PHYSICAL REVIEW D VOLUME 41, NUMBER 2 15 JANUARY 1990

Extended supersymmetry in the space-time $R \times S^3$

Diptiman Sen
Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560 012, India
(Received 20 July 1989)

Following recent work on $N=1$ rigid supersymmetry in the curved space S^3, models with extended supersymmetry are examined. Unusual superalgebras with $N=2$ and $N=3$ appear. These reduce to appropriate superalgebras in Minkowski space if the curvature is taken to zero. It is shown that possible anomalies in some of the bosonic currents cancel between the various spin-$\frac{1}{2}$ fermions present in these theories. Extended supersymmetry is therefore anomaly-free in $R \times S^3$.

I. INTRODUCTION

Studies of supersymmetric theories in curved space-time are useful for three reasons. First, the curvature can act as an infrared regulator. This may cure the problem of zero-energy modes which, for example, makes it difficult to calculate the Witten index in supersymmetric theories in flat space. In $R \times S^3$, where space is a compact three-sphere and time is flat, the index can be calculated and shown to be nonzero in a large class of theories, some of which contain chiral fermions.[1,2] This implies that certain theories cannot break supersymmetry in Minkowski space. Second, the algebra of fermionic and bosonic charges (called the superalgebra) is different from that in flat space in mathematically interesting ways. Some of the symmetries of flat space may be broken by the curvature. Finally, a study of supersymmetry in curved space may have cosmological applications.

Rigid $N=1$ supersymmetry in $R \times S^3$ has been studied recently and it presents several novel features.[1] The most mysterious of these is the role played by a U(1) symmetry called R invariance.[3] It is found (a) that the bosonic charge R appears as an integral part of the superalgebra, namely, as a term in the anticommutator of two supersymmetry generators, and (b) that supersymmetry is a good symmetry only if all the cubic and quartic interaction terms are R invariant, and if the triangle anomalies in the R current coming from the various spin-$\frac{1}{2}$ fermions in gauge theories cancel among each other.

These observations lead us to ask whether something similar happens in theories with extended supersymmetry. It is shown here that an even larger number of internal bosonic symmetries appear as a part of the superalgebra, and that the anomalies possible in these bosonic currents can be made to cancel.

In Sec. II we briefly review $N=1$ supersymmetry[1] since this forms the basis of everything to follow. In Sec. III, it is shown that theories with $N=2$ supersymmetry can be formulated in $R \times S^3$. The corresponding superalgebra is presented and $N=2$ gauge and matter multiplets are discussed. In Sec. IV we show that the $N=4$ super-Yang-Mills theory in Minkowski space can also exist in $R \times S^3$ but that one of the four supersymmetries is broken by the curvature of S^3. Only an $N=3$ superalgebra is therefore retained. In all cases, the $R \times S^3$ algebra reduces to a subalgebra of the corresponding (extended) superalgebra in Minkowski space if the radius ρ of S^3 is taken to infinity. The $N=1$, 2, and 3 algebras are special cases of the family of superunitary algebras[4] SU(M/N) that is sketched in the Appendix. In Sec. V we conclude by pointing out some directions for future investigations.

II. $N=1$ RIGID SUPERSYMMETRY (REF. 1) IN $R \times S^3$

The group of symmetries of S^3 is SO(4) which is homomorphic of SU(2)$\times$SU(2). The symmetry group of $R \times S^3$ therefore has seven generators: namely, the Hamiltonian H which translates in time and two sets of SU(2) generators L^i and M^i, where $i=1,2,3$. H will be found to be a central charge in any supersymmetric theory. It commutes with all generators, bosonic, and fermionic. The algebra of L^i and M^i is normalized to be

$$\begin{aligned}&[L^i,L^j]=-2ia\epsilon^{ijk}L^k ,\\ &[M^i,M^j]=-2ia\epsilon^{ijk}M^k,\quad [L^i,M^j]=0 ,\end{aligned} \tag{1}$$

where $a=\rho^{-1}$ is the inverse radius of the sphere S^3. Note that L^i and M^i have dimensions of mass. Define $P^i=\frac{1}{2}(L^i-M^i)$ and $J^i=-(1/2a)(L^i+M^i)$. These have the right algebra to reduce to translations and rotations in Minkowski space in the limit $a \to 0$. (The three boosts cannot be recovered in this limit because they are absent in $R \times S^3$ to begin with.)

