Anharmonic Grassmann oscillator. II

R. Delbourgo, L. M. Jones,* and M. White
Physics Department, University of Tasmania, Hobart, Australia 7005
(Received 11 September 1989)

A properly Hermitian anharmonic Grassmannian oscillator involves higher powers of quadratics in coordinates x and momenta p, as well as the scaling operator $x \cdot p$, and the Hamiltonian looks very different from the conventional bosonic case. Because these fermionic multinomials act on a finite space, it is possible to solve such problems completely. We do so for a number of typical examples.

In an earlier paper,[1] bearing the same title and hereafter referred to as I, we studied the quantum mechanics[2,3] of an oscillator system in which the basic coordinates were taken to be anticommuting. For the case of N Sp(2) coordinate doublets, the unperturbed oscillator with SU(N) symmetry has as its Hamiltonian (with k summed from 1 to N)

$$H_0=p_k^1p_k^2+x_k^1x_k^2 \ . \tag{1}$$

We tried to make an analogy as far as possible with the bosonic (commuting) system by permitting anharmonic higher-order, but necessarily finite, polynomial interactions among the anticommuting x coordinates; and we obtained the energy levels and eigenfunctions of the Hamiltonians for a variety of cases. These energy eigenvalues are real for a selected range of coupling constants; however (as noted in Ref. 5 of I) those perturbed Hamiltonians are in fact not Hermitian under the standard scalar product for this system.

In this paper we study the Hermitian extensions of the same unperturbed Hamiltonian. Because the Hermitian conjugation corresponds to $x \to \partial/\partial x$, with the usual factor-reversal rule, we cannot add perturbations which are higher powers of $x_k^1x_k^2$ without also adding the same power of $p_k^1p_k^2$. This makes the Hamiltonian look rather frightening, because of the higher powers of derivatives. Nevertheless, if we restrict our consideration to Hamiltonians which are both Sp(2) and SU(N) invariant, there is a very limited set of possible perturbing operators. Because of this, a remarkable simplification can be achieved in the study of the perturbed energy levels.

For proper orientation, let us begin with the case of just one fermionic oscillator. As usual

$$2H=[A^\dagger,A] \ \text{ and } \ \{A,A^\dagger\}=1 \ .$$

With the usual replacement $A=(\Xi+i\Pi)/\sqrt{2}, A^\dagger=(\Xi-i\Pi)/\sqrt{2}$ we see that $2H=i[\Xi,\Pi]$. Hence we should not expect the operator $\Xi^2+\Pi^2$ (which is basically the whole dynamics in the bosonic case) to be of importance in the fermionic case. For the fermionic case, all the dynamics is contained in the scaling operator $\Xi\Pi$.

Now take two fermionic oscillators in order to make a correspondence with the Sp(2) formalism. Let $A,B,A^\dagger,B^\dagger$ stand for the normal creation and annihilation operators. Consider a pair of coordinates and momenta which are defined through

$$A^\dagger=(x^1-ip^1)/\sqrt{2}, \quad A=(x^2+ip^2)/\sqrt{2} \ , \tag{2a}$$

$$B^\dagger=i(x^2-ip^2)/\sqrt{2}, \quad B=i(x^1+ip^1)/\sqrt{2} \ . \tag{2b}$$

These are properly Hermitian conjugates in spite of appearances; the point is that $(x^1)^\dagger=\partial/\partial x^1=ip_1=ip^2$ and $(p^1)^\dagger=-ix^2$, etc. The reason for such a decomposition is that when we adopt the conventional Hamiltonian $2H=[A^\dagger,A]+[B^\dagger,B]$ we find it can be rewritten in an explicitly Sp(2)-invariant form

$$H=x^1x^2+p^1p^2=\eta_{ij}(x^ix^j+p^ip^j)/2 \tag{3}$$

as has been advocated in I.

