Approximate computation of the small-fluctuation determinant around a sphaleron

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We compute, in the high-temperature limit, the determinant of small fluctuations around the sphaleron configuration of electroweak theory using the approximate method of Diakonov, Petrov, and Yung. For the ratio λ/g^2 of scalar four-point coupling λ to gauge coupling g^2 near unity, we find that the determinant is of order 1 in agreement with previous computations. For λ/g^2 large corresponding to a strongly coupled Higgs phase, or for λ/g^2 very small tending to the Coleman-Weinberg limit, we find that the determinant strongly suppresses the rate of baryon-number-changing processes.

I. INTRODUCTION

In electroweak theory at high temperature, $T\sim 1$ TeV, sphaleron field configurations mediate baryon-number-changing processes.[1-6] This sphaleron configuration is a classical solution of the electroweak equations of motion which is static but classically unstable. It has been shown how to relate the sphaleron and small fluctuations around the sphaleron to a computation of the rate of baryon-number-changing processes[4,5] using techniques of nonequilibrium statistical mechanics developed by Langer and by Affleck.[7-9] Previously, this computation has been performed for the ratio λ/g^2 of scalar four-point coupling λ to gauge coupling g^2 near unity. It was found that baryon-number nonconservation occurs at a rate sufficiently large to be relevant for cosmology.[4]

If baryon number is violated at a significant rate at temperatures of the order of the electroweak scale, then surely such processes must affect the results of computations of baryogenesis which take place at higher-energy scales.[10,11] In particular, any $B+L$ excess will be strongly modified. It is also possible that the baryon number of the Universe might arise at the electroweak scale as a result of sphaleron-induced transitions.[12,13] If so, then it may be possible to compute the baryon asymmetry using a theory which will be thoroughly tested in the near future.

A key ingredient in the computation of the rate of baryon-number-changing processes is the determinant of small fluctuations around the sphaleron field configuration. This determinant may be estimated for $\lambda/g^2\sim 1$. After the translation and rotation zero modes of the sphaleron have been extracted, the determinant may be scaled in such a way that it is a pure number κ, and is therefore assumed to be of order 1. While this is a plausible result for $\lambda/g^2=1$ [as confirmed in studies of solvable $(1+1)$-dimensional models], it is possible that this number could have a strong dependence on λ/g^2. For the extreme limits where $\lambda/g^2\rightarrow\infty$, corresponding to strongly coupled Higgs particles, and for the Coleman-Weinberg limit, $\lambda/g^2\rightarrow 0$, the number κ associated with the determinant might become singular.

We have therefore attempted an approximate computation of the determinant of small fluctuations around the sphaleron. To do this, we shall employ the approximate method of Diakonov, Petrov, and Yung.[14] The validity of this approximation is discussed in detail by one of the authors in a separate study,[15] where the method is applied to a solvable $(1+1)$-dimensional model.[16] There the approximation scheme of Diakonov-Petrov-Yung can be carried to very high order and appears to agree well with the exact results.

The results of our computations confirm the conjecture of Arnold and McLerran that the number κ is in fact of order 1 for $\lambda/g^2=1$ (Ref. 4.) At this value of λ/g^2, we find that $\kappa\simeq 1.3$. Our calculations are also in accord with the recent computation of Akiba, Kikuchi, and Yamagida who use $\lambda/g^2=\frac{1}{8}$ (corresponding to $M_H=M_W$), and find that $\kappa=0.79$ when angular momentum excited modes are ignored.[17] For $\lambda/g^2=\frac{1}{8}$, we obtain $\kappa\simeq 0.60$.

The most interesting consequence of our computations is that we find a very strong dependence on λ/g^2. In Fig. 1 we plot our results for $\ln\kappa$. In Fig. 2 we present our results, up to a factor of 2 $[T(\mathrm{TeV})]^{-1}\alpha_3^{-7}e^{-E_{\rm sp}/T}$, for the fractional rate F of baryon-number transitions relative to the expansion rate of the Universe:

$$\left|\frac{1}{N_B}\frac{dN_B}{dT}\bigg/\frac{1}{R}\frac{dR}{dT}\right|\equiv F\frac{2}{T}\alpha_3^{-7}e^{-E_{\rm sp}/T}\ . \tag{1}$$

[The quantity $\alpha_3(T)$ is the effective fine-structure constant of electroweak theory in the high-temperature limit. See discussion below.] Evidently, at temperatures of order 1 TeV, the expansion rate of the Universe is about 9 to 10 orders of magnitude less than a typical inverse thermal time $1/t\sim T$. Therefore, if we take the results of our computation seriously, then we see that, in accord with previous computations, the rate of baryon-number-changing processes exceeds the expansion rate of the Universe for $T\gg M_W$.

We should also note that our computations are only

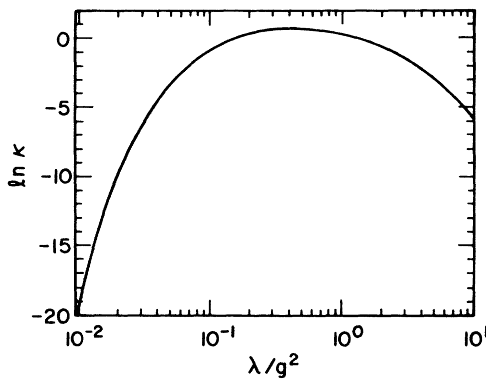

FIG. 1. A plot of $\ln\kappa$, where κ is the determinant of small fluctuations around the sphaleron solution with zero modes removed, as a function of λ/g^2.

valid in the range of temperatures $M_W(T) \ll T \ll M_W(T)/\alpha_W$. Our computations break down well before the Boltzmann suppression factor is of order 1, and the rate at high temperatures may be more suppressed than we present here.

We should also note that in the limit of small λ/g^2 corresponding to the Coleman-Weinberg limit, the approximate calculation we present breaks down. In this limit terms of higher order than those which we compute become important, and the suppression shown in Fig. 1 may be entirely an artifact of this approximation.

Although the results of our computations suggest that there may be strong suppression of the rate of baryon-number-changing processes for small and large λ/g^2, these computations are based on an approximation which has been carried out only to first order. The computation of the determinant of small fluctuations with no approximations is clearly required to verify the results presented here. Nevertheless, that portion of our calculation associated with the contribution of zero modes is exact for all λ/g^2. Working in a background gauge, we also prove that, even after the removal of zero modes, the determinant is a gauge-invariant function of the external sphaleron field.

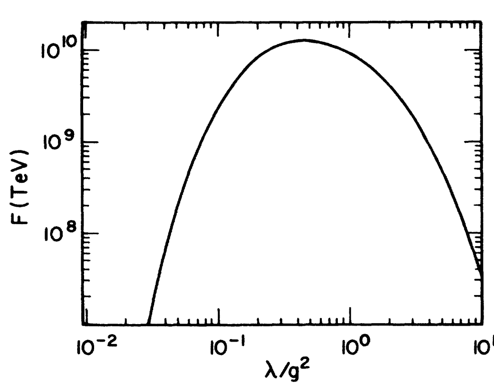

FIG. 2. A plot of the fractional rate F of baryon-number transitions relative to the expansion rate of the Universe [cf. Eq. (1)].

The outline of this paper is as follows. In the second section we review the sphaleron computation, and isolate those terms which must be computed. Here we discuss how the theory reduces to a three-dimensional theory in the high-temperature limit, and show explicitly how the masses and coupling constants of the theory acquire a temperature dependence consistent with effective potential analysis.

In the third section, we expand the action of the effective three-dimensional theory (e.g., the energy) to first order in small fluctuations around a given classical configuration. Gauge degrees of freedom are fixed with the $R_{\xi=1}$ background gauge condition. The zero-frequency modes are identified and their contribution to the rate of baryon-number change is evaluated. We also review the computation of the negative-frequency mode.

In the fourth section, we use the method of Diakonov-Petrov-Yung to compute the determinant of small fluctuations with the zero-frequency modes removed.[14]

In the fifth and final section, we gather together all the factors to present results for baryon-number-changing rates for various λ/g^2. In the range where the computation is consistent, we compare these rates with the expansion rate of the Universe to determine if there is a range of temperatures where the rate is sufficiently large to be cosmologically relevant.

In the Appendix, we collect formulas necessary to implement the method of Diakonov, Petrov, and Yung. This includes the evaluation of certain gauge-invariant functionals of the background sphaleron field.

II. THE DETERMINANT

As has been much discussed,[6] to compute the rate of baryon-number generation at high temperature in a reliable way, we must work in the temperature range

$$M_W(T) \ll T \ll M_W(T)/\alpha_W \ . \tag{2}$$

The lower limit in this equation validates the use of high-temperature expansion techniques, and is equivalent to the condition that high-temperature transition processes proceed at a rate faster than the low-temperature instanton-induced process.[18,19] The upper limit is equivalent to the criteria that weak-coupling expansion techniques may be used to compute the rate. For temperatures larger than this upper limit, the magnetic screening mass gap becomes so small that infrared effects invalidate the use of weak-coupling expansions.[19,20] The temperature-dependent mass in this equation is

$$M_W(T) = M_W(0)(1 - T^2/T_c^2)^{1/2} \ . \tag{3}$$

The zero-temperature W-boson mass is given in terms of the vacuum expectation value of the Higgs field, v, and the SU(2) weak coupling, g as $M_W(0) = gv/2$. The critical temperature for symmetry restoration is

$$T_c = v\left[\frac{1}{1+\frac{3}{8}g^2/\lambda}\right]^{1/2} . \tag{4}$$

In all of the following analysis, we shall study only pure SU(2) gauge theory. This corresponds to electroweak theory in the limit that the Weinberg angle vanishes:

$$\Theta_W = 0 \ . \tag{5}$$

The finite-temperature action in this case is

$$S = \frac{1}{g^2}\int_0^\beta dt \int d^3x \left[\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + (D\Phi)^\dagger(D\Phi) + \lambda(\Phi^\dagger\Phi - v^2/2)^2\right] , \tag{6}$$

where $\beta \equiv 1/T$. Corrections arising from nonzero Θ_W can be systematically accounted for by a perturbative expansion in Θ_W, and are not expected to lead to large effects.