$N=1$ supersymmetry has four fermionic generators, denoted by Q_α and $\bar{Q}_\alpha$, with $\alpha=1,2$. Theories in $R \times S^3$ can be defined in either one of two coordinate frames. In one of these, parallel transports on S^3 are generated by the L^i, whereas in the other they are generated by the M^i. Choosing the first coordinate frame in all calculations, we obtain the algebra

$$\begin{aligned}&[L^i,Q_\alpha]=a\sigma^i_{\alpha\beta}Q_\beta ,\quad [L^i,\bar{Q}_\alpha]=-a\bar{Q}_\beta\sigma^i_{\beta\alpha} ,\\ &[M^i,Q_\alpha]=[M^i,\bar{Q}_\alpha]=0 ,\end{aligned} \tag{2}$$

where σ^i are the three Pauli matrices. In terms of P^i and J^i, these become

$$[P^i,Q_\alpha]=\frac{a}{2}\sigma^i_{\alpha\beta}Q_\beta\ ,\quad [J^i,Q_\alpha]=-\tfrac{1}{2}\sigma^i_{\alpha\beta}Q_\beta \tag{3}$$

and similar ones for $\bar{Q}_\alpha$. These reduce to a familiar algebra in Minkowski space if $a\to 0$.

The most interesting part of the superalgebra comes from the anticommutator of two Q's and $\bar{Q}$'s. Two Q's anticommute, as do two $\bar{Q}$'s. A Q with a $\bar{Q}$ gives

$$\{Q_\alpha,\bar{Q}_\beta\}=2\delta_{\alpha\beta}(H+aR)+2\sigma^i_{\alpha\beta}L^i\ , \tag{4}$$

where R is a Hermitian charge satisfying

$$[R,Q_\alpha]=-Q_\alpha\ ,\quad [R,\bar{Q}_\alpha]=\bar{Q}_\alpha\ . \tag{5}$$

R generates an interval U(1) symmetry and it commutes with H, L^i, and M^i. Its presence is required in Eq. (4) in order to satisfy a Jacobi identity because the Q's are not invariant under parallel transport on S^3 [Eq. (2)]. The algebra in (4) reduces to the one familiar in Minkowski space (more accurately, to the subalgebra which does not contain boosts) if we substitute $L^i=P^i-aJ^i$ and then let $a\to 0$.

The presence of R in (4) has several interesting consequences. All interaction terms must be R invariant in order to be supersymmetric. Second, the total R anomaly coming from the various fermions in a gauge theory must vanish; otherwise supersymmetry will be broken. If either of these two conditions is violated, supersymmetry is broken by terms of order a and a^2. If the R anomaly is not zero, it means that the current corresponding to R, namely, $R^m=(R^0,R^i)$. is not conserved. Then the change $R=\int d\Omega\, R^0(t,\Omega)$, where $d\Omega$ denotes the volume element of S^3, is not constant in time. Thus $dR/dt=-i[H,R]$ is not zero, and by applying a Jacobi identity to Eq. (4), $[H,Q_\alpha]$ cannot be zero either. (This assumes that the bosonic space-time symmetries H, L^i, and M^i are not broken.) Hence supersymmetry is broken if the R anomaly is nonzero.

Requiring that the R anomaly should vanish and that interactions should respect R invariance imposes certain constraints on $N=1$ supersymmetric theories in $R\times S^3$.

There are two kinds of field multiplets possible with $N=1$ supersymmetry. A matter (chiral) multiplet (A,ψ,F) contains a complex scalar field A, a Weyl spinor ψ, and a complex auxiliary field F. The R charge acts on this multiplet thus

$$[R,A]=rA,\quad [R,\psi]=(r-1)\psi,\quad [R,F]=(r-2)F\ , \tag{6}$$

where r is a real number characterizing the multiplet. One interesting fact about supersymmetry in $R\times S^3$ is that the quadratic part of the Lagrangian describing the physical fields A and ψ explicitly knows about the parameter r. In addition to the expected (Minkowskian) form of the Lagrangian

$$\mathcal{L}=\partial_0A^*\partial_0A-\partial_iA^*\partial_iA+\bar{\psi}(i\partial_0+i\sigma^i\partial_i)\psi+F^*F\ , \tag{7}$$

where $\partial_i=iL^i$, there are the pieces

$$\mathcal{L}_a=a(r+1)\bar{\psi}\psi+2ia(r-1)A^*\partial_0A+a^2r(r-2)A^*A\ . \tag{8}$$

These terms of order a and a^2 are required by supersymmetry. Equation (8) will be frequently referred to later.

