Construction of four-dimensional strings

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A systematic study of the general properties of four-dimensional heterotic strings is carried out. These are strings constructed from (22;10) self-dual lattices. The constraints from supercurrent, chirality, and spacetime supersymmetry are discussed. Each of them helps to nail down the self-dual lattice constructed by the gluing method. Eleven general theorems are proved in Sec. IV, on the existence or absence of tachyons, adjoint-representation Higgs particles, higher-spin massless objects, and possible gauge groups in the theory. The questions of chirality and shadow matter are also investigated. For example, it is shown that a string containing the standard-model group and one generation of fermions, as well as 16 dimensions of shadow matter, is necessarily nonchiral no matter how many generations of fermions or what kinds of Higgs particles are present. The theorems proved in this paper are for a specific choice of the supercurrent, that which is used in the $\mathbf{Z}_3$ orbifold, though similar theorems can be proved for other supercurrents.

I. INTRODUCTION

String theory[1-4] is the only known grand unified theory where quantum gravity can be meaningfully incorporated. And, because of that, the theory is most easily formulated at the Planck scale. From that scale down, as the universe cools, dynamics intervene to give rise to spontaneous symmetry breakings and phase transitions. Our inability to deal with the dynamical issues makes it difficult to predict anything from first principles at the present energies. For phenomenological applications, it is necessary to bypass these difficult dynamical questions and choose directly, among the allowed four-dimensional vacua, the ones that describe the real world. This is a perfectly valid approach, the difficulty being that it has proven to be very difficult to find phenomenologically correct models. Whether this is due to the inherent scarcity of such models, or just our inability to find them, is a question well worth studying. In a recent paper,[5] we began such a study in the limited context of strings that can be constructed from self-dual lattices. This includes all complex-fermionic constructions[2] as well as covariant[3] and noncovariant[2] lattice strings. The emphasis of the previous paper was on the construction and properties of self-dual lattices. In this paper we will continue the study by focusing on four-dimensional lattice strings. Besides being self-dual, we must also choose the lattice that can lead to the construction of a supercurrent and a chiral theory. It turns out that the requirement of the existence of such a supercurrent, and the chirality, place very strong restrictions on the self-dual lattices. As a result, certain gauge groups are definitely ruled out, and the possible massless spectrum for an allowed gauge group is highly restricted. This is unlike the usual four-dimensional grand unified theories where almost all gauge groups are allowed, and that other than weak restrictions coming from anomaly cancellation and the like, just about all possible spectra are permitted. See Sec. V for more details.

The restrictions mentioned could depend on the supercurrent chosen. Unfortunately, a complete list of supercurrents constructable from lattice bosons is not known, though special solutions are available.[6] Among these known supercurrents, some of them can never lead to a four-dimensional chiral theory and we reject those on phenomenological grounds. Of the remaining ones, we choose to discuss in this paper only the one introduced in Sec. III. This is the supercurrent used in the $\mathbf{Z}_3$ orbifold at a particularly symmetric moduli. Although the restrictions obtained in Secs. III–V may be specific to this particular form of the supercurrent, the method employed is general enough to be usable for other supercurrents as well. A detailed discussion of those, however, will be postponed until a future publication.

Four-dimensional lattice strings suffer a drawback in that the gauge group is of rank 22, which is definitely too large for phenomenological applications. There are, however, at least two possible ways out of this difficulty. We can treat the lattice string as a starting point, and then proceed to break down the symmetry by introducing marginal operators to move it along a flat direction. Alternatively we might try to arrange some of these 22 ranks to be in a *shadow group,* thereby effectively reducing the rank of the observed gauge group. By definition, a shadow group is one in which the observed particles have zero quantum numbers, and that any massless particles with nontrivial quantum numbers in the shadow group have no quantum numbers in the observed gauge group. Thus matter in the observed gauge group and matter in the shadow group can interact only through gravity, and the shadow group is effectively decoupled and unobservable at the present energies. In this paper, we will not discuss systematically the first option. We will, however, prove some general theorems concerning the shadow-group scenario. These theorems are to be found in Sec. V.