We ignore fermions here since we are interested in the high-temperature limit. In this limit, fermions decouple except for the fermion zero mode which produces a shift in the baryon number when topological charge is changed, and generates effective temperature-dependent masses for the Higgs boson, W and Z. The first effect is straightforward to include, and only performs the bookkeeping needed to relate topological charge to baryon number. The second effect is more subtle, and affects the proper reduction of the theory at high temperatures to an effective three-dimensional theory. Later in this section, we shall explicitly make this reduction to the three-dimensional theory. It will be trivial there to see how fermions modify this reduction.

For the range of temperatures where weak-coupling methods may be applied it can be convincingly argued that the rate per unit volume of sphaleron-induced baryon-number-changing processes may be computed semiclassically.[7-9] The semiclassical computation involves finding a static, unstable solution to the classical equations of motion: the sphaleron. The energy of this solution gives the minimal height of the energy barrier between two local minima of the free energy which differ in baryon number by N_F units, where N_F is the number of weak doublet families. The sphaleron is an unstable static solution since it corresponds to the top of an energy barrier, and the rate of decay in small fluctuations around the sphaleron is denoted by ω_-.

The formula for the rate of decay is

$$\Gamma = \frac{\omega_- \beta}{\pi}\,\mathrm{Im}F \ , \tag{7}$$

where F is the free energy of the system. In the semiclassical limit, the imaginary part of the free energy may be computed. The result[4] for the rate per unit volume is

$$\Gamma/V = \frac{\omega_-}{2\pi}\mathcal{N}_{\mathrm{tr}}(\mathcal{N}V)_{\mathrm{rot}}\left[\frac{\alpha_W T}{4\pi}\right]^3 \alpha_3^{-6} e^{-\beta E_{\mathrm{sp}}}\kappa \ , \tag{8}$$

where $\alpha_W = g^2/4\pi \simeq \frac{1}{29}$ is the weak fine-structure constant.

Let us briefly discuss the various elements that go into this formula.

The factor $e^{-\beta E_{\mathrm{sp}}}$ is the Boltzmann suppression factor for thermal activation over the energy barrier between sectors of different baryon number. As discussed above, the minimal height of this energy barrier is given by the classical energy of sphaleron E_{sp}. In the range of temperatures considered, Eq. (2), that is to say, $T \ll E_{\mathrm{sp}}$, all classical trajectories connecting sectors of different baryon number must pass through (or very close to) the sphaleron configuration.

The sphaleron mass E_{sp} may be computed knowing the sphaleron classical solutions. First, we rescale coordinates and fields according to

$$\mathbf{r} \to \boldsymbol{\xi}/gv, \quad A \to vA, \quad \Phi \to v\Phi \ . \tag{9}$$

Then in terms of these dimensionless rescaled fields, the sphaleron solutions are

$$A_0 = 0 \ ,$$
$$\mathbf{A} = 2\frac{f(\xi)}{\xi}\hat{\boldsymbol{\xi}}\times\boldsymbol{\tau} \ , \tag{10}$$
$$\Phi = \sqrt{2}h(\xi)\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\tau}u_{-1/2} \ ,$$

where

$$\xi = |\boldsymbol{\xi}| = gvr \ . \tag{11}$$

The profile functions $f(\xi)$ and $h(\xi)$ satisfy the coupled equations

$$-\frac{d^2f}{d\xi^2} + \frac{2}{\xi^2}f(1-f)(1-2f) - \tfrac{1}{4}h^2(1-f) = 0 \ ,$$
$$-\frac{d^2h}{d\xi^2} - \frac{2}{\xi}\frac{dh}{d\xi} + \frac{2}{\xi^2}h(1-f)^2 + \frac{\lambda}{g^2}(h^2-1)h = 0 \ , \tag{12}$$

subject to the boundary conditions

$$\lim_{\xi\to 0}\frac{f(\xi)}{\xi} = \lim_{\xi\to 0}h(\xi) = 0 \ ,$$
$$\lim_{\xi\to\infty}f(\xi) = \lim_{\xi\to 0}h(\xi) = 1 \ . \tag{13}$$

We have solved these equations for the sphaleron profile functions and find that the energy of the sphaleron is

$$E_{\mathrm{sp}} = B2M_W(T)/\alpha_W \ , \tag{14}$$

where B is a dimensionless function of λ/g^2. B is displayed in Fig. 3, and for all values of λ/g^2 is of order 1.

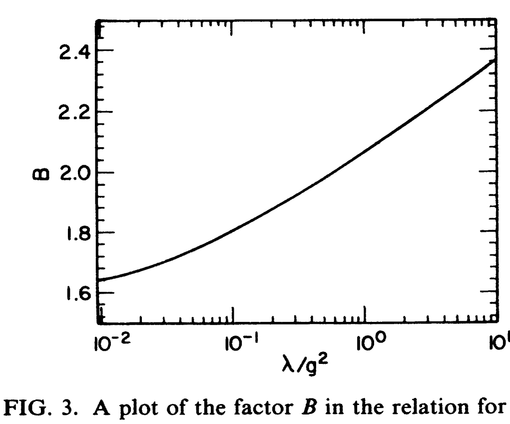

FIG. 3. A plot of the factor B in the relation for the sphaleron energy $E_{\mathrm{sp}} = B2M_W(T)/\alpha_W$, as a function of λ/g^2.

The quantities $\mathcal{N}_{\text{tr}}$ and $\mathcal{N}_{\text{rot}}$ are normalization integrals for the translation and rotation zero modes. We shall specify these quantities in complete detail below, but a generic rule for constructing them may be given: If the coefficient of the generator of a symmetry is denoted by χ, then the associated zero-mode normalization integral is

$$\mathcal{N}=\left[\frac{1}{2\pi}\int(\chi)^2\right]^{1/2} . \tag{15}$$

As we work with the dimensionless coordinates and fields given by (9), $\mathcal{N}_{\text{tr}}$ and $\mathcal{N}_{\text{rot}}$ are dimensionless functionals of the profile functions $f(\xi)$ and $h(\xi)$.

There are two additional factors in the transition rate (8) associated with the normalization of the zero modes. One is the volume of the rotation group V_{rot}, which is $8\pi^2$. There are also the factors of α_3^{-6}, where $\alpha_3=g_3^2/4\pi$. Here g_3 is the coupling constant which scales the strength of interactions in the three-dimensional gauge theory obtained by taking the high-temperature limit of (6):

$$g_3^2=\frac{g^2T}{2M_W(T)} . \tag{16}$$

Finally there is the quantity κ defined as the square-root of the inverse determinant of small fluctuations around the sphaleron multiplied by the square-root of the determinant of small fluctuations in the vacuum at finite temperature:

$$\kappa=\text{Im}\frac{\det^{1/2}(\delta^2S/\delta\phi^2)|_{\phi=\phi_{\text{vac}}}}{\det^{1/2}(\delta^2S/\delta\phi^2)|_{\phi=\phi_{\text{sp}}}} . \tag{17}$$

In this equation, ϕ generically represents the scalar and vector fields of electroweak theory. It is also understood that the zero-frequency modes of $\delta^2S/\delta\phi^2|_{\phi=\phi_{\text{sp}}}$ are to be deleted in the evaluation of the determinant.

The determinants in the above equation are to be evaluated at finite temperature. Since we are interested in the high-temperature limit, $T\gg M_W$, we wish to reduce the theory to an effective three-dimensional gauge theory. This is nontrivial because the three-dimensional gauge theory which results is an ultraviolet-divergent (albeit super-renormalizable) theory. The ultraviolet singularities of this three-dimensional theory arise in part from true ultraviolet singularities of the four-dimensional finite-temperature theory. However, spurious divergences also occur if one improperly reduces the theory from four to three dimensions. In the latter case, the so-called ultraviolet singularities are actually cut off at a scale on the order of the temperature.

Naively, the argument for dimensional reduction arises from the following observation: In propagators at finite temperatures there are the Matsubara frequencies

$$\begin{aligned}&\omega=2n\pi T \text{ bosons} ,\\ &\omega=(2n+1)\pi T \text{ fermions} .\end{aligned} \tag{18}$$

where n is an integer. At high temperatures, these frequency factors make the propagator small and so reduce their contribution to Feynman diagrams unless $\omega=0$. This is impossible for fermions, and therefore fermions decouple. And in the case of bosons, it would appear that all but the static, $n=0$ modes can be dropped in evaluating Matsubara sums.

This last argument must be modified however. There are contributions which arise from values of $\omega\sim T$. For example, in the mass insertions shown in Fig. 4, the internal sum over Matsubara frequencies is cut off at ω of this order. Consequently, a mass-squared correction is generated which is of order T^2. It is this mass insertion which is responsible for symmetry restoration in electroweak theory.[21-23] To properly reduce to a three-dimensional theory these singular mass insertions must be summed.[24]

The standard way to perform this summation is to do these singular terms explicitly and include them in the definition of the effective potential around which loop corrections are performed. By including them in the effective potential, they must be deleted in the loop corrections, rendering the loop corrections reducible to three dimensions. We shall explicitly demonstrate this in the one-loop approximation in the next few paragraphs.