A gauge multiplet in the Wess-Zumino gauge also has three fields (A^m,λ,D) describing a gauge field A^m, a Weyl spinor λ, and a real auxiliary field D. These have R charges 0, 1, and 0, respectively. Thus,

$$[R,\lambda]=\lambda,\quad [R,\bar{\lambda}]=-\bar{\lambda}\ . \tag{9}$$

The Lagrangian for λ has the curvature-dependent piece

$$\mathcal{L}_a=a\bar{\lambda}\lambda\ , \tag{10}$$

which is the same that a matter fermion in Eq. (8) would have if r for that multiplet was zero. This observation proves to be important when $N=2$ supersymmetry is discussed.

The parameters r of the various matter multiplets are constrained by the requirements that the interactions be R invariant and that the R anomaly should vanish. The possibility of an anomaly arises in gauge theories from one-loop triangle diagrams, one vertex of which is the current R^m and the other two vertices go to the gauge field A^m. Suppose that r_x is the R charge of the fermion x circulating in the triangle, and C_T is the character of the representation T of the gauge group to which x belongs (namely,

$$\mathrm{tr}\lambda^a_T\lambda^b_T=C_T\delta^{ab}\ ,$$

where λ^a_T are the gauge group matrices in representation T). Then the R anomaly due to x is given by

$$\partial_0R^0+\partial_iR^i=\frac{g^2}{16\pi^2}r_xC_T\epsilon_{mnlp}F^{amn}F^{alp}\ ,$$

where g is the gauge coupling. So the required condition is that $\sum_x r_xC_T=0$. Similar conditions must hold for the $N=2$ and $N=3$ algebras as discussed later.

The algebra of the generators $(L^i,R,Q,\bar{Q})$ is known in the literature[4] as the superunitary algebra SU(2/1), the 2 referring to the SU(2) group L^i and the 1 referring to $N=1$ supersymmetry. The Hamiltonian H is a central charge with respect to this algebra. In the following sections, we discover that the $N=2$ and $N=3$ extended supersymmetries are described by the superalgebras SU(2/2) and SU(2/3), respectively.

III. $N=2$ SUPERSYMMETRY

In this section, we first write down the $N=2$ superalgebra and then discuss the (non-Abelian) gauge and matter multiplets of $N=2$ supersymmetry.

The superalgebra contains eight fermionic generators $Q_{a\alpha}$ and $\bar{Q}_{a\alpha}$, where $a=1,2$ denotes the internal index and $\alpha=1,2$ as before. Three bosonic operators B^i generating an internal SU(2) group also appear. These satisfy

$$\begin{aligned}&[B^i,B^j]=-2i\epsilon^{ijk}B^k,\quad [B^i,Q_{a\alpha}]=\sigma^i_{ab}Q_{b\alpha}\ ,\\ &[B^i,\bar{Q}_{a\alpha}]=-\bar{Q}_{b\alpha}\sigma^i_{ba}\ .\end{aligned} \tag{11}$$

The B^i commute with H, L^i, and M^i. The anticommutator of two Q's or two $\bar{Q}$'s can produce two bosonic central charges Z and its Hermitian $\bar{Z}$:

$$\{Q_{a\alpha},Q_{b\beta}\}=\epsilon_{ab}\epsilon_{\alpha\beta}Z,\ \ \{\bar{Q}_{a\alpha},\bar{Q}_{b\beta}\}=\epsilon_{ab}\epsilon_{\alpha\beta}\bar{Z}\ . \tag{12}$$

Finally Q and $\bar{Q}$ produce

$$\{Q_{a\alpha},\bar{Q}_{b\beta}\}=2\delta_{ab}\delta_{\alpha\beta}(H+aS)+2\delta_{ab}\sigma^i_{\alpha\beta}L^i-2a\delta_{\alpha\beta}\sigma^j_{ab}B^j\ , \tag{13}$$

where S is another central charge. Once again, all this reduces to (a subalgebra of) the $N=2$ superalgebra in Minkowski space if $a\to 0$.