We now wish to consider the most general Hamiltonian which is Hermitian as well as Sp(2) and SU(N) invariant. Starting with $N=1$, we see that the only Hermitian combinations with Sp(2) invariance are

$$p^1p^2+x^1x^2=\frac{\partial^2}{\partial x^2\partial x^1}+x^1x^2 \ , \tag{4a}$$

$$i(p^1p^2-x^1x^2)=i\left[\frac{\partial^2}{\partial x^2\partial x^1}-x^1x^2\right] \tag{4b}$$

and

$$i(x^1p^2-x^2p^1)=x^1\frac{\partial}{\partial x^1}+x^2\frac{\partial}{\partial x^2} \ . \tag{4c}$$

For N greater than 1, we find that the only Hermitian operators with both SU(N) and Sp(2) symmetry are powers of the three operators (again a sum over k is assumed)

$$H_0=(p_k^1p_k^2+x_k^1x_k^2) \ , \tag{5a}$$

$$I_0=i(p_k^1p_k^2-x_k^1x_k^2) \ , \tag{5b}$$

$$D=i(x_k^1p_k^2-x_k^2p_k^1) \ . \tag{5c}$$

(Note that D is effectively the scaling operator when it acts on the space of x monomials.)

The commutation relations between the three operators of Eqs. (5) are

$$[H_0,I_0]=2i(D-N),\quad [D,H_0]=2iI_0\ ,$$
$$[D,I_0]=-2iH_0 \tag{6}$$

so if we identify

$$J_1=H_0/2,\quad J_2=I_0/2,\quad \text{and}\quad J_3=(D-N)/2 \tag{7}$$

we recover the usual angular momentum algebra $\mathbf{J}\times\mathbf{J}=i\mathbf{J}$. Finally, then, we can consider the full Hamiltonian to be a general Hermitian function of the **J** operators, and the problem begins to appear tractable.

To understand the role of this spin algebra, look at the $N=1$ case where the operators are represented differentially by

$$J_1\to[\partial^2/(\partial x^2\partial x^1)+x^1x^2]/2\ ,$$
$$J_2\to i[\partial^2/(\partial x^2\partial x^1)-x^1x^2]/2\ , \tag{8}$$
$$J_3\to(x^1\partial/\partial x^1+x^2\partial/\partial x^2-1)/2\ .$$

If we apply these to the monomials $x^1x^2,x^1,x^2,1$ we see that in the bosonic subspace (consisting of x^1x^2 and 1), the operators reduce to the Pauli matrices, whereas on the fermionic subspace (x^1 and x^2) they vanish. As this is a reversal of traditional roles, we shall call the **J** "quasispin" operators from now on. The $N=1$ space of states consists of an SU(2) doublet (the two bosonic levels) and a pair of SU(2) singlets (the two fermionic levels).

Since the free Hamiltonian H_0 is simply proportional to J_1, we can easily understand why it is that the (1) doublet eigenfunctions are $u_\pm(x)=(1\pm x^1x^2)/\sqrt{2}$, with $E=\pm1$, and (2) singlet eigenfunctions are $u^i(x)=x^i$, with $E=0$. Note also that D has two eigenvalues 0 and 2, which are associated with quasispin up and down states of J_3. Because interaction terms consist of polynomials in J and because no higher powers than 1 need be taken when $N=1$, we can write the full H as a linear combination $\mathbf{n}\cdot\mathbf{J}$. Since we can always rotate this into J_1, the $N=1$ problem is uninteresting.

Next, let us have a brief look at $N=2$. The eigenfunctions of H_0 are products of quartets which break up into SU(2) multiplets as follows:

$$([1/2]+2[0])^2=[1]+4[1/2]+5[0]\ .$$

The quasispin-1 states are just the symmetrical combinations we encountered in I, namely,

$$|++\rangle,\quad (|+-\rangle+|-+\rangle)/\sqrt{2},\quad |--\rangle\ .$$

The four quasispin-$\frac{1}{2}$ states are $|\pm i\rangle$ and $|i\pm\rangle$, while the quasispin 0 states consist of the products $|ij\rangle$ plus the antisymmetric combination $(|+-\rangle-|-+\rangle)/\sqrt{2}$.

By using the commutation relations among the J, the full Hamiltonian can always be expressed as

$$H=H_0+\sum_{a,b,c}C_{abc}J_+^aJ_1^bJ_-^c,\quad J_\pm\equiv J_2\pm iJ_3\ , \tag{9}$$

where a, b, and $c=0,1,2$, and $C^*_{abc}=C_{abc}$ for Hermiticity. Since the **J** operators act individually within each quasispin, and will not change one quasispin to another, even the most general possible combination of C_{abc} will yield a Hamiltonian matrix which is fairly simple in form in the quasispin basis.