We employ in this paper the methods discussed in the preceding paper[5] to construct self-dual lattices. Unlike the methods usually adopted,[2,3] where specific boundary conditions are used to discriminate different lattice strings, the present methods allow us to construct strings encompassing specific gauge groups and massless spectra of particles, and incorporating other desirable constraints such as chirality and the choice of a particular supercurrent. It is for this reason we call the present procedure a "bottom-up construction." Mathematically, the present construction makes heavy use of the "gluing method" rather than just the "shifting method" used elsewhere. These methods will be briefly reviewed and further developed in Sec. II. The constraint placed on the lattice by the supercurrent will be discussed in Sec. III, and that by chirality and/or supersymmetry will be discussed in Sec. IV. General theorems resulting from the various desirable restrictions will be discussed in Sec. V.

II. GLUING AND SHIFTING CONSTRUCTIONS OF SELF-DUAL LATTICES

A four-dimensional heterotic string can be constructed from 22 left-moving, and 10 right-moving, compactified bosonic coordinates. Their momenta p are 32-component vectors in a vector space with an orthonormal metric whose signatures are $((+1)^{22};(-1)^{10})$. One of the ten right movers, say, the tenth, is the bosonized coordinate of the real fermions ψ_i $(i=1,2)$. In the light-cone gauge, these two ψ_i are the world-sheet supersymmetric partners of the uncompactified transverse spacetime coordinates X_i. For that reason, the transverse spacetime group SO(2) is a symmetry group of this tenth coordinate. Let v_0 be a vector weight of this SO(2) with its other 31 components set equal to zero. Then in order for modular invariance to be obeyed, and the correct connection between spin and statistics to be established, the "shifted momenta" $q=p-v_0$ must form a self-dual lattice Λ obeying

$$q^2+2q\cdot v_0\in 2\mathbf{Z} \; . \tag{2.1}$$

Before we describe the gluing construction[5,7] of Λ, it is necessary to review some standard lattices and their notations. We will use the symbol of a Lie algebra to denote its root lattice. For example, the lattice E_8 means the root lattice of the Lie algebra E_8. The roots of simply laced algebras are normalized to have norm (length squared) two. Z_l is used to denote the l-dimensional orthonormal Euclidean lattice, and Z the one-dimensional one. If Γ is any integral lattice, then $\Gamma^{(l)}$ is used to denote the *scaled lattice* defined by $\Gamma^{(l)}=\{\sqrt{l}x|x\in\Gamma\}$. For $\Gamma=Z$, $Z^{(l)}$ will often be abbreviated as simply (l).

The notation $|\Gamma|$ is used to denote the volume squared of the unit cell of the lattice Γ. A lattice is self-dual if and only if $|\Gamma|=1$.

The dual lattice Γ^* of a root lattice Γ is the weight lattice of that Lie algebra. These weights are divided into different equivalent classes (conjugacy classes); elements within each class differ from each other by elements of Γ. The number of conjugacy classes is given by $|\Gamma|$, which is equal to $r+1,4,3,2,1,1$, respectively, for $\Gamma=A_r$, D_r, E_6, E_7, E_8, and Z_r. The conjugacy classes $[i]$ are labeled from $i=0$ to $|\Gamma|-1$ according to a convention of Conway and Sloane.[7] Class [0] always represents the root lattice. The other classes and their properties are given in Table I. The fundamental weight from which class $[i]$ is defined is indicated by the class symbol $[i]$ in the Dynkin diagrams Fig. 1.

