Berry's connection for non-Abelian, chiral fermions

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We study the set of Hamiltonians corresponding to second-quantized chiral fermions interacting with non-Abelian background gauge fields in $1+1$ dimensions. The spectral projections define a Hilbert bundle over the space of gauge-field configurations with a natural, flat, connection. The projection of this connection onto a one-dimensional, nondegenerate eigenstate defines the Berry connection. We compute this connection and calculate the first Chern number of this subbundle over noncontractible two-spheres in the group of gauge transformations.

In the Hamiltonian formalism anomalies manifest themselves through central extensions of the Lie algebra of the group of gauge transformations.[1] This means that the group of gauge transformations $\mathcal{G}^d$ is only projectively realized. The phases arising in this projective representation are not removable by a redefinition of the states. This is equivalent to the existence of a Berry connection[2] with nonzero curvature. In Ref. 1, Nelson and Alvarez-Gaumé established the nonvanishing of the Berry curvature in anomalous gauge theories through topological considerations. Sonoda subsequently calculated Berry's phase in $(3+1)$-dimensional theories.[3] He sets up the Hamiltonian formalism, but had to resort to effective-action methods in the actual calculation. His expression for the Berry phase furthermore suffers from regularization ambiguities. These are connected to the fact that a fixed-time formalism for fermions in an external gauge field, a key ingredient in the Hamiltonian approach, does not exist in dimensions greater than $1+1$. This also is related to the nonexistence of a unitary realization of the automorphisms of the field algebra corresponding to gauge transformations.[4] Such a realization is necessary, at least locally, for the existence of the putative Hilbert bundle of fermionic Fock spaces over the gauge orbits. The meaning of the deductions of Nelson and Alvarez-Gaumé in Ref. 1 remains therefore unclear for spatial dimensions greater than one.

In $1+1$ dimensions one does not have to confront this problem as gauge transformations are generally unitarily implementable, if the point at infinity is identified.[5] It is also possible to construct well-defined operators at fixed time out of fermion bilinears.[6] To have nonremovable phases it is clear that in the case of perturbative anomalies nonzero Berry curvature is sufficient, it is not necessary to have nontrivial global topology of the gauge orbit. Indeed, in the perturbative anomaly of the U(1) gauge group, the Berry curvature is given by the Schwinger term,[7] but the connected components of the gauge orbit are topologically trivial.

In this paper we consider the case of a non-Abelian gauge group G. Here the gauge orbit has noncontractible two-spheres. This case exemplifies the constructions contemplated by Nelson and Alvarez-Gaumé.[1] We show by explicit computation that the vacuum state forms a nontrivial complex line bundle over these two-spheres. These are the Dirac monopole bundles. We show that the monopole number is exactly given by the topological winding number characterizing an associated map of S^3 into G.

The non-Abelian charge densities for second-quantized massless free Weyl fermions in $1+1$ dimensions, in the Schrödinger representation, satisfy the Kac-Moody algebra[8]

$$[\rho_a(x),\rho_b(y)]=if_{abc}\rho_c(y)\delta(x-y)+\frac{1}{4\pi i}\delta_{ab}\delta'(x-y)\ , \tag{1}$$

where the generators are normalized to

$$\mathrm{tr}(T_aT_b)=\tfrac{1}{2}\delta_{ab}\ . \tag{2}$$

A gauge transformation

$$g(x)=\exp[i\phi_a(x)T_a]\equiv e^{i\phi} \tag{3}$$

is unitarily implemented on the Hilbert space by the operator

$$U(g)=\exp\left[i\int dx\ \phi_a(x)\rho_a(x)\right]\ , \tag{4}$$

which is defined through the spectral representation. Note that the specific definition of the operator ρ^a at this point is irrelevant, since the addition of a c-number term results only in a phase for U. The spectral projections at the point on the gauge orbit parametrized by g are simply given by

$$|E,A^g\rangle\langle E,A^g|=U^\dagger(g)|E,A\rangle\langle E,A|U(g)\ . \tag{5}$$

The corresponding (flat) connection is given by

$$\mathcal{A}=U^\dagger dU\ . \tag{6}$$

Consider now a two-parameter family of gauge transformations $g(\lambda_1,\lambda_2)$. The vacuum at the point g is

$$|0,A^g\rangle=U^\dagger(g)|0,A\rangle\ . \tag{7}$$

The Berry connection[2] is

$$\mathcal{B}_i = \langle 0, A^g | \partial_i | 0, A^g \rangle = \langle 0, A | U^\dagger(g) \partial_i U(g) | 0, A \rangle = \langle 0, A | \mathcal{A}_i | 0, A \rangle . \quad (8)$$

The corresponding curvature is

$$F_{ij} = \partial_i \mathcal{B}_j - \partial_j \mathcal{B}_i . \quad (9)$$

Using the fact that the full non-Abelian curvature is flat,

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2 = 0 \quad (10)$$

which gives explicitly

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_j] = 0 , \quad (11)$$

we get

$$F_{ij} = -\langle 0, A | [\mathcal{A}_i, \mathcal{A}_j] | 0, A \rangle . \quad (12)$$