One might also worry about the renormalization of the coupling constants of the theory, which effectively makes them temperature dependent. We can see however that such effects are not important in the temperature range in which we are interested to the order in the loop expansion to which we compute. Recall that the sphaleron Boltzmann factor is of order

$$e^{-2BM_W(T)/\alpha_W/T} . \tag{19}$$

The one-loop corrections are independent of coupling, but there may be singular logarithms of T which arise and effectively replace

$$\alpha_W\rightarrow\alpha_W(T) \tag{20}$$

in the previous equation. This generates a correction of order

$$e^{-2BM_W(T)\ln[T/M_W(T)]/T} , \tag{21}$$

which, in the range of temperatures, $T\gg M_W(T)$, may be ignored relative to the terms being computed. We see therefore that in making the dimensional reduction, we shall not have to be careful about corrections of coupling constants in the four-dimensional theory which are ultra-

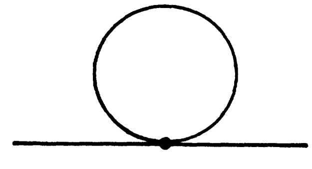

FIG. 4. A mass insertion at finite temperature.

violet singular.

In order to see how to treat the finite-temperature mass insertions in one loop, it is useful to employ the Schwinger proper-time representation of the ratio of determinants:

$$\frac{\det K}{\det K_0}=\exp\left[-\mathrm{Tr}\int_0^\infty \frac{dt}{t}(e^{-tK}-e^{-tK_0})\right] . \tag{22}$$

Here Tr denotes a functional trace. In our application, K and K_0 will be operators of the form

$$K=-D(A)^2+V , \tag{23}$$

$$K_0=K(A\to 0,\ V\to V_0)=-\partial^2+V_0 , \tag{24}$$

where $D_\mu(A)=\partial_\mu-igA_\mu$ is the covariant derivative in some representation of the gauge group, V is a potential function of background fields, and V_0 is the same function V evaluated in the vacuum. The integral in (22) is also assumed to be regularized, as may be arranged, for example, by cutting off the integration near $t=0$. The singularity near $t=0$ arises from ultraviolet singularities.

The functional Tr can be evaluated in a complete set of basis states, which we choose to be plane waves:

$$\mathrm{Tr}e^{-tK}=\mathrm{tr}\int d^4x\int\frac{d^4p}{(2\pi)^4}e^{-ipx}e^{-tK}e^{ipx} . \tag{25}$$

In this equation, the uncapitalized factor tr denotes a trace over spin and gauge group indices. At finite temperature, the integration over p^0 is in fact a sum over Matsubara frequencies, $\omega_n=2\pi nT$:

$$\int\frac{dp^0}{2\pi}=T\sum_n . \tag{26}$$

Bringing the plane-wave factors through the operator e^{-tK} yields

$$\begin{aligned}\mathrm{Tr}e^{-tK}&=\mathrm{tr}\int d^4x\int\frac{d^4p}{(2\pi)^4}e^{-tK(\partial\to\partial+ip)}\mathbb{1} ,\\ &=\mathrm{tr}\int d^4x\int\frac{d^4p}{(2\pi)^4}e^{-tp^2}e^{-t\delta K}\mathbb{1} ,\end{aligned} \tag{27}$$

where we have defined the operator $\delta K(p,x)$ via

$$K(\partial\to\partial+ip)=p^2+\delta K(p,x) . \tag{28}$$

If K is given by (23), then

$$\delta K(p,x)=-2ip\cdot D(A)-D(A)^2+V . \tag{29}$$

Similar expressions for $\mathrm{Tr}e^{-tK_0}$ and $\delta K_0(p,x)$ may be obtained by setting $A\to 0$ and $V\to V_0$ in the above formulas.

Taking the logarithm of (22), our expression for $\ln(\det K/\det K_0)$ now reads

$$\ln\left[\frac{\det K}{\det K_0}\right]=-\int_0^\infty\frac{dt}{t}\mathcal{O}(t) , \tag{30}$$

where

$$\mathcal{O}(t)=\int d^4x\int\frac{d^4p}{(2\pi)^4}e^{-tp^2}\mathrm{tr}(e^{-t\delta K}-e^{-t\delta K_0})\mathbb{1} . \tag{31}$$

To isolate the potentially ultraviolet singular terms, we begin by expanding the exponentials $e^{-t\delta K}$ and $e^{-t\delta K_0}$ in powers of t, keeping the factor of e^{-tp^2} intact. Next we collect terms of equal order in the t expansion. In doing so, we must take into account the fact that, due to the exponential factor e^{-tp^2}, factors of momenta p_μ are implicitly of order $t^{-1/2}$ in the ultraviolet limit.

The term of lowest order cancels between K and K_0. This implies that $\mathcal{O}(t)$ has a t expansion of the form

$$\mathcal{O}(t)\sim\frac{1}{t}\mathcal{O}_1+\mathcal{O}_2+t\mathcal{O}_3+\cdots . \tag{32}$$

Only the terms of order t^{-1} and t^0 give potentially singular contributions to $\ln(\det K/\det K_0)$. The term of order t^0 yields only a logarithmic divergence, which is important for generating running coupling constants. However, as we have argued above, these order t^0 terms are irrelevant for the computation of the high-temperature limit in the range of temperatures that we consider, and so we ignore them. We see therefore that the terms of order t^0 and higher may be immediately dimensionally reduced to the three-dimensional theory by excluding all but the $n=0$ term in the sum over Matsubara frequencies.

The term of order t^{-1} is trickier however. Its contribution to $\ln(\det K/\det K_0)$ is

$$\begin{aligned}&-\int_0^\infty\frac{dt}{t}\int d^4x\int\frac{d^4p}{(2\pi)^4}e^{-tp^2}\mathrm{tr}\left[(-t)\delta K+\frac{(-t)^2}{2!}[-2ip\cdot D(A)]^2\right]\mathbb{1}\\ &\qquad=\int_0^\infty dt\int d^4x\int\frac{d^4p}{(2\pi)^4}e^{-tp^2}\mathrm{tr}\{-D(A)^2+2t[p\cdot D(A)]^2+V\}\mathbb{1} ,\end{aligned} \tag{33}$$

minus the above taking $A\to 0$ and $V\to V_0$. Here we have specialized to the operators K and K_0 given by Eqs. (23) and (24). If we further assume that the potentials V and V_0 are independent of x^0 and p_μ and that the gauge-field component A_0 is zero (as is the case for the sphaleron), then the covariant derivatives cancel in the expression above and we are left with

$$\beta\int d^3x\,\mathrm{tr}(V-V_0)\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2} \tag{34}$$

after integrating over t. This expression is evidently quadratically divergent. We may absorb this quadratic divergence by redefining the Higgs vacuum expectation value; that is, the bare value of v is made divergent so that the resulting renormalized value is finite. There are also the finite-temperature corrections which are ultraviolet finite. After subtracting the ultraviolet singular term, the resulting integrations over p and sum over Matsubara frequencies give a factor proportional to T.

It is this finite term of order t^{-1} which cannot be dimensionally reduced to the three-dimensional theory since, obviously, its contribution is undefined if we take $T\to\infty$. However, the strategy for dealing with this finite term is clear: If we include this term in the effective potential, then we can drop the corresponding order t^{-1} term in the expression above for $\ln(\det K/\det K_0)$. Its only effect (to one-loop order) is to generate a temperature dependence for the W-boson mass, as we may see by expanding

$$M_W(T)/T=M_W(0)/T[1-T^2/2T_c^2+O(T^4/T_c^4)] \ . \tag{35}$$

The higher-order terms in this expansion are not included in our one-loop evaluation. The one-loop term is the only term of order T, and we have therefore proven that its only effect is to generate the proper effective mass for particles in the effective potential.

III. SMALL FLUCTUATIONS AROUND THE SPHALERON

To compute κ, we must determine the spectrum of small fluctuations around the sphaleron field configuration. We must then isolate the zero and negative-frequency modes, since these contributions are handled specially in the computation of the baryon-number transition rate.

We begin with the SU(2) electroweak action in three dimensions corresponding to the dimensionally reduced theory:

$$S=\frac{1}{g_3^2}\int d^3\xi[\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a+(D\Phi)^\dagger(D\Phi)+\lambda(\Phi^\dagger\cdot\Phi-\tfrac{1}{2})^2] \ , \tag{36}$$

where fields and coordinates have been rescaled according to (9). We are interested in expanding this around the sphaleron background field, which we denote as A and Φ. We define the small fluctuation fields a and η via

$$\delta A=g_3 a, \quad \delta\Phi=g_3\eta \ . \tag{37}$$

To perform the analysis, it is useful to work in the background $R_{\xi=1}$ gauge, given by

$$D_\mu(A)a^\mu+i(\Phi^\dagger\tau\eta-\eta^\dagger\tau\Phi)=0 \ . \tag{38}$$

It is easy to show that the Faddeev-Popov determinant for this gauge is

$$\Delta_{\mathrm{FP}}=\det[-D(A)^2+\tfrac{1}{2}\Phi^\dagger\Phi] \ . \tag{39}$$

The computation of the gauge-fixed action to first order in small fluctuations is tedious, and the result is

$$\delta S=\int d^3\xi\left[\tfrac{1}{2}(D_\mu a_\nu)(D^\mu a^\nu)+\epsilon_{abc}F^{\mu\nu}_a a^b_\mu a^c_\nu+\tfrac{1}{4}\Phi^\dagger\Phi a_\mu a^\mu+2i(\eta^\dagger a_\mu\cdot\tau D^\mu\Phi-D^\mu\Phi^\dagger a_\mu\cdot\tau\eta)\right.$$
$$\left.+(D_\mu\eta)^\dagger(D^\mu\eta)+2\frac{\lambda}{g^2}(\Phi^\dagger\Phi-\tfrac{1}{2})\eta^\dagger\eta+\tfrac{1}{2}\eta^\dagger\eta\Phi^\dagger\Phi+\left[\frac{\lambda}{g^2}-\frac{1}{8}\right](\Phi^\dagger\eta+\eta^\dagger\Phi)^2\right] \ . \tag{40}$$