The number of classes of the dual lattice $\Gamma^{(l)*}$ of a scaled lattice (l is a positive integer) $\Gamma^{(l)}$ is l^r times the number of classes of Γ^*, where r is the dimension of the lattice Γ. If λ_i is a representative vector of the ith class of Γ^*, then the representative vectors of the classes of $\Gamma^{(l)*}$ can be taken to be $(\lambda_i+\sum_{j=1}^{r}m_j\alpha_j)/\sqrt{l}$, where α_j is a basis of Γ and the integers m_j each may range from 0 to $l-1$. In the case of $(l)=Z^{(l)}$, $r=1$ and it has l classes. These classes will be denoted by the integer m_1 above, as in $[m_1]$. The norm of its representative is m_1^2/l.

We will often identify *Euclidean* lattices which differ from one another simply by a rotation, for they would give rise to the same string physics. If we should want to make this point clear, that a (definite or indefinite) lattice

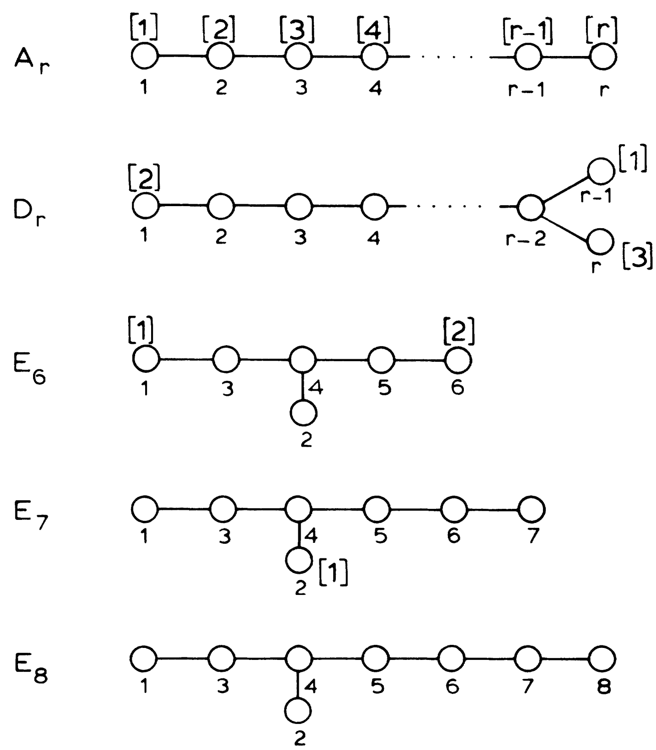

FIG. 1. Dynkin diagrams of simply laced simple Lie algebras. The class symbols [i] are put beside the fundamental weight λ_i representing that class. The labels i not between square brackets are for the simple roots α_i. Note that in this way of labeling, λ_i is not always the dual of α_i.

TABLE I. Glues for simple-Lie-algebra root lattices. The weights (glues) are expressed in orthonormal bases in column 2. Weight λ_i is a representative vector in glue class $[i]$. The additive properties of the glue classes are shown in column 3, and the dot products $[i]\cdot[j]\equiv\lambda_i\cdot\lambda_j$ (mod 1) are shown in column 4.

Algebra	Weights	Additive property	Dot products
		$([i]+[0]=[i])$	
A_r	$\lambda_i=[(i)^{i'},(-i')^{i}]/(r+1)$ $i'\equiv r+1-i$	$[i]+[j]=[i+j]$ (mod $r+1$)	$[i]\cdot[j]=ij'/(r+1)$ $(i\le j)$
D_r		$[i]+[j]=[i+j]$ (mod 4; r odd)	
	$\lambda_1=[(1/2)^r]$ $\lambda_2=[1,0^{r-1}]$ $\lambda_3=[(1/2)^{r-1},(-1/2)]$	$[i]+[i]=[0]$ $[i]+[j]=[k]$ ($i,j,k=$permutation of 1,2,3) (r even)	$[1]^2=[3]^2=r/4$ $[2]^2=1$ $[1]\cdot[3]=(r-2)/4$ $[1]\cdot[2]=[3]\cdot[2]=\frac{1}{2}$
E_6	$\lambda_1=\frac{2}{3}[0^5,(-1)^2,1]$	$[i]+[j]=[i+j]$	$[1]^2=[2]^2=\frac{4}{3}$
	$\lambda_2=\frac{1}{3}[0^4,3,(-1)^2,1]$	(mod 3)	$[1]\cdot[2]=\frac{2}{3}$
E_7	$\lambda_1=\frac{1}{2}[1^6,(-2),2]$	$[1]+[1]=[0]$	$[1]^2=\frac{7}{2}$

