is only 2.5%. For $r - r_0 > 0.31$, the density is smaller in magnitude than one-tenth of that of the center and well reproduced by the exponential tail of the tanh function at high r. Second, it is interesting to observe that Eq. (5.11) takes the basic form which was deduced by Parke.[7] The difference lies in P for which Parke assumes the Landau-Zener[8] probability P_x which can be given for the tanh distribution (2.4) as

$$P_x = \exp\left[-\frac{\pi l}{2}\tan^2 2\theta \frac{VG}{|V-G|}\right]. \tag{6.1}$$

Both methods give precisely the same result for the adiabatic region. For the region where no level crossing occurs, i.e., $V(1-y_0) < G$ or $x_0 > 2V(1-y_0)/\cos 2\theta$, P tends naturally to zero, but P_x is not valid unless we are putting it to zero. For the nonadiabatic region, $P \to \cos^2\theta$ whereas $P_x \to 1$ in the limit $x_0 \to 0$. This behavior of P together with $\cos 2\theta_m \to -1$ yields naturally the expression of a vacuum oscillation to (5.11). The discrepancy in (5.11) when P_x is used in the limit $x_0 \to 0$ becomes larger for large $\sin^2 2\theta$. It is less than 3% for $\sin^2 2\theta < 0.1$, but goes over 10% for $\sin^2 2\theta > 0.4$. This may happen for the linear approximation because in this method a nonadiabatic effect remains large even when the level crossing occurs near the solar surface. Third, Haxton[9] treated a finite Landau-Zener approximation. As his result cannot be met with a compact form of elementary function, his method should be useful when no solvable form can be fitted for the electron density in the differential equations.

[1]S. P. Mikheyev and A. Yu. Smirnov, Nuovo Cimento **9C**, 17 (1986); L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978).
[2]D. Nötzold, Phys. Rev. D **36**, 1625 (1987).
[3]T. Kaneko, Prog. Theor. Phys. **78**, 532 (1987); S. Toshev, Phys. Lett. B **196**, 170 (1987); M. Ito, T. Kaneko, and M. Nakagawa, Prog. Theor. Phys. **79**, 13 (1988); S. T. Petcov, Phys. Lett. B **200**, 373 (1988).
[4]J. N. Bahcall and R. K. Ulrich, Rev. Mod. Phys. **60**, 287 (1988).
[5]V. Barger, R. J. N. Phillips, and K. Whisnant, Phys. Rev. D **34**, 980 (1986).
[6]For example, *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdelyi (McGraw-Hill, New York, 1953), Vol. 1.
[7]S. J. Parke, Phys. Rev. Lett. **57**, 1275 (1986).
[8]L. D. Landau, Phys. Z. Sowjetunion **2**, 46 (1932); C. Zener, Proc. R. Soc. London **A137**, 696 (1932).
[9]W. C. Haxton, Phys. Rev. D **35**, 2352 (1987).