Since the fluctuation fields a^a_μ and η are static (recall that we are retaining only $n=0$ Matsubara modes) and $A_0=0$, the a_0 fields decouple from the rest, giving a contribution to the action of

$$\delta S=\int d^3\xi\tfrac{1}{2}a_0[-D(A)^2+\tfrac{1}{2}\Phi^\dagger\Phi]a_0+\cdots \ . \tag{41}$$

Functional integration over these fields yields a factor of $\Delta_{\mathrm{FP}}^{-1/2}$ which partially cancels the Faddeev-Popov determinant. The remaining terms of the fluctuation action can be written in a simple form if we introduce the real 13-dimensional vector

$$\Psi=\begin{bmatrix} a^a_i \\ \tilde\eta \end{bmatrix} , \tag{42}$$

where $\tilde\eta$ is the four-vector constructed from the real and imaginary parts of η according to

$$\tilde\eta=\sqrt{2}\begin{bmatrix}\mathrm{Re}\eta \\ \mathrm{Im}\eta\end{bmatrix} . \tag{43}$$

Analogously,

$$\phi=\sqrt{2}\begin{bmatrix}\mathrm{Re}\Phi \\ \mathrm{Im}\Phi\end{bmatrix} . \tag{44}$$

is the four-vector corresponding to the complex spinor Φ. One then finds that

$$\delta S=\int d^3\xi\tfrac{1}{2}\Psi^T[-\mathcal{D}(A)^2+\mathcal{V}]\Psi \tag{45}$$

(a superscript T denotes transpose) where $\mathcal{D}_i=\partial_i-iA^a_iT^a$ is the covariant derivative in the (reducible) $3\oplus3\oplus3\oplus2\oplus\bar{2}$ representation of SU(2) generated by

$$T^a=\mathrm{diag}(t^a_{\mathrm{adj}},t^a_{\mathrm{adj}},t^a_{\mathrm{adj}},t^a) \ , \tag{46}$$

where

$$(t^a_{\mathrm{adj}})_{bc}=-i\epsilon_{abc}, \quad t^1=-\tfrac{1}{2}\sigma_2\otimes\sigma_1 \ ,$$
$$t^2=\tfrac{1}{2}\sigma_0\otimes\sigma_2, \quad t^3=-\tfrac{1}{2}\sigma_2\otimes\sigma_3 \ . \tag{47}$$

Here σ_k are the Pauli matrices and σ_0 is the unit 2×2 matrix. In Eq. (45), $\mathcal{V}$ is the 13-dimensional real symmetric matrix,

$$\mathcal{V}=\begin{bmatrix} V^{ab}_{ij} & V^{aT}_i \\ V^b_j & W \end{bmatrix} , \tag{48}$$

whose components are given by the following functions of the background fields A^a_i and ϕ:

$$V_{ij}^{ab}=\delta^{ab}\delta_{ij}\tfrac{1}{4}\phi^T\phi+2\epsilon^{abc}F_{ij}^c(A)\ ,$$
$$V_i^a=2it^aD_i(A)\phi\ , \tag{49}$$
$$W=\tfrac{1}{4}\phi^T\phi+\frac{\lambda}{g^2}(\phi^T\phi-1)+2\left[\frac{\lambda}{g^2}-\frac{1}{8}\right]\phi\phi^T\ .$$

In terms of these quantities, the factor κ introduced in Sec. II is given by

$$\kappa=\mathrm{Im}\left[\frac{\det(-\partial^2+\mathcal{V}_0)}{\det'[-\mathcal{D}(A_{\mathrm{sp}})^2+\mathcal{V}_{\mathrm{sp}}]}\frac{\Delta_{\mathrm{FP,sp}}}{\Delta_{\mathrm{FP,vac}}}\right]^{1/2}\ , \tag{50}$$

where $\mathcal{V}_{\mathrm{sp}}$ and $\mathcal{V}_0$ are potentials $\mathcal{V}$ evaluated in the sphaleron and vacuum configurations, respectively. The prime denotes the deletion of the zero eigenmodes of the operator, $-\mathcal{D}(A_{\mathrm{sp}})^2+\mathcal{V}_{\mathrm{sp}}$.

As discussed in Sec. II, the zero-frequency modes, which correspond to translations and rotations of the sphaleron configuration, give important contributions to the total rate for baryon-number violation. Because of the rescaling (37) of the fluctuation fields, the first contribution so introduced is a factor of $1/g_3$ for each of the six zero modes:

$$(g_3)^{-6}=(gv)^{-3}\left[\frac{\alpha_W T}{4\pi}\right]^3\alpha_3^{-6}\ . \tag{51}$$

There are also contributions due to the normalization of zero modes. To evaluate these, we introduce collective coordinates q_α for each mode; generically, we have

$$\Psi=\Psi_\alpha q_\alpha+\cdots\ , \tag{52}$$

where Ψ_α are the zero-mode eigenfunctions satisfying

$$[-\mathcal{D}(A_{\mathrm{sp}})^2+\mathcal{V}_{\mathrm{sp}}]\Psi_\alpha=0\ . \tag{53}$$

The ellipsis $(\cdots)$ in (52) denotes summation over nonzero modes orthogonal to Ψ_α. In terms of the components of Ψ, this reads

$$a_i^a=\psi_{i\alpha}^a q_\alpha+\cdots,\quad \tilde{\eta}=\tilde{\psi}_\alpha q_\alpha+\cdots\ . \tag{54}$$

The eigenfunctions Ψ_α are orthonormalized according to

$$\begin{aligned}2\pi\delta_{\alpha\alpha'}&=\int d^3\xi\,\Psi_\alpha^T\Psi_{\alpha'}\\&=\int d^3\xi(\psi_{i\alpha}^a\psi_{i\alpha'}^a+\tilde{\psi}_\alpha^T\tilde{\psi}_{\alpha'})\\&=\int d^3\xi(\psi_{i\alpha}^a\psi_{i\alpha'}^a+\psi_\alpha^\dagger\psi_{\alpha'}+\psi_{\alpha'}^\dagger\psi_\alpha)\ .\end{aligned} \tag{55}$$

In the last line, we have converted back to complex spinor notation using the definition

$$\tilde{\psi}_\alpha=\sqrt{2}\begin{bmatrix}\mathrm{Re}\psi_\alpha\\ \mathrm{Im}\psi_\alpha\end{bmatrix}\ . \tag{56}$$

The orthonormality condition (55) ensures that, up to factors of $\sqrt{2\pi}$, the Jacobian associated with the change of coordinates is unity:

$$[d\Psi]=[da][d\tilde{\eta}]=\prod_\alpha\frac{dq_\alpha}{\sqrt{2\pi}}\prod(\cdots)\ . \tag{57}$$

Normalization factors are acquired in relating the canonical coordinates q_α above to the "natural" coordinates of each zero mode. Take translations, for example. Under the shift $\xi\to\xi+\epsilon$, the fields transform according to

$$\begin{aligned}a_k^a&=\delta A_k^a+\delta_\Lambda A_k^a=\epsilon_j\partial_j A_k^a+(D_k\Lambda_\epsilon)^a\ ,\\ \eta&=\delta\Phi+\delta_\Lambda\Phi=\epsilon_j\partial_j\Phi+i\Lambda_\epsilon\Phi\ ,\end{aligned} \tag{58}$$

where we have supplemented the naive transformation with a gauge transformation parametrized by $\Lambda_\epsilon(\xi)$. The latter is determined by requiring that the $R_{\xi=1}$ gauge condition (38) is maintained. This implies that

$$\Lambda_\epsilon=-\epsilon_j A_j\ , \tag{59}$$

so that the transformations given above now have the simple form

$$a_k^a=\epsilon_j F_{jk}^a,\quad \eta=\epsilon_j D_j\Phi\ . \tag{60}$$

We now introduce a normalization factor N_{tr} relating the canonical coordinates q_j to the natural coordinates ϵ_j:

$$q_j=N_{\mathrm{tr}}\epsilon_j,\quad j=1,2,3\ . \tag{61}$$

Then by equating (54) with (60), and applying the orthonormality condition (55), we obtain

$$\begin{aligned}2\pi\delta_{jj'}=N_{\mathrm{tr}}^{-2}\int d^3\xi[&F_{ij}^aF_{ij'}^a+(D_j\Phi)^\dagger(D_{j'}\Phi)\\&+(D_{j'}\Phi)^\dagger(D_j\Phi)]\end{aligned} \tag{62}$$

or, after contracting the indices j and j',

$$N_{\mathrm{tr}}^2=\frac{1}{6\pi}\int d^3\xi[F_{ij}^aF_{ij}^a+2(D_j\Phi)^\dagger(D_j\Phi)] \tag{63}$$
$$=\tfrac{2}{3}\int_0^\infty d\xi\left[16\left[\frac{df}{d\xi}\right]^2+\frac{32f^2(1-f)^2}{\xi^2}+\left[\xi\frac{dh}{d\xi}\right]^2+2h^2(1-f)^2\right]\ . \tag{64}$$

In the last line, we have evaluated the expression in terms of the profile functions $f(\xi)$ and $h(\xi)$ defining the sphaleron configuration. Note that (63) is invariant with respect to gauge transformations of the background field, as it should be.