Λ_1 differs from the lattice Λ_2 only by a Euclidean rotation of its left-hand part and a Euclidean rotation of its right-hand part, then we will write $\Lambda_1\approx\Lambda_2$ and call the two lattices *equivalent.*

We will now return to the construction of self-dual lattices by gluing. Given an integral, but not self-dual, *base lattice* Λ_0, one may try to enlarge it into a self-dual lattice Λ by adding to it enough new lattice vectors g_i. These newly added lattice vectors of Λ are called *glues.* Grouping them into equivalent *glue classes* $[g_i]$, so that two of them in the same class differ only by a vector in Λ_0, then Λ is an integral lattice if these glue classes $[g_i]$ generate an additive group G with $g_i\in\Lambda_0^*$ and with integral dot products for all $g_i\cdot g_j$. Such a group is called a *glue group.* The enlarged lattice Λ is given by the union of these glue classes $[g_i]$. This lattice will also be denoted as $\Lambda_0[G]$, or $\Lambda_0[g_1,g_2,\dots]$, to emphasize its dependences on Λ_0 and the glues. It has a smaller unit-cell volume than that of Λ_0; indeed the size of its cell volume shrinks with the increasing size of the glue group G. Theorem 3.1 in Ref. 5 gives the quantitative relationship for this shrinking. It is reproduced below.

Theorem 2.1. Let Λ_0 be a base lattice, and $G=\mathbf{Z}_{n_1}\times\mathbf{Z}_{n_2}\times\cdots\times\mathbf{Z}_{n_B}$ a glue group. Then $\Lambda_0[G]$ is integral, and $|\Lambda_0[G]|=|\Lambda_0|/(\prod_{j=1}^{B}n_j)^2$.

To make Λ self-dual and $|\Lambda|=1$, we need to have a large enough glue group G. A necessary and sufficient condition for this to happen is given by the theorem below, which is Theorem 3.2 of Ref. 5:

Theorem 2.2. Let the glue group G be isomorphic to $\mathbf{Z}_{n_1}\times\mathbf{Z}_{n_2}\times\cdots\times\mathbf{Z}_{n_B}$, then $\Lambda=\Lambda_0[G]$ is self-dual iff $|\Lambda_0|=(\prod_{i=1}^{B}n_i)^2$.

This theorem is extremely useful in the construction of self-dual lattices.[5] It tells us glues of what *orders* (i.e., the numbers n_i in the theorem) to look for. On the other hand, it does not tell us whether a large enough glue group G can be found to make $\Lambda_0[G]$ self-dual. We will define a base lattice Λ_0 to be *self-dualizable* if such a glue group can be found. According to this theorem, it is necessary to have $|\Lambda_0|$ to be the square of an integer in order for Λ_0 to be self-dualizable. This condition, however, is not sufficient as we shall see.

Among the root lattices of the simply laced simple Lie algebras, the only one that is self-dual is E_8. But glues can be found, using Theorem 2.2, to complete the base lattices $\Lambda_0=D_{4k}$ or $\Lambda_0=A_{r^2-1}$ into self-dual lattices.[5,7] The glue group for the former is generated by the glue class [1] *or* [3], and the resulting self-dual lattice is denoted as D_{4k}^+. The glue group for the latter is generated by the glue class $[r]$, and the resulting self-dual lattice will be denoted by $A_{r^2-1}^+$.