Now using the formula

$$\mathcal{A}_i = U^\dagger \partial_i U = i \int_0^1 dt\, U^\dagger(t) \left[\partial_i \int dx\, \phi_a(x) \rho_a(x) \right] U(t) , \quad (13)$$

where

$$U(t) = \exp \left[it \int dx\, \phi_a(x) \rho_a(x) \right] , \quad (14)$$

and the additional formula[6]

$$U^\dagger(t) (\rho_a T_a) U(t) = e^{it\phi} \left[\rho_a T_a - \frac{1}{4\pi i} \partial_x \right] e^{-it\phi} , \quad (15)$$

we find

$$\mathcal{A}_i = 2i \int_0^1 dt \int dx\, \mathrm{tr}(\partial_i \phi e^{it\pi} T_a e^{-it\pi}) \rho_a(x) + \mathcal{B}_i = 2 \int dx\, \mathrm{tr}[g^{-1}(x) \partial_i g(x) T_a] \rho_a(x) + \mathcal{B}_i . \quad (16)$$

The c-number Berry connection $\mathcal{B}_i$ is given by

$$\mathcal{B}_i = -\frac{1}{2\pi} \int_0^1 dt \int dx\, \mathrm{tr}(\partial_i \phi e^{it\phi} \partial_i e^{-it\phi}) . \quad (17)$$

To calculate the Berry curvature, it is easier to use (12) instead of the explicit expression for the connection. We get

$$F_{ij} = -4 \int dx\, dy\, \mathrm{tr}[g^{-1}(x) \partial_i g(x) T_a] \mathrm{tr}[g^{-1}(y) \partial_j g(y) T_b] \langle 0, A | [\rho_a(x), \rho_b(y)] | 0, A \rangle = \frac{1}{2\pi i} \int dx\, \mathrm{tr}\{ \partial_x [g^{-1}(x) \partial_i g(x)] g^{-1}(x) \partial_j g(x) \} . \quad (18)$$

A similar formula such as the first one in (18) has already been obtained by Sonoda[3] in the (3+1)-dimensional case. However, his expression involves the anomalous part of the commutator of the full Gauss operator with functional derivatives with respect to the gauge field. This anomalous part is regularization dependent which leaves the result ambiguous. Our formula involves only the anomalous part of the current-current commutator: i.e., the Schwinger term. This allows an explicit calculation of the Berry connection and curvature. Equations (16)–(18) are our main result. They have been obtained in a true operator formulation of the theory.

We can write (18) also as

$$B = i \tfrac{1}{2} \epsilon^{ij} F_{ij} = \frac{1}{4\pi} \int dx\, [\epsilon^{ij} \mathrm{tr}(g^{-1} \partial_x g g^{-1} \partial_i g g^{-1} \partial_j g) + \epsilon^{ij} \partial_i \mathrm{tr}(g^{-1} \partial_x g g^{-1} \partial_j g)] . \quad (19)$$

The second term is exact; hence, its integral over any boundaryless two-surface vanishes. The integral of B over any such surface is the first Chern number of the corresponding bundle. This is also the magnetic charge enclosed. Hence we obtain

$$m = \frac{1}{4\pi} \int d^2\lambda\, B = \frac{1}{16\pi^2} \int d^2\lambda\, dx\, \epsilon^{ij} \mathrm{tr}(g^{-1} \partial_x g g^{-1} \partial_i g g^{-1} \partial_j g) = \frac{1}{48\pi^2} \int d^3x\, \epsilon^{ijk} \mathrm{tr}(g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g) , \quad (20)$$

where we have assembled the space coordinate x and the two parameters into the vector $(x_i) \equiv (x, \lambda_1, \lambda_2)$. This last integral is topological invariant, depending only on the homotopy class of the mapping of the two-manifold into $\mathcal{G}^1$. It is quantized in half-integer units, which is exactly the Dirac-quantization condition. If we focus on two-spheres, the homotopy classes define $\Pi_2(\mathcal{G}^1)$. Every mapping of S^2 into $\mathcal{G}^1$ automatically defines a mapping from $S^2 \times S^1$ into G. If $\Pi_2(G) = \Pi_1(G) = 0$, each such mapping is homotopic to a mapping which may be interpreted as a mapping from S^3 into G. The invariant integral on the right-hand side of (20) measures exactly the winding number of this mapping. Of course every mapping of S^3 in G directly defines an element of $\Pi_2(\mathcal{G}^1)$. Hence the homotopy classes are in a one-to-one relation.

Thus we have shown by explicit computation that the vacuum state forms a nontrivial monopole bundle over noncontractible two-spheres in the gauge orbit. This is an explicit realization of the fermionic Hilbert bundle structure deduced by Nelson and Alvarez-Gaumé.[1]

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