The final step is to actually perform the integration over the translation zero modes. We have

$$\int d^3q=N_{\mathrm{tr}}^3\int d^2\epsilon=(gv)^3(\mathcal{N}V)_{\mathrm{tr}}\ , \tag{65}$$

where

$$\mathcal{N}_{\rm tr}=N_{\rm tr}^3 \ . \tag{66}$$

The factor of $(gv)^3$, obtained above in converting from dimensionless units to the three-space volume $V_{\rm tr}$, will cancel with a similar factor in Eq. (51) to give expression (8) for the rate per unit volume Γ/V.

Derivation of the contribution due to the rotational zero modes follows similarly, with one slight twist. Under $\xi \to \epsilon \times \xi$, the fields transform according to

$$\begin{aligned} a_k^a &= \delta A_k^a + \delta_\Lambda A_k^a \\ &= \epsilon_j \epsilon_{jkl} A_l^a + \epsilon_j \epsilon_{jmn} \xi_m \partial_n A_k^a + (D_k \Lambda_\epsilon)^a \ , \\ \eta &= \delta \Phi + \delta_\Lambda \Phi \\ &= \epsilon_j \epsilon_{jmn} \xi_m \partial_n \Phi + i \Lambda_\epsilon \Phi \ . \end{aligned} \tag{67}$$

This time, however, the compensating gauge transformation is given by

$$\Lambda_\epsilon = -\epsilon_{ijk}\epsilon_i \xi_j A_k + \epsilon_i \tilde{\Lambda}_i \ , \tag{68}$$

where the residual transformation $\tilde{\Lambda}_i$ solves the inhomogeneous equation

$$[-D(A)^2 + \tfrac{1}{2}\Phi^\dagger \Phi]\tilde{\Lambda}_i = -\epsilon_{ijk} F_{jk} \ . \tag{69}$$

Note that this equation is invariant with respect to gauge transformations of the background fields if we require that $\tilde{\Lambda}_i$ transforms covariantly:

$$\delta \tilde{\Lambda}_i = -i[\Lambda, \tilde{\Lambda}_i] \ . \tag{70}$$

With the choice (68) for the compensating gauge transformation, the transformation of fields now reads

$$\begin{aligned} a_k^a &= \epsilon_{jmn}\epsilon_j \xi_m F_{nk}^a + \epsilon_j (D_k \tilde{\Lambda}_j)^a \ , \\ \eta &= \epsilon_{jmn}\epsilon_j \xi_m D_n \Phi + i\epsilon_j \tilde{\Lambda}_j \Phi \ . \end{aligned} \tag{71}$$

The associated normalization factor is

$$N_{\rm rot}^2 = \frac{1}{6\pi}\int d^3\xi \{ (\xi^2 \delta_{ik} - \xi_i \xi_k)[F_{ij}^a F_{kj}^a + 2(D_i\Phi)^\dagger (D_k \Phi)] + \epsilon_{jkl}\tilde{\Lambda}_j^a F_{kl}^a \} \ . \tag{72}$$

Therefore, the contribution to the rate due to the rotational zero modes is the factor $(\mathcal{N}V)_{\rm rot}$, where

$$\mathcal{N}_{\rm rot} = N_{\rm rot}^3 \tag{73}$$

and $V_{\rm rot} = 8\pi^2$ is the volume of the rotation group.

To evaluate $N_{\rm rot}$ for the sphaleron, we must solve the equations (69) for $\tilde{\Lambda}_i$. This is accomplished with the ansatz

$$\tilde{\Lambda}_i^a = 4(\delta_{ia} - \hat{\xi}_i \hat{\xi}_a) P(\xi) + 4\hat{\xi}_i \hat{\xi}_a Q(\xi) \ , \tag{74}$$

where $P(\xi)$ and $Q(\xi)$ solve the following linear, coupled, inhomogeneous equations:

$$\begin{aligned} \frac{1}{\xi}\frac{df}{d\xi} &= \left[-\frac{d^2}{d\xi^2} - \frac{2}{\xi}\frac{d}{d\xi} + \frac{2+4f(f-1)}{\xi^2} + \frac{h^2}{4} \right] P + \frac{2(2f-1)}{\xi^2} Q \ , \\ \frac{2f(1-f)}{\xi^2} &= \left[-\frac{d^2}{d\xi^2} - \frac{2}{\xi}\frac{d}{d\xi} + \frac{4+8f(f-1)}{\xi^2} + \frac{h^2}{4} \right] Q + \frac{4(2f-1)}{\xi^2} P \ . \end{aligned} \tag{75}$$

From the known boundary conditions of $f(\xi)$ and $h(\xi)$, we deduce that

$$P \to p_0 + p_2 \xi^2, \quad Q \to p_0 - 2p_2 \xi^2 \tag{76}$$

as $\xi \to 0$ (p_0 and p_2 are constants), and

$$P, Q \sim \frac{e^{-\xi/2}}{\xi} \tag{77}$$

as $\xi \to \infty$. Our final expression for $N_{\rm rot}$ is

$$N_{\rm rot}^2 = \tfrac{32}{3}\int_0^\infty d\xi \left[\frac{1}{2}\left[\xi \frac{df}{d\xi} \right]^2 + 2f^2(1-f)^2 + \tfrac{1}{8}(\xi h)^2(1-f)^2 - 2\left[\xi \frac{df}{d\xi} \right] P - 2f(1-f)Q \right] \ . \tag{78}$$

The full normalization factors $\mathcal{N}_{\rm tr}$ and $\mathcal{N}_{\rm rot}$ are shown in Fig. 5. We find that the product $\mathcal{N}_{\rm tr}\mathcal{N}_{\rm rot}$ is relatively insensitive to λ/g^2, varying from 84 to 92 over the whole range of λ/g^2 considered.

In addition to extracting the zero modes from κ, we also have to give special consideration to the negative-frequency mode of the operator, $-\mathcal{D}(A_{\rm sp})^2 + \mathcal{V}_{\rm sp}$, before we can apply the method of Diakonov, Petrov, and Yung to be determinant of this operator. This negative mode, with a mass of oscillation we shall denote by $-\omega_-^2$ [in

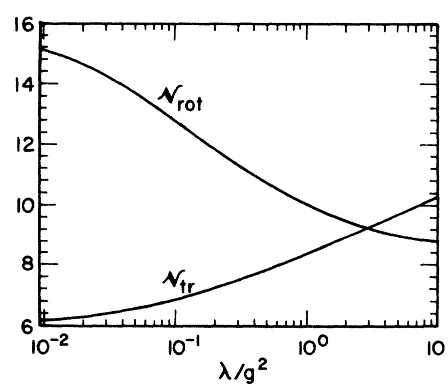

FIG. 5. The zero-mode normalization factors $\mathcal{N}_{\rm tr}$ and $\mathcal{N}_{\rm rot}$ as a function of λ/g^2.

units of $(gv)^2$], exists since the sphaleron is not a minimum, but merely an extremum of the static SU(2) Yang-Mills-Higgs energy functional. This is the unique negative mode for the range of couplings that we consider, $10^{-2} \le \lambda/g^2 \le 10$ (Ref. 25).

The channel in which the negative mode occurs has been identified by Akiba, Kikuchi, and Yanagida.[17] It is given by the ansatz

$$a_i^a = (\delta_{ia} - \hat{\xi}_i \hat{\xi}_a) B(\xi) + \hat{\xi}_i \hat{\xi}_a C(\xi),$$
$$\eta = -\eta^* = \frac{i}{\sqrt{2}} u_{-1/2} H(\xi) . \tag{79}$$

The eigenvalue/eigenfunction equations corresponding to this ansatz are

$$-\omega_-^2 B = \left[-\frac{d^2}{d\xi^2} - \frac{2}{\xi}\frac{d}{d\xi} + \frac{2+12f(f-1)}{\xi^2} + \frac{h^2}{4} \right] B + \left[-\frac{4}{\xi}\frac{df}{d\xi} + \frac{2(2f-1)}{\xi^2} \right] C + \frac{h(1-f)}{\xi} H ,$$
$$-\omega_-^2 C = \left[-\frac{d^2}{d\xi^2} - \frac{2}{\xi}\frac{d}{d\xi} + \frac{4+8f(f-1)}{\xi^2} + \frac{h^2}{4} \right] C + \left[-\frac{8}{\xi}\frac{df}{d\xi} + \frac{4(2f-1)}{\xi^2} \right] B + \frac{dh}{d\xi} H , \tag{80}$$
$$-\omega_-^2 H = \left[-\frac{d^2}{d\xi^2} - \frac{2}{\xi}\frac{d}{d\xi} + \frac{2f^2}{\xi^2} + \frac{h^2}{4} + \frac{\lambda}{g^2}(h^2-1) \right] H + \frac{2h(1-f)}{\xi} B + \frac{dh}{d\xi} C .$$

The boundary conditions on B, C, and H are

$$B \to b_0 + b_2 \xi^2, \quad C \to b_0 - 2b_2 \xi^2, \quad H \to h_0 \tag{81}$$

as $\xi \to 0$ (b_0, b_2, and h_0 are constants), while

$$B, C, H \sim \frac{e^{-q\xi}}{\xi} \tag{82}$$

as $\xi \to \infty$, where $q = (\frac{1}{4} + \omega_-^2)^{1/2}$. Figure 6 gives a plot of the eigenvalue ω_-^2 over several decades of λ/g^2. Our value of $\omega_-^2 \simeq 0.56$ at $\lambda/g^2 = \frac{1}{8}$ agrees with the value obtained by Akiba, Kikuchi, and Yanagida.[17]

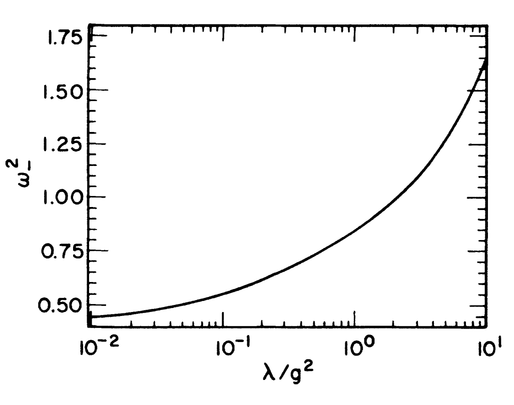

FIG. 6. The negative-mode frequency ω_-^2 [in units of $(gv)^2$] as a function of λ/g^2.