Thus A_{r^2-1} and D_{4k} are self-dualizable. Now the determinant $|\Lambda_0|$ of $\Lambda_0=A_2\oplus A_2$ is 3^2, the square of an integer. However it is *not* self-dualizable. The necessary and sufficient conditions for an integral lattice to be self-dualizable have been worked out and will be discussed in a separate publication.[8] Some of these results will be used in Sec. V to determine the allowed gauge groups.

Given a base lattice Λ_0 that is self-dualizable, meaning that there is a glue group G such that $\Lambda_0[G]$ is self-dual, there is still the question of whether this glue group is compatible with the desired massless spectrum $\mathcal{S}$ (see the end of this section for a discussion between the glue group and the massless spectrum). The following theorem, which is Theorem 3.3 of Ref. 5, assures us that they may always be made compatible:

Theorem 2.3. Let Λ_0 be a self-dualizable base lattice. Let $G'\subset\Lambda_0^*$ be any set of vectors with pairwise integral dot products. Then there exists a self-dual gluing of Λ_0 containing G'.

There is another method to build self-dual lattices. This is the *shifting method.* One starts here with a *self-dual* lattice Λ_1 and a *shift group* Ω to construct another self-dual lattice Λ_2. To emphasize its dependence on Λ_1 and Ω, this new lattice Λ_2 will also be denoted by $\Lambda_1(\Omega)$. It is important not to mix up this notation of shifting with the notation in gluing. The latter is denoted using a pair of *square* brackets, as in $\Lambda_0[G]$. The shifting method was reviewed and its properties developed in Ref. 5. In the following, we shall recapitulate it in a slightly different, and conceptually perhaps simpler, approach than that used in Ref. 5.

A shift group Ω is a finite Abelian additive group generated by the *shift vectors* ω_i of *order* n_i, $1\le i\le A$. This means that $Q_i\equiv n_i\omega_i\in\Lambda_1$, and that $\Omega\simeq \mathbf{Z}_{n_1}\times\cdots\times\mathbf{Z}_{n_A}$. The shift vectors must satisfy the conditions

$$\begin{aligned} &Q_i\cdot Q_i\equiv n_iL_{ii}\in\epsilon_in_i\mathbf{Z}\ ,\\ &Q_i\cdot Q_j\equiv D_{ij}L_{ij}\in D_{ij}\mathbf{Z}\quad (i\neq j)\ , \end{aligned} \tag{2.2}$$

where ϵ_i is defined to be 1 for n_i odd and 2 for n_i even, and $D_{ij}=(n_i,n_j)$ is the largest common divisor of n_i and n_j.

Given a shift group Ω with generators ω_i, vectors $r_i\in\Lambda_1$ can be found (Lemma B.2 of Ref. 5) such that $r_i\cdot Q_j=\delta_{ij}$ (mod n_j). Let us partition the vectors q in Λ_1 into $\prod_{i=1}^{A}n_i$ sectors specified by the integers $m=\{m_1,m_2,\ldots,m_A\}$ $(0\le m_i<n_i)$, and the condition $q\cdot Q_i=m_i$ (mod n_i). No sector is empty for $q_m=\sum_{i=1}^{A}m_ir_i$ is always a member of that sector. Denote by $\Lambda_{1,m}$ the sector labeled by the set m. It is easy to see that $\Lambda_{1,m}=\Lambda_{1,0}+q_m$, where $\Lambda_{1,0}$ is the *unshifted sector* with all $m_j=0$. The new lattice $\Lambda_2=\Lambda_1(\Omega)$ is constructed out of the union of all sectors $\Lambda_{1,m}$, each shifted by a suitable vector $s_m\in\Omega$, so determined as to make Λ_2 integral. It turns out[5] that this lattice $\Lambda_2=\Lambda_1(\Omega)=\cup_m\{\Lambda_{1,m}+s_m\}$ is self-dual if it is integral.