It is instructive to see how this $N=2$ algebra reduces to the $N=1$ algebra if the internal SU(2) indices a, b in Eqs. (11)–(13) are all put equal to 1 (or all equal to 2). If $a=b=1$, then there is a U(1) charge R_1 satisfying Eqs. (4) and (5) with $Q_{1\alpha}$ and $\bar{Q}_{1\alpha}$, where $R_1=S-B^3$ from Eqs. (11) and (13). If $a=b=2$, then again there is an algebra such as Eqs. (4) and (5) but the U(1) charge now is given by $R_2=S+B^3$. Obviously, an anomaly-free $N=2$ supersymmetric theory must have no anomaly in either R_1 or R_2. Thus the anomalies in S and B^3 [and therefore in B^1 and B^2 by SU(2) symmetry] must individually vanish.

We now discuss the form of the field multiplets.[5] The $N=2$ gauge multiplet is obtained by taking a non-Abelian gauge multiplet of $N=1$ and coupling it to a matter multiplet of $N=1$ in the adjoint representation. The discussion simplifies if all the $N=1$ auxiliary fields are first eliminated through the equations of motion. The physical fields are (A^m,A,λ,ψ). In terms of one $N=1$ supersymmetry, the gaugino λ has R_1 charge 1, and hence the matter fermion ψ must have R_1 charge -1 if the R_1 anomaly is to cancel. This means that the coefficient of the curvature-dependent term in the Lagrangian for ψ [Eq. (8) with $r=0$] agrees with that for λ [Eq. (10)]. This is good because (ψ,λ) must form a doublet under the internal SU(2). That is, under the second supersymmetry, ψ must be the photino with R_2 charge 1 and λ must be the matter fermion with R_2 charge equal to -1.

Putting all this together, we see that the S charges of both ψ and λ are zero whereas the B_3 charges equal $+1$ and -1, respectively. Clearly, the anomalies in both of these currents vanish. In fact the anomalies in all three of the B^i vanish because (ψ,λ) form a doublet and $\mathrm{tr}B^i=0$. Thus the supersymmetries $Q_{a\alpha}$ and $\bar{Q}_{a\alpha}$ are also anomaly-free.

It can be similarly shown that the bosonic fields (A^m,A) are neutral under S and singlets under the B^i. Finally, the central charges $(Z,\bar{Z})$ vanish for the entire $N=2$ gauge multiplet.

We turn next to the $N=2$ matter multiplet (called the hypermultiplet). This is obtained by taking two $N=1$ matter multiplets (A_1,ψ_1,F_1) and (A_2,ψ_2,F_2). Under $Q_{1\alpha}$ and $\bar{Q}_{1\alpha}$, this is how the various field transform. Under $Q_{2\alpha}$ and $\bar{Q}_{2\alpha}$, the fields transform as the two $N=1$ multiplets (A_1^*,ψ_2,F_1^*) and (A_2^*,ψ_1,F_2^*). Under the first $N=1$ supersymmetry, the U(1) charges for the six fields $(A_1,\psi_1,F_1,A_2^*,\bar{\psi}_2,F_2^*)$ must, from Eq. (6), be of the form $R_1=(r_1,r_1-1,r_1-2,-r_2,1-r_2,2-r_2)$, respectively, where r_1 and r_2 are two numbers. Under the second $N=1$ supersymmetry, the R_2 charges for the same six fields must, from Eq. (6), be given by $(-r_2',r_1'-1,2-r_2',r_1',1-r_2',r_1'-2)$ where r_1' and r_2' are two other numbers. However, because the quadratic parts of the Lagrangian for the fields A_a and ψ_a explicitly known about the parameters denoting the U(1) charges [Eq. (8)], there must be certain relations between the four numbers r_1, r_2, r_1', and r_2'. Examining the Lagrangian of A_1 (or A_2), we see that r_1' must equal r_1 and $r_2'=r_2$. That is, the R_1 and R_2 charges must be equal for ψ_1 (and also for ψ_2). Similarly inspecting the Lagrangian of A_1 (or A_2), we find that $r_1-1=1-r_2$. Thus the R_1 charges of A_1 and A_2 (or the R_2 charges of A_1^* and A_2^*) add up to 2. Since $R_1=S-B^3$ and $R_2=S+B^3$, we find that the S charge of all the six fields put together above must equal r_1-1. Since these six fields transform among each other under the various supersymmetries, we see that $[S,Q_{a\alpha}]=[S,\bar{Q}_{a\alpha}]=0$. The six conjugate fields $(A_1^*,\bar{\psi}_1,F_1^*,A_2,\psi_2,F_2)$ all have an S charge equal to $1-r_1$. Under the action of the B^i, $(A_1A_2^*)$, $(A_2A_1^*)$, $(F_1F_2^*)$, and $(F_2F_1^*)$ form doublets whereas the ψ_a and $\bar{\psi}_a$ form singlets.