IV. APPROXIMATE CALCULATION OF THE SPHALERON FUNCTIONAL DETERMINANT

The remaining task is the evaluation of κ, which we have reduced to the ratio (50) of several functional determinants. Each of these determinants has the generic form

$$\det K = \det[-D(A)^2 + V] , \tag{83}$$

where A_i^a is the background gauge field, $D_i(A) = \partial_i - iT^a A_i^a$ is the covariant derivative in some representation R of the gauge group G, and V is a potential function of the background fields transforming under G according to some linear combination of the singlet and $R \otimes \bar{R}$ representations. For example, in the case of the Faddeev-Popov determinant (39), R is the adjoint representation of SU(2) with $(T^a)_{bc} = -i\epsilon_{abc}$, while V is the pure singlet field, $\Phi^\dagger \Phi/2$.

In some special circumstances, specifically in cases where the background fields have a simple, analytic form, it may be feasible to determine the complete spectrum of K and thereby construct $\det K$ exactly from the product of its eigenvalues. Examples of background fields amendable to this exact treatment are the vacuum configuration ($A_i^a = V = 0$) and the fields describing a single instanton.[26] In the case of the sphaleron, however, the gauge potential A_μ^a and Higgs field Φ are analytically known only up to the profile functions $f(\xi)$ and $h(\xi)$, which are numerically determined. Thus, in this case, the direct evaluation of $\det K$ must proceed via numerical solution of the eigenvalue equation

$$K\Psi = [-D(A)^2 + V]\Psi = \omega^2 \Psi . \tag{84}$$

Recently, Akiba, Kikuchi, and Yanagida have made progress on this arduous task, but thus far their calculation of $\det K$ has been limited to spherically symmetric modes and a few angular momentum excited modes.[17]

An alternative scheme for estimating the ratio,

$$\frac{\det K}{\det K_0} , \tag{85}$$

has been developed by Diakonov, Petrov, and Yung[14] (DPY). Here K_0 is some operator that acts in the same space of functions as the operator K. If K_0 is sufficiently simple that the determinant of K_0 can be evaluated by some other means, then we can turn the estimate of (85) into an estimate of $\det K$. Actually we shall not need to take this last step, since the quantity κ depends only on ratios of the form (85), where K_0 is the operator K evaluated in the vacuum configuration

$$K_0 = K(A \to 0,\ V \to V_0) = -\partial^2 + V_0 . \tag{86}$$

The method of DPY is based on the Schwinger proper-time representation of the ratio:

$$\frac{\det K}{\det K_0} = \exp\left[-\mathrm{Tr} \int_0^\infty \frac{dt}{t} (e^{-tK} - e^{-tK_0}) \right] , \tag{87}$$

where Tr denotes a functional trace. As discussed above, the functional trace can be resolved over a complete set of plane-wave states plus a residual trace (tr) over group and spinor indices. Defining $\delta K(p,x)$ according to (28), we obtain

$$\ln\left[\frac{\det K}{\det K_0}\right] = -\int_0^\infty \frac{dt}{t}\int d^dx \int \frac{d^dp}{(2\pi)^d} e^{-tp^2} \times (e^{-t\delta K} - e^{-t\delta K_0})\mathbf{1} \;, \tag{88}$$

where

$$\begin{aligned} \delta K(p,x) &= -2ip\cdot D(A) - D(A)^2 + V \;, \\ \delta K_0(p,x) &= -2ip\cdot\partial - \partial^2 + V_0 \;, \end{aligned} \tag{89}$$

and d is the dimension of space. (We shall take $d=3$ later on in this section.) Keeping the factor of e^{-tp^2} intact, the factors of $e^{-t\delta K}$ and $e^{-t\delta K_0}$ are expanded in powers of t and the resulting expression is integrated over momenta p. This produces the series

$$\ln\left[\frac{\det K}{\det K_0}\right] = -\int_0^\infty \frac{dt}{t}\frac{1}{(4\pi t)^{d/2}}\sum_{p=1}^{\infty}\frac{(-t)^p}{p!}\mathcal{O}_p \;, \tag{90}$$

where $\mathcal{O}_p$ are gauge-invariant functionals of the background fields. The first few of these functionals are

$$\begin{aligned} \mathcal{O}_1 &= \int d^dx\, \mathrm{tr}(V - V_0) \;, \\ \mathcal{O}_2 &= \int d^dx\, \mathrm{tr}(V^2 - V_0^2 - \tfrac{1}{6}F_{ij}F_{ij}) \;, \\ \mathcal{O}_3 &= \int d^dx\, \mathrm{tr}\Big[V^3 - V_0^3 + \tfrac{1}{2}[D_i, V]^2 - \tfrac{1}{2}[\partial_i, V_0]^2 \\ &\qquad - \tfrac{1}{2}F_{ij}F_{ij}V - \tfrac{1}{10}D_iF_{ik}D_jF_{jk} \\ &\qquad - \frac{i}{30}F_{ij}[F_{jk}, F_{ki}] \Big] \;. \end{aligned} \tag{91}$$

Up to this point, our manipulations have been exact. The DPY method as applied to (90) consists of two approximations. First, the t expansion is truncated to some finite order $p_{\max}$. Such a truncation is, of course, dictated by practical necessity. Second, the upper limit of the t integration is cut off at some infrared scale δ^2, which produces an expression for $\det K_0$ as a polynomial $P(\delta)$ in δ. The effect of this cutoff is minimized by evaluating $P(\delta)$ at that value δ_0 where $P(\delta)$ is extremal:

$$\left.\frac{dP}{d\delta}\right|_{\delta=\delta_0} = 0 \;. \tag{92}$$

Zeros of $dP/d\delta$ always exist for positive-definite operators K and K_0 since $\mathrm{Tr}e^{-tK}$ and $\mathrm{Tr}e^{-tK_0}$, when expanded in powers of t, produce series whose terms oscillate in sign. Actually, there may be several solutions to (92), but to the order in the t expansion to which we calculate, we shall not be plagued by this inherent ambiguity of the method. The value $P(\delta_0)$ thus constitutes the DPY estimate of $\det K/\det K_0$.

Several adjustments of the DPY method must be made before it can be used to estimate the ratios

$$\frac{\det'[-\mathcal{D}(A_{\rm sp})^2 + \mathcal{V}_{\rm sp}]}{\det(-\partial^2 + \mathcal{V}_0)} \tag{93}$$

and

$$\frac{\Delta_{\rm FP,sp}}{\Delta_{\rm FP,0}} = \frac{\det[-D(A_{\rm sp})^2 + \frac{1}{2}\Phi_{\rm sp}^\dagger\Phi_{\rm sp}]}{\det(-\partial^2 + \frac{1}{4})} \tag{94}$$

that appear in the expression (50) for κ. The first issue to be dealt with is the linear divergence of the $p=1$ term of Eq. (90) when $d=3$. Cutting off the lower limit of the t integral at $1/\Lambda^2$, we obtain a contribution to $\ln(\det K/\det K_0)$ from this term of

$$\frac{\Lambda}{4\pi^{3/2}}\mathcal{O}_1 = \frac{\Lambda}{4\pi^{3/2}}\int d^3x\, \mathrm{tr}(V - V_0) \;, \tag{95}$$

which diverges as $\Lambda \to \infty$. However, this linear divergence was anticipated in our discussion in Sec. II. It is an artifact of improperly reducing the theory to three dimensions in the high-temperature limit and reflects the quadratic divergence of the underlying four-dimensional theory. To see this, note that if we isolate the $n=0$ contribution of (34), we obtain

$$\int d^3x\, \mathrm{tr}(V - V_0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2} \;, \tag{96}$$

which is linearly divergent. By redefining the vacuum expectation value v of the Higgs field to absorb the quadratic divergence and large-T behavior of the corresponding term in the four-dimensional theory, we have already taken the contribution of this term into account. Consequently, we shall simply drop this term from the t expansion (90) of $\ln(\det K/\det K_0)$.

We must also deal with the negative and zero eigenmodes of the operator, $K_{\rm sp} = -\mathcal{D}(A_{\rm sp})^2 + \mathcal{V}_{\rm sp}$. If $-\omega_-^2$ is the eigenvalue corresponding to the negative-frequency eigenmode, then these give contributions to $\mathrm{Tr}e^{-tK_{\rm sp}}$ of

$$\mathrm{Tr}e^{-tK_{\rm sp}} = e^{t\omega_-^2} + 6\cdots \;, \tag{97}$$

and hence result in divergent contributions to $\det K_{\rm sp}$. Actually we have already accounted for the zero eigenmodes above by separately computing their contribution to the rate Γ. We are thus only interested in the reduced determinant, $\det' K_{\rm sp}$, which excludes the zero-eigenmode contribution. In the Schwinger formalism, we can implement this reduction by subtracting 6 from the functional trace (97). We can also mimic this procedure for the negative eigenmode by subtracting the term $e^{t\omega_-^2}$ from (97) and by explicitly including its contribution to κ as the factor $1/\omega_-$.

These subtractions, however, must be compensated for by an equal number of subtractions from $\det K_0$. The reason for this is that the Schwinger proper-time representation of $\det K/\det K_0$ is valid only if the operators K and K_0 act in the same functional space or, in other words, possess the same number of eigenvalues. This is easy to see from (87) since, if there is a mismatch between the number of eigenvalues of K and K_0, then this expression is logarithmically divergent in the exponential.