We proceed to describe how s_m is determined. If $q_m\in\Lambda_{1,m}$ and $q'_m\in\Lambda_{1,m'}$, then $q_m+q'_m\in\Lambda_{1,m+m'}$. Thus for Λ_2 to be a lattice we must have $s_m+s'_m=s_{m+m'}$, up to a vector in $\Lambda_{1,0}$. In other words, s_m is linear in all the m_j's. In fact, the vectors s_m cover the whole shift group Ω (Theorem 4.2 of Ref. 5), so there must be an m for which $s_m=\omega_i$ for each i. Let the m_j for this m be denoted M_{ij}, and the corresponding vector q_m in $\Lambda_{1,m}$ simply be denoted as $q(i)=\sum_j M_{ij}r_j$. Given an i, our task is to find the sector parameter m for which the shift vector is $s_m=\omega_i$. In order for Λ_2 to be integral, we need to have $[q(i)+\omega_i]\cdot[q(j)+\omega_j]\in\mathbf{Z}$ for all i and j. This requires $M_{ij}/n_j+M_{ji}/n_i+\omega_i\cdot\omega_j\in\mathbf{Z}$. Using (2.2), this requirement becomes

$$n'_iM_{ij}+n'_jM_{ji}+L_{ij}=0\ (\mathrm{mod}\ E_{ij})\ , \tag{2.3}$$

where $n'_i=n_i/D_{ij}$, $n'_j=n_j/D_{ij}$, and $E_{ij}=n'_in'_jD_{ij}$ is the least common multiple of n_i and n_j.

It is now clear why the quantities L_{ij} defined in (2.2) have to be integers: the rest of the quantities in (2.3) are. It is also clear why the diagonal quantity L_{ii} has to be even when n_i is even. Setting $i=j$ in (2.3), the equation has a solution only when either L_{ii} is even, or when $E_{ii}=n_i$ is odd.

With these conditions, Eq. (2.3) can always be solved[5,9] for M_{ij} and M_{ji}. The solution is not unique but is parametrized by the *discrete torsion* parameters $Q_{ij}=-Q_{ji}$, each taking values freely between 0 and $D_{ij}-1$.

This completes the description of the construction by shifting. For completeness, a comparison of the notations used here and in Ref. 5 is listed below. Equation (2.3) here is identical to Eq. (C19) in Ref. 5. Since $M_{ij}=q(i)\cdot n_j\omega_j$ here and $M_{ij}=\omega_i\cdot n_j\zeta_j$ in Ref. 5, we obtain $q(i)\cdot\omega_j=\omega_i\cdot\zeta_j$, which connects the quantity $q(i)$ used here and the quantity ζ_i used in Ref. 5.

An interesting and important question is, whether we can start from any self-dual lattice Λ_1 and be able to get to all other self-dual lattices Λ_2 of the same dimension by shifting. The answer is no. For example, all the (22;10) lattices $\Lambda_1=\Lambda_0[G]$ defined below are self-dual, but they are not mutually shiftable. Let the base lattice be $\Lambda_0=(\Lambda_{0L};\Lambda_{0R})$, with Λ_{0L} given by the direct sum of a $Z^{(p)}$ and 21 Z's, and Λ_{0R} given by the direct sum of a $Z^{(p)}$ and $9Z$'s. p is a prime number. The glue group G is generated by the glue vector $g=(1^{22};1^{10})$. It can be shown that we can never shift from a Λ_1 of a given p to that of another p. The $\mathbf{Z}_3$ orbifold discussed later belongs to the class that can be shifted from the above lattice Λ_1 with $p=3$. If the lattices are Euclidean and self-dual, then one can always shift between them. This and other related aspects will be discussed in a forthcoming publication.[8]

So much for the construction of self-dual lattices. We would like to construct heterotic strings by constructing self-dual lattices Λ of dimensions (22;10), viz., a 32-dimensional lattice with 22 positive signatures and 10 negative signatures. Moreover, we would like to tailor it to possess a given gauge group $\bar{\mathcal{G}}$ and to contain a given massless spectrum $\mathcal{S}$. We will next discuss the constraint put on by these two requirements on the construction of the self-dual lattice.