If the form of the interaction is restricted in some way, there is greater degeneracy between the states. As an example, suppose that the interaction is purely given by

$$V=\mu[(p_k^1p_k^2)^2+(x_k^1x_k^2)^2]/2=\mu(J_1^2-J_2^2)\ .$$

Then in a basis in which J_3 is diagonalized, viz., symmetrized x monomials, the Hamiltonian reduces to just

$$H=\begin{bmatrix}0&\sqrt{2}&\mu\\ \sqrt{2}&0&\sqrt{2}\\ \mu&\sqrt{2}&0\end{bmatrix}$$

on triplets and $H=0$ on doublets and singlets. It is then quite easy to work out resulting eigenfunctions and eigenvalues. The important point is that the $|jm\rangle$ states are collectively disturbed in the same manner; thus the triplets (levels no longer separated by unity) will split off from the doublets and singlets, which in this example remain undisturbed at their free-level values.

Proceed to any N. The Hamiltonian H will be assumed to be a general Hermitian function of the total quasispin operators **J**, beginning to first order in **J** with the free value, J_1. We should remark that the eigenvalues of the scaling operator D are $0,1,2,\ldots,2N$ because the maximum x monomial is $x_1^1x_1^2\cdots x_N^1x_N^2$, implying that the maximum quasispin value is $j=N/2$. This is entirely consonant with the direct product of our N quartets.

It only remains to work out the multiplicity factors; these come through the rule (easily derived via recurrence relations)

$$[1/2]^n=\sum_{j=0}^{n|2}c_j^n[n/2-j]\ , \tag{10a}$$

where $n|2$ signifies the integer part of $n/2$, and

$$c_j^n=n!(n-2j+1)/j!(n-j+1)!\ . \tag{10b}$$

Actually what we require is a decomposition formula which incorporates the quasispin singlets, namely,

$$([1/2]+2[0])^n=\sum_{j=0}^{n|2}d_j^n[n/2-j]+\sum_{j=0}^{(n-1)|2}e_j^n[(n-1)/2-j]\ , \tag{11}$$

wherein the multiplicity factors read

$$d_j^n=(n-2j+1)(2n+1)!/j(2j-1)!(2n-2j+2)!$$
$$j>0,\quad d_0^n=1 \tag{12a}$$
$$e_j^n=2(n-2j)(2n+1)!/(2j+1)!(2n+1-2j)!\ . \tag{12b}$$

The d and e coefficients carry all the necessary information since they count the number of states which enjoy the same mixing in the most general situation. Or, in other words, within a given quasispin representation j' there may be $2j'+1$ different energy levels but all representations of a given quasispin will have the same $2j'+1$ values.

If it happens that the model Hamiltonian has a very specific form, then extra degeneracy will arise. To illustrate, consider a problem where there is but one interaction (the highest power in coordinates and momenta):

$$H=H_0+V=H_0+\mu[(p_k^1p_k^2)^N+(x_k^1x_x^2)^N]/2N! \\ =2J_1+\mu[(J_1+iJ_2)^N+(J_1-iJ_2)^N]/2N! \ . \tag{13}$$

The "potential" can only mix states with the highest quasispin, i.e., $j=N/2$, leaving all the other levels unaffected because there are too many powers of the raising and lowering quasispin operators. We can therefore be certain that it is only the $N+1$ levels associated with the symmetrized products of free doublet wave function $u_\pm$ that will be disturbed by V. The magnitude of the effect on those states can be readily calculated in a basis where J_3 is diagonalized. Clearly, perturbations which are powers of D (i.e., J_3) are easily included.

In summary, we see that any Hermitian Hamiltonian invariant under Sp(2) and SU(N) can be made block diagonal by use of the appropriate quasispin states. Representations with the same quasispin will have the same energy splittings. The calculations for any given Hamiltonian are greatly simplified by the use of the **J** operators.

L.M.J. was supported in part by the Australian-American Educational Foundation, and by U.S. Department of Energy Contract No. DE-AC02-76ER01195-Task P.

*Permanent address: Physics Department, University of Illinois, 1110 W. Green St., Urbana, Illinois 61801.

[1]R. Delbourgo, L. M. Jones, and M. White, Phys. Rev. D **40**, 2716 (1989).

[2]R. Delbourgo, Int. J. Mod. Phys. **3**, A 591 (1988).

[3]R. Marnelius, Chalmers University report, 1989 (unpublished).