The central charges Z and $\bar{Z}$ are not zero for the $N=2$ matter multiplet if the two $N=1$ multiplets are coupled by a quadratic interaction, i.e., a mass term given by $m(A_1F_2+A_2F_1-\psi_1\psi_2)+\mathrm{H.c.}$ (This interaction is allowed by both the $N=1$ R invariances, R_1 and R_2, because $r_1+r_2=2$.) One finds that $[Z,A_1]=-2F_2^*$ which is proportional to $2mA_1$ through the equations of motion. Similarly we find that $[Z,A_1^*]=2F_2$, $[Z,A_2]=2F_1^*$, and $[Z,A_2^*]=2F_1$. From these, it is clear that Z is Hermitian ($Z=\bar{Z}$). The action of Z on the ψ's and F's can be similarly verified to be nontrivial.

We now couple the $N=2$ matter multiplet to the $N=2$ gauge multiplet. The matter fermions ψ_1 and ψ_2 can then contribute to the anomalies of S and B^i. Fortunately, these cancel because the fermions are singlets under the B^i and have equal and opposite S charges r_i-1 and $1-r_i$.

This concludes our discussion of $N=2$ supersymmetry in $R\times S^3$. In Minkowski space, there exists an off-shell representation for the $N=2$ gauge multiplet also.[5] Presumably this can be carried over to $R\times S^3$, but we have not checked this. We have not examined the possibility of constructing an $N=2$ superspace either.

IV. $N=3$ SUPERSYMMETRY

In Minkowski space, the minimal multiplet which has $N=3$ supersymmetry also automatically has $N=4$ supersymmetry. It is obtained by coupling a massless $N=2$ matter multiplet in the adjoint representation to a (non-Abelian) $N=2$ gauge multiplet. On following this procedure in $R\times S^3$, we find that at most three supersymmetries can be maintained and one supersymmetry must necessarily be broken (by terms of order a and a^2).

Let us first discuss the $N=3$ superalgebra. There is an internal SU(3) symmetry group generated by eight bosonic operators C^i. These satisfy

$$[C^i, C^j] = -2if^{ijk}C^k \,, \tag{14}$$

where f^{ijk} are the structure constants of SU(3). Introduce the Gell-Mann matrices λ^i normalized so that $[\lambda^i,\lambda^j]=2if^{ijk}\lambda^k$ and $\mathrm{tr}\lambda^i\lambda^j=2\delta^{ij}$. There are 12 fermionic generators in the algebra, $Q_{a\alpha}$ and $\bar{Q}_{a\alpha}$, where $a=1,2,3$ and $\alpha=1,2$. These satisfy

$$[C^i, Q_{a\alpha}] = \lambda^i_{ab}Q_{b\alpha}, \quad [C^i, \bar{Q}_{a\alpha}] = -\bar{Q}_{b\alpha}\lambda^i_{ba} \,. \tag{15}$$

Two Q's or two $\bar{Q}$'s anticommute. The nonzero anticommutator is

$$\{Q_{a\alpha}, \bar{Q}_{b\beta}\} = 2\delta_{ab}\delta_{\alpha\beta}\left[H - \frac{a}{3}R\right] + 2\delta_{ab}\sigma^i_{\alpha\beta}L^i - 2a\,\delta_{\alpha\beta}\lambda^j_{ab}C^j \,, \tag{16}$$

where R is a U(1) charge satisfying Eq. (5) with all the Q's and $\bar{Q}$'s. R commutes with all other generators and the C^i commute with H, L^i, and M^i. No central charges (besides H) appear. This algebra reduces to (a subalgebra of) the Minkowski space $N=3$ superalgebra if $a\to 0$.

We can check that this gives the $N=2$ algebra of Sec. III if the internal indices a,b are restricted to 1 and 2. Identifying B^1, B^2, and B^3 of the $N=2$ algebra with C^1, C^2, and C^3, and comparing Eq. (13) with Eq. (16), we see that

$$S = -\frac{R}{3} - \frac{C^8}{\sqrt{3}} \,. \tag{17}$$

This is indeed a central charge as far as $Q_{1\alpha}$, $Q_{2\alpha}$ and their conjugates are concerned because it commutes with all these eight generators due to Eqs. (5) and (15).