Consequently, we will subtract seven (six zero modes plus one negative eigenmode) eigenmodes from the operator K_0. The choice of which eigenmodes to subtract is somewhat arbitrary but we have found the following choice to be convenient: First, to compensate the six zero modes we shall modify the integrand of (87) as follows:

$$\mathrm{Tr}(e^{-tK}-e^{-tK_0}) \rightarrow \mathrm{Tr}(e^{-tK}-e^{-tK_0})-6+6e^{-t\epsilon}-6e^{-t\epsilon} , \tag{98}$$

where ϵ is an eigenvalue of K_0 in units of $(gv)^2$. (This implies $\epsilon \geq \frac{1}{4}$ since this is the value of M_W^2 in these units.) The terms $-6+6e^{-t\epsilon}$ are expanded to order t^2 while the integral of the last term is replaced by $-6\ln\epsilon$, reflecting the fact that these eigenmodes of K_0 are being treated separately:

$$\det K_0 \rightarrow \epsilon^6 \det' K_0 . \tag{99}$$

Second, the negative eigenmode is isolated by adding to $\mathrm{Tr}(e^{-tK}-e^{-tK_0})$ the terms

$$0=(-e^{t\omega_-^2}+e^{-t\omega_-^2})-(-e^{t\omega_-^2}+e^{-t\omega_-^2}) . \tag{100}$$

The first set of terms is expanded to order t^2 while the second set is integrated to give a factor of unity to κ.

With these modifications of the integrand of Eq. (87) in place, the DPY method may be applied with no further ado. The series in t is truncated at order t^2 and the t-integral is cut off at an infrared scale of δ^2. This produces the polynomial expression for $\ln\kappa$:

$$\begin{aligned}\ln\kappa &\simeq P(\delta)\\ &=3\ln\epsilon+C_1\delta+C_2\delta^2+C_3\delta^3+C_4\delta^4 ,\end{aligned} \tag{101}$$

where the coefficients C_i, $i=1,\ldots,4$ are given by

$$\begin{aligned}C_1&=\frac{1}{2(4\pi)^{3/2}}(\mathcal{O}_{2,\mathrm{sp}}-\mathcal{O}_{2,\mathrm{FP}}) ,\\ C_2&=-\omega_-^2-3\epsilon ,\\ C_3&=-\frac{1}{18(4\pi)^{3/2}}(\mathcal{O}_{3,\mathrm{sp}}-\mathcal{O}_{3,\mathrm{FP}}) ,\\ C_4&=\tfrac{3}{4}\epsilon^2 .\end{aligned} \tag{102}$$

The quantities $\mathcal{O}_{2,\mathrm{sp}}$ and $\mathcal{O}_{3,\mathrm{sp}}$ are the functionals $\mathcal{O}_i$, $i=2,3$ given by Eq. (91) evaluated for the operators defined by the ratio (93), while $\mathcal{O}_{2,\mathrm{FP}}$ and $\mathcal{O}_{3,\mathrm{FP}}$ are the same functionals evaluated for the ratio of Faddeev-Popov determinants (94). The explicit evaluation of these quantities leads to some rather lengthy formulas, and so is relegated to the Appendix.

Examining our expression for $\ln\kappa=P(\delta)$, we observe that since the coefficient C_4 is always positive we are always assured of finding a minimum for $P(\delta)$ at some finite value of δ. Indeed, since P is a quartic polynomial there could be two such minima. Nevertheless, for all values of λ/g^2 that we examined, the second minimum always occurred at negative values of δ and thus is unphysical. Therefore, by evaluating $P(\delta)$ at the unique physical minimum $\delta_0(\lambda/g^2)$, we have obtained an estimate for $\ln\kappa$ which, for $\epsilon=1$, we plot in Fig. 1.

V. CONCLUSIONS

We have used the method of Diakonov, Petrov, and Yung to compute the rate of baryon-number change in electroweak theory for arbitrary values of the ratio λ/g^2,

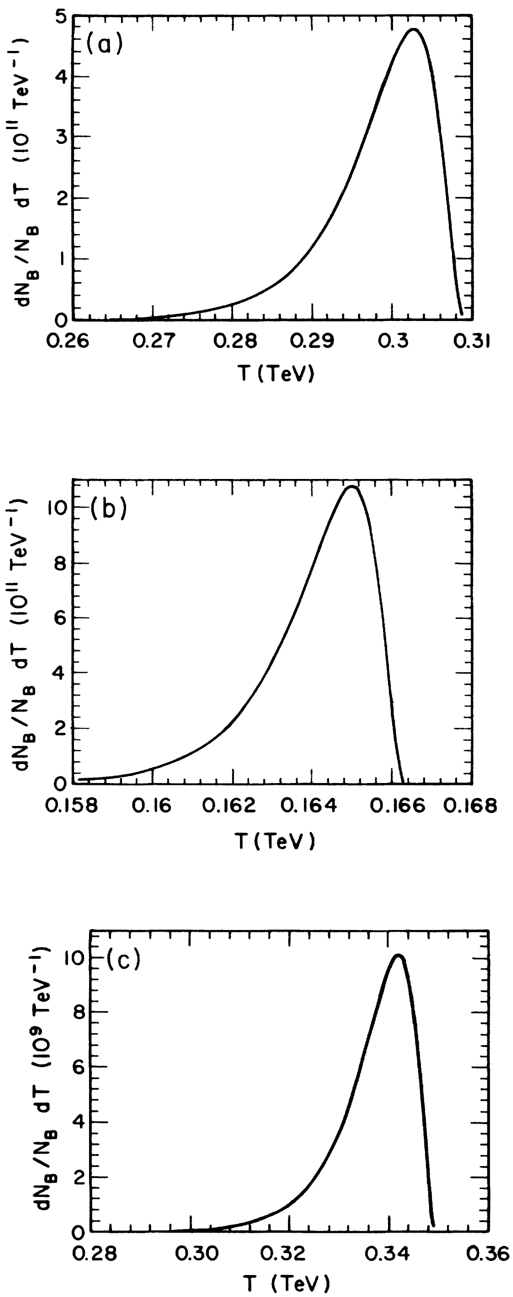

FIG. 7. The rate of baryon-number change per unit baryon number as a function of temperature. (a) $\lambda/g^2=1.0$, (b) $\lambda/g^2=0.1$, (c) $\lambda/g^2=5.0$.

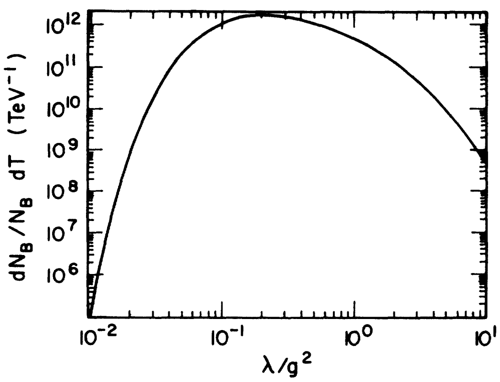

FIG. 8. The maximum rate of baryon-number violation as a function of λ/g^2.

or alternatively the ratio of M_H/M_W. We find a somewhat strong dependence upon the ratio of λ/g^2. For large and small values of this ratio, the rate is rapidly suppressed. For small values of the ratio, our method of computation becomes unreliable as higher-order terms neglected in our expansion are surely important.

In Figs. 7(a)–7(c), the rate of baryon-number change per unit baryon number is computed as a function of temperature for various values of λ/g^2. The natural scale in all of these figures is of order 10^9–10^{11} TeV^{-1}. Consequently, if these computations are correct, then baryon-number conservation is violated at a rate sufficient for baryogenesis in the early Universe.

In Fig. 8 we attempt to gain some knowledge about how the rate varies by plotting its maximum as a function of λ/g^2. Recall that perturbation theory has surely broken down well before one reaches a temperature where there is a maximum in the rate. Here the condition $T \ll M_W(T)/\alpha$ is no longer satisfied. The maximum may be and probably is an artifact of the perturbative treatment at this temperature. Nevertheless, the height of this maximum provides a useful measure of the overall scale of baryon-number violation. Going to the region where perturbation theory is reliable costs roughly only an order of magnitude or so in the rate.

Figure 8 demonstrates the main point of our computations. For $0.1 \le \lambda/g^2 \le 1$, the rate is essentially unsuppressed by the proper computation of the determinant of small fluctuations. For smaller values, our computations are certainly not trustworthy. To reliably estimate the rate in all regions, a proper and exact numerical computation of the determinant is called for. This work is currently under way.[27]

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APPENDIX

In this appendix, we explicitly evaluate certain gauge-invariant functionals of the background sphaleron fields encountered in the proper-time (or t) expansion of the quantity $\ln(\det K/\det K_0)$ [cf. Eq. (90)]. The coefficients $\mathcal{O}_p$, $p=1,2,3$, of this t expansion, given by Eq. (91), define the functionals of interest.