We can now discuss the field multiplet of $N=3$ super-Yang-Mills theory. Denote the $N=2$ gauge multiplet by $(A^m, A, \lambda_1, \lambda_2)$ and the $N=2$ matter multiplet (without its auxiliary fields) in the adjoint representation by $(A_1, A_2, \psi_1, \psi_2)$. For the latter, we must make a choice for the parameter r_1 appearing at the end of Sec. III. For arbitrary r_1, the theory will only have $N=2$ supersymmetry in $R\times S^3$, and nothing more.

However, for the special choices $r_1=0$ or 2, the theory will have an additional supersymmetry giving $N=3$. Let us choose $r_1=0$ henceforth and show that this is true. Let us rename $\psi_1\equiv\lambda_3$, $\psi_2\equiv\lambda_4$, and $A\equiv A_3$. The order a and a^2 terms in the quadratic part of the Lagrangian for the A's and λ's is then

$$\mathcal{L}_a = 3a\bar{\lambda}_4\lambda_4 + a\sum_{a=1}^{3}\bar{\lambda}_a\lambda_a - 2ia\sum_{a=1}^{3}A_a^*\partial_0 A_a \,. \tag{18}$$

Clearly, the Lagrangian for λ_4 differs by a term of order a from the other λ's. There is no choice of r_1 which can make all the four fermions look the same, but $r_1=0$ or 2 makes three of them look identical at the quadratic level. The three A's also look identical then.

Let us now look at some of the interaction terms (cubic and quadratic) in order to make it plausible that $(\lambda_1\lambda_2\lambda_3)$ and $(A_1A_2A_3)$ form triplets under the internal-symmetry group SU(3), whereas A^m and λ_4 are singlets. The potential has terms which are (a) cubic in the A's and λ's of the Yukawa type and (b) quartic in the A's. Dropping all the gauge group indices and numerical coefficients, there are three kinds of terms in the potential:

$$\begin{aligned} V_1 &= g(A_1^*\lambda_2\lambda_3 + A_2^*\lambda_3\lambda_1 + A_3^*\lambda_1\lambda_2) + \text{H.c.} \,, \\ V_2 &= g(A_1\lambda_1\lambda_4 + A_2\lambda_2\lambda_4 + A_3\lambda_3\lambda_4) + \text{H.c.} \,, \\ V_3 &= g^2(A_1^*A_1 + A_2^*A_2 + A_3^*A_3)^2 \,, \end{aligned} \tag{19}$$

where g is the gauge coupling. This potential is SU(3) invariant if $(A_1A_2A_3)$ are in the 3 (fundamental) representation of SU(3) and $(\lambda_1\lambda_2\lambda_3)$ are in the $\bar{3}$ (antifundamental) representation. (These assignments will be explicitly demonstrated a little later.) Then $\sum_{a=1}^3 A_a^*A_a$, $\sum_{a=1}^3 A_a\lambda_a\lambda_4$, and $\sum_{a,b,c=1}^3 \epsilon^{abc}A_a^*\lambda_b\lambda_c$ are all SU(3) invariant. One can check that the other interaction terms, which involve the gauge field A^m, are also SU(3) invariant.

Once again, the $N=3$ supersymmetry will be broken if there are anomalies in the currents corresponding to the charges C^i and R. The C^i anomalies vanish because $\mathrm{tr}C^a=0$ on $(\lambda_1\lambda_2\lambda_3)$ and λ_4 is an SU(3) singlet. The R anomaly is more interesting. We will show that it vanishes by determining the R charges of all the fields.

The simplest way of doing this is to work out how the various fields fall into $N=1$ supersymmetry multiplets under the three supersymmetries $(Q_{1\alpha},\bar{Q}_{1\alpha})$, $(Q_{2\alpha},\bar{Q}_{2\alpha})$, and $(Q_{3\alpha},\bar{Q}_{3\alpha})$ which we denote by $n=1$, 2, and 3. We find

$$\begin{aligned} n=1&: \ (A^m\lambda_1)(A_3\lambda_2)(A_2\lambda_3)(A_1^*\lambda_4) \,, \\ n=2&: \ (A^m\lambda_2)(A_1\lambda_3)(A_3\lambda_1)(A_2^*\lambda_4) \,, \\ n=3&: \ (A^m\lambda_3)(A_2\lambda_1)(A_1\lambda_2)(A_3^*\lambda_4) \,. \end{aligned} \tag{20}$$

For each of these supersymmetries, the algebra of Eqs. (4) and (5) will hold with the three U(1) charges denoted here by R_1, R_2, and R_3, respectively. From Eq. (16),

$$\begin{aligned} R_1 &= -\frac{R}{3} - C^3 - \frac{C^8}{\sqrt{3}}, \quad R_2 = -\frac{R}{3} + C^3 - \frac{C^8}{\sqrt{3}} \,, \\ R_3 &= -\frac{R}{3} + 2\frac{C^8}{\sqrt{3}} \,. \end{aligned} \tag{21}$$

Note that Eq. (17) follows from this because the central charge of the $N=2$ superalgebra is given by $S=\frac{1}{2}(R_1+R_2)$.

From Eq. (20), the charges R_1, R_2, and R_3 for all fields can be computed and hence the values of C^3, C^8, and R can be obtained. We discover that the R charges of $(A_1A_2A_3A^m)$ and $(\lambda_1\lambda_2\lambda_3\lambda_4)$ are given by (2,2,2,0) and $(1,1,1,-3)$, respectively. Therefore, the interaction terms in Eq. (19) are all R invariant. Further, the R anomaly cancels between the four fermions. $N=3$ supersymmetry is therefore anomaly-free in $R\times S^3$.

The C^3 charges for the same eight fields are found to be $(1,-1,0,0)$ and $(-1,1,0,0)$. The C^8 charges are found to be $1/\sqrt{3}(1,1,-2,0)$ and $1/\sqrt{3}(-1,-1,2,0)$. This clearly shows that $(A_1A_2A_3)$ are in the 3 representation of SU(3), $(\lambda_1\lambda_2\lambda_3)$ are in the $\bar{3}$ representation, and λ_4 and A^m are singlets.

At this point, one can ask the following question. The $N=4$ superalgebra in Minkowski space does not allow a U(1) charge which has an algebra such as Eq. (5) with all the 16 fermionic supersymmetry generators. Where does the U(1) charge R of Eq. (16) come from? On examining the R charges of the four fermions, it is clear that $\mathrm{tr}R=0$ and therefore that R can actually be an SU(4) generator. But it looks like a U(1) generator on the 12 fermionic generators of $N=3$ supersymmetry. Evidently, the SU(4) internal symmetry of the $N=4$ superalgebra in Minkowski space has broken down to SU(3)×U(1) due to the curvature of S^3. The supersymmetries generated by $Q_{4\alpha}$ and $\bar{Q}_{4\alpha}$ of Minkowski space are broken by terms of order a and a^2 on S^3. The four fermions λ_a form an irreducible representation of SU(4).

V. OUTLOOK

It would be interesting to know whether the $N=2$ and $N=3$ theories in $R\times S^3$ share all the remarkable finiteness properties of the $N=2$ and $N=4$ theories in Minkowski space.[5] One way to approach this topic is to develop the language of superspace in $R\times S^3$ and set up perturbation theory in terms of superfields. Then one can attempt to construct nonrenormalization arguments, analogous to those in Minkowski space.

Physically, it is somewhat mysterious why certain bosonic internal-symmetry generators appear in the anticommutators of some of the supersymmetry generators in S^3. It may be possible to understand this by considering the various (extended) superconformal algebras in Minkowski space and examining which elements of each algebra continue to generate exact symmetries in the curved space-time background called $R\times S^3$. It is known that such a prescription can be used to derive the (extended) superalgebra in anti–de Sitter space[6] which has certain similarities with $R\times S^3$. For example, both spaces are characterized by a dimensional constant which determines the curvature.

Finally, the possibility of defining supergravity in $R\times S^3$ needs to be examined.

ACKNOWLEDGMENTS

I thank M. Mehta for some stimulating comments.