The first set of such quantities is associated with the operators

$$K=-\mathcal{D}(A_{\rm sp})^2+\mathcal{V}_{\rm sp},\quad K_0=-\partial^2+\mathcal{V}_0 \tag{A1}$$

found in the ratio (93). Here $\mathcal{D}_i$ is the covariant derivative in the representation (47) of SU(2) and $\mathcal{V}_{\rm sp}$ and $\mathcal{V}_0$ are matrix potentials $\mathcal{V}$, Eq. (48), evaluated in sphaleron and vacuum background fields, respectively. For the moment we shall express $\mathcal{V}_{\rm sp}$ in terms of generic background fields as given by Eq. (49). Then the relevant gauge-invariant functionals are constructed from

$$\mathrm{tr}(\mathcal{V}-\mathcal{V}_0)=(12x+6)(\Phi^\dagger\Phi-\tfrac{1}{2})\ , \tag{A2}$$

$$\mathrm{tr}(\mathcal{V}^2-\mathcal{V}_0^2)=8F^a_{ij}F^a_{ij}+12(D_i\Phi)^\dagger(D_i\Phi)+(48x^2+6x+3)(\Phi^\dagger\Phi-\tfrac{1}{2})^2+(24x^2+3x+3)(\Phi^\dagger\Phi-\tfrac{1}{2})\ , \tag{A3}$$

$$\begin{aligned}\mathrm{tr}(\mathcal{V}^3-\mathcal{V}_0^3)=\;&48iF^a_{ij}(D_i\Phi)^\dagger\tau^a(D_j\Phi)+12\Phi^\dagger\Phi F^a_{ij}F^a_{ij}+8\epsilon^{abc}F^a_{ij}F^b_{jk}F^c_{ki}-\tfrac{3}{4}(8x-1)[\partial_i(\Phi^\dagger\Phi)]^2\\&+[15(4x+1)\Phi^\dagger\Phi-18x](D_i\Phi)^\dagger(D_i\Phi)+(240x^3+18x^2+\tfrac{9}{2}x+\tfrac{3}{2})(\Phi^\dagger\Phi-\tfrac{1}{2})^3\\&+(216x^3+9x^2+\tfrac{9}{2}x+\tfrac{9}{4})(\Phi^\dagger\Phi-\tfrac{1}{2})^2+(72x^3+\tfrac{9}{8}x+\tfrac{9}{8})(\Phi^\dagger\Phi-\tfrac{1}{2})\ ,\end{aligned} \tag{A4}$$

$$\begin{aligned}\mathrm{tr}[\mathcal{D}_i,\mathcal{V}]^2=\;&8(D_iF_{jk})^a(D_iF_{jk})^a+12(D_iD_j\Phi)^\dagger(D_iD_j\Phi)\\&+(32x^2-8x+\tfrac{1}{2})\Phi^\dagger\Phi(D_i\Phi)^\dagger(D_i\Phi)+(40x^2+8x+\tfrac{23}{8})[\partial_i(\Phi^\dagger\Phi)]^2\ ,\end{aligned} \tag{A5}$$

$$\mathrm{tr}F_{ij}F_{ij}\mathcal{V}=[(3x+\tfrac{27}{8})\Phi^\dagger\Phi-x]F^a_{ij}F^a_{ij}\ , \tag{A6}$$

$$\mathrm{tr}(\mathcal{D}_iF_{ik})(\mathcal{D}_jF_{jk})=7(D_iF_{ik})^a(D_jF_{jk})^a\ , \tag{A7}$$

$$\mathrm{tr}(-i)F_{ij}[F_{jk},F_{ki}]=7\epsilon^{abc}F^a_{ij}F^b_{jk}F^c_{ki} \ . \tag{A8}$$

Note we have converted the background field ϕ back to the complex spinor notation Φ using Eq. (44). In the formulas above, and for the remainder of this appendix, x denotes the ratio λ/g^2.

We must construct analogous quantities for the operators

$$K=-D(A_{\rm sp})^2+V_{\rm sp}, \quad K_0=(-\partial^2+\tfrac{1}{4})\mathbb{1} \ , \tag{A9}$$

appearing in the ratio (94) of Faddeev-Popov determinants. Here $\mathbb{1}$ denotes the 3×3 unit matrix. This time the covariant derivative D_i is in the adjoint representation of SU(2) while the matrix $V_{\rm sp}$ is the potential

$$V=\tfrac{1}{2}\Phi^\dagger\Phi\mathbb{1} \tag{A10}$$

evaluated in the sphaleron background. We define $V_0=\frac{1}{4}\mathbb{1}$ as the matrix V in the vacuum. We then have

$$\mathrm{tr}(V-V_0)=\tfrac{3}{2}(\Phi^\dagger\Phi-\tfrac{1}{2}) \ , \tag{A11}$$

$$\mathrm{tr}(V^2-V_0^2)=\tfrac{3}{4}(\Phi^\dagger\Phi-\tfrac{1}{2})^2+\tfrac{3}{4}(\Phi^\dagger\Phi-\tfrac{1}{2}) \ , \tag{A12}$$

$$\mathrm{tr}(V^3-V_0^3)=\tfrac{3}{8}(\Phi^\dagger\Phi-\tfrac{1}{2})^3+\tfrac{9}{16}(\Phi^\dagger\Phi-\tfrac{1}{2})^2 + \tfrac{9}{32}(\Phi^\dagger\Phi-\tfrac{1}{2}) \ , \tag{A13}$$

$$\mathrm{tr}[D_i,V]^2=\tfrac{3}{4}[\partial_i(\Phi^\dagger\Phi)]^2 \ , \tag{A14}$$

$$\mathrm{tr}F_{ij}F_{ij}V=\Phi^\dagger\Phi F^a_{ij}F^a_{ij} \ , \tag{A15}$$

$$\mathrm{tr}(D_iF_{ik})(D_jF_{jk})=2(D_iF_{ik})^a(D_jF_{jk})^a \ , \tag{A16}$$

$$\mathrm{tr}(-i)F_{ij}[F_{jk},F_{ki}]=2\epsilon^{abc}F^a_{ij}F^b_{jk}F^c_{ki} \ . \tag{A17}$$

It remains to evaluate the right-hand side of Eqs. (A2)–(A8) and (A11)–(A17) in the sphaleron background given by Eq. (10). This lengthy chore is shortened somewhat by judicious use of the equations of motion and the Jacobi identity. This leads to the identities

$$\int d^3\xi(D_i\Phi)^\dagger(D_i\Phi)=-2x\int d^3\xi\,\Phi^\dagger\Phi(\Phi^\dagger\Phi-\tfrac{1}{2}) \ , \tag{A18}$$

$$\begin{aligned}\int d^3\xi(D_iD_j\Phi)^\dagger(D_iD_j\Phi)=\int d^3\xi\{&4x^2\Phi^\dagger\Phi(\Phi^\dagger\Phi-\tfrac{1}{2})^2\\ &+\tfrac{1}{2}\Phi^\dagger\Phi(D_i\Phi)^\dagger(D_i\Phi)\\ &-\tfrac{1}{8}[\partial_i(\Phi^\dagger\Phi)]^2\\ &+\tfrac{1}{4}\Phi^\dagger\Phi F^a_{ij}F^a_{ij}\} \ ,\end{aligned} \tag{A19}$$

$$\begin{aligned}-i\int d^3\xi(D_i\Phi)^\dagger F^a_{ij}\tau^a(D_j\Phi)=\int d^3\xi\{&\tfrac{1}{2}\Phi^\dagger\Phi(D_i\Phi)^\dagger(D_i\Phi)\\ &-\tfrac{1}{8}[\partial_i(\Phi^\dagger\Phi)]^2\\ &+\tfrac{1}{8}\Phi^\dagger\Phi F^a_{ij}F^a_{ij}\} \ ,\end{aligned} \tag{A20}$$

$$\begin{aligned}\int d^3\xi(D_iF_{ik})^a(D_jF_{jk})^a=\int d^3\xi\{&\Phi^\dagger\Phi(D_i\Phi)^\dagger(D_i\Phi)\\ &-\tfrac{1}{4}[\partial_i(\Phi^\dagger\Phi)]^2\} \ ,\end{aligned} \tag{A21}$$

$$\begin{aligned}\int d^3\xi(D_iF_{jk})^a(D_iF_{jk})^a=2\int d^3\xi[&(D_iF_{ik})^a(D_jF_{jk})^a\\ &-\epsilon^{abc}F^a_{ij}F^b_{jk}F^c_{ki}] \ .\end{aligned} \tag{A22}$$

Consequently, the number of integrals we need to evaluate is reduced to eight. These are given by the following functionals of the profile functions $f(\xi)$ and $h(\xi)$:

$$\int\frac{d^3\xi}{4\pi}(1-2\Phi^\dagger\Phi)^n=\int_0^\infty d\xi\,\xi^2(1-h^2)^n, \quad n=1,2,3 \ , \tag{A23}$$

$$\int\frac{d^3\xi}{4\pi}[\partial_i(\Phi^\dagger\Phi)]^2=\int_0^\infty d\xi(\xi hh')^2 \ , \tag{A24}$$

$$\int\frac{d^3\xi}{4\pi}F^a_{ij}F^a_{ij}=16\int_0^\infty d\xi\left[(f')^2+\frac{2f^2(1-f)^2}{\xi^2}\right] \ , \tag{A25}$$

$$-\frac{1}{4!}\int\frac{d^3\xi}{4\pi}\epsilon^{abc}F^a_{ij}F^b_{jk}F^c_{ki}=\int_0^\infty d\xi\frac{4f(1-f)(f')^2}{\xi^2} \ , \tag{A26}$$

$$\begin{aligned}\int\frac{d^3\xi}{4\pi}&\Phi^\dagger\Phi(D_k\Phi)^\dagger(D_k\Phi)\\ &=\int_0^\infty d\xi[\tfrac{1}{4}(\xi hh')^2+\tfrac{1}{2}h^4(1-f)^2] \ ,\end{aligned} \tag{A27}$$

$$\int\frac{d^3\xi}{4\pi}\Phi^\dagger\Phi F^a_{ij}F^a_{ij}=8\int_0^\infty d\xi\left[(hf')^2+\frac{2h^2f^2(1-f)^2}{\xi^2}\right] \ , \tag{A28}$$

In these expressions, a prime denotes $d/d\xi$.

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