APPENDIX

The SU(M/N) superunitary algebra[4] is briefly discussed here. Let M^i, $i=1,2,\ldots,M^2-1$, and N^i, $i=1,2,\ldots,N^2-1$ be Hermitian bosonic generators of SU(M) and SU(N), respectively. Thus,

$$\begin{aligned}&[M^i,M^j]=-im^{ijk}M^k\,,\\&[N^i,N^j]=-in^{ijk}N^k,\quad [M^i,N^j]=0\,,\end{aligned}\tag{A1}$$

where m^{ijk} and n^{ijk} are the structure constants of SU(M) and SU(N). Let $(\mu^i)_{\alpha\beta}$ and $(\nu^i)_{ab}$ be $M\times M$ and $N\times N$ matrices generating SU(M) and SU(N). They satisfy (A1), but without the minus signs on the right-hand sides, and they are normalized so that $\mathrm{tr}\mu^i\mu^j=2\delta^{ij}$ and $\mathrm{tr}\nu^i\nu^j=2\delta^{ij}$. The following relations prove very useful when verifying some Jacobi identities later:

$$\begin{aligned}&\sum_i \mu^i_{\alpha\beta}\mu^i_{\gamma\delta}=2\delta_{\alpha\delta}\delta_{\beta\gamma}-\frac{2}{M}\delta_{\alpha\beta}\delta_{\gamma\beta}\,,\\&\sum_j \nu^j_{ab}\nu^j_{cd}=2\delta_{ad}\delta_{bc}-\frac{2}{N}\delta_{ab}\delta_{cd}\,.\end{aligned}\tag{A2}$$

The $2MN$ fermionic generators $Q_{a\alpha}$ and $\bar{Q}_{a\alpha}$ transform as the (anti)fundamental representations (M,N) and $(\bar{M},\bar{N})$, respectively, of SU(M)×SU(N). Thus,

$$\begin{aligned}&[M^i,Q_{a\alpha}]=\mu^i_{\alpha\beta}Q_{a\beta},\quad [M^i,\bar{Q}_{a\alpha}]=-\bar{Q}_{a\beta}\mu^i_{\beta\alpha}\,,\\&[N^j,\bar{Q}_{a\alpha}]=\nu^j_{ab}Q_{b\alpha},\quad [N^j,\bar{Q}_{a\alpha}]=-\bar{Q}_{b\alpha}\nu^j_{ba}\,,\end{aligned}\tag{A3}$$

We define a U(1) generator R such that

$$[R,Q_{a\alpha}]=-Q_{a\alpha},\quad [R,\bar{Q}_{a\alpha}]=\bar{Q}_{a\alpha}\tag{A4}$$

and $[R,M^i]=[R,N^i]=0$.

The anticommutator of the Q's and $\bar{Q}$'s can then be found purely from the Jacobi identities:

$$\begin{aligned}\{Q_{a\alpha},\bar{Q}_{b\beta}\}&=2\delta_{ab}\delta_{\alpha\beta}\left[S+\left[\frac{2}{N}-\frac{2}{M}\right]R\right]\\&\quad+2\delta_{ab}\mu^i_{\alpha\beta}M^i-2\delta_{\alpha\beta}\nu^j_{ab}N^j\,.\end{aligned}\tag{A5}$$

Here S is an Hermitian central charge which commutes with all generators. Note that R does not appear in (A5) if $M=N$. This is the case for $N=2$ supersymmetry in $R\times S^3$ since $M=2$ there.

Central charges can appear in the anticommutators $\{Q,Q\}$ and its Hermitian $\{\bar{Q},\bar{Q}\}$ if $M=N=2$:

$$\{Q_{a\alpha},Q_{b\beta}\}=\epsilon_{ab}\epsilon_{\alpha\beta}Z,\quad \{\bar{Q}_{a\alpha},\bar{Q}_{b\beta}\}=\epsilon_{ab}\epsilon_{\alpha\beta}\bar{Z}\,.\tag{A6}$$

Then $[R,Z]=-2Z$ and $[R,\bar{Z}]=2\bar{Z}$.

The above discussion is valid for all $M,N\geq 1$.

---

[1] D. Sen, Nucl. Phys. **B284**, 201 (1987).

[2] D. Sen, Phys. Rev. D **39**, 1795 (1989).

[3] A. Salam and J. Strathdee, Nucl. Phys. **B87**, 85 (1975); P. Fayet, *ibid.* **B90**, 104 (1975).

[4] P. van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981); P. G. O. Freund and I. Kaplansky, J. Math. Phys. **17**, 228 (1976).

[5] For reviews of extended supersymmetry and references, see M. F. Sohnius, Phys. Rep. **128**, 39 (1985); P. West, *Introduction to Supersymmetry and Supergravity* (World Scientific, Singapore, 1986).

[6] S. J. Gates, Jr., M. T. Grisaru, M. Roček, and W. Siegel, *Superspace or One Thousand and One Lessons in Supersymmetry* (Benjamin/Cummings, Reading, MA, 1983).