Let us recall two facts. The momentum $p=(p_L;p_R)$ of a massless state must satisfy $p_L^2=0$ or 2, and $p_R^2=1$. Moreover, the rank of the gauge group for the four-dimensional strings constructed this way is always 22. Consequently if $\mathcal{G}$ is the largest semisimple group in $\bar{\mathcal{G}}$ and it is of rank r, the gauge group must be of the form $\bar{\mathcal{G}}=\mathcal{G}\times\mathrm{U}(1)^{22-r}$, with the U(1) groups generated by the left-moving bosonic oscillators.

Let $(\Lambda_L;\Lambda_R)$ be the maximum sublattice of Λ which can be expressed as a direct sum of a 22-dimensional Euclidean (i.e., definite) lattice Λ_L and a 10-dimensional Euclidean lattice Λ_R. We will assume throughout this paper that these two sublattices exist. Because of (2.1) the lattice Λ_L has to be even. According to a theorem of Witt,[5,7] the semisimple root lattice $\mathcal{G}$ of the gauge group is just the sublattice of Λ_L generated by its norm-2 vectors. Hence $\Lambda_L\supset\mathcal{G}$. We shall show in the next section that the right-hand lattice Λ_R contains the lattice Λ_{1R} defined in Theorem 3.1.

The following theorem, proved in the Appendix, is useful in relating Λ_L to Λ_R. In four-dimensional string applications, Λ_R is largely determined by the supercurrent and supersymmetry, whereas Λ_L is related to the gauge group. A relation between the two, therefore, helps to nail down possible gauge groups. See Sec. V for more details.

Theorem 2.4. Let Λ be an indefinite self-dual lattice with the maximal definite sublattices Λ_L and Λ_R as defined above. Then $|\Lambda_L|=|\Lambda_R|$.

We will always choose our base lattice Λ_0 to be a direct sum of a left and a right part: $\Lambda_0=(\Lambda_{0L};\Lambda_{0R})$. A glue with zero components in Λ_{0R} will be called a *left-hand glue,* and a glue with zero components in Λ_{0L} will be called a *right-hand glue.* If Λ_{0L} (Λ_{0R}) is a *proper* subset of Λ_L (Λ_R), then left-hand (right-hand) glues must exist to glue Λ_{0L} (Λ_{0R}) to Λ_L (Λ_R). The choice of Λ_{0L} is not unique. One possibility is to choose it to be of the form $\Lambda_{0L}=\mathcal{G}\oplus\mathcal{H}\equiv(\mathcal{G},\mathcal{H})$, where $\mathcal{H}$ is an even lattice [so that (2.1) is satisfied] of dimension $22-r$ spanned by vectors of norm larger or equal to 4 (so that $\mathcal{H}$ cannot be included in $\mathcal{G}$) (Ref. 5). $\mathcal{H}$ contains the "hidden quantum numbers" and is presently undetectable as a gauge group. The advantage of this Λ_{0L} is to make the gauge group $\mathcal{G}$ very explicit. The disadvantage is that it gives no clue as to what $\mathcal{H}$ should be. Moreover, for classification purposes, $\mathcal{G}$ is not *a priori* known. It is, therefore, desirable to be able to find some sort of "standard" forms for the base lattices Λ_0, at least for the purpose of classifications. In this regard the following theorem is very useful. We shall call the base lattice used below the *orthogonal base lattice.*

Theorem 2.5. Let Γ be any Euclidean integral lattice of dimension n. Then there exists positive integers $m_1,m_2,\ldots,m_n$ such that Γ can be obtained by gluing $Z^{(m_1)}\oplus Z^{(m_2)}\oplus\cdots\oplus Z^{(m_n)}$. Moreover, (i) $\prod_{i=1}^{n} m_i/|\Gamma|$ must be the square of an integer; (ii) if μ is the minimum norm of Γ, then each $m_i\geq\mu$; (iii) if Γ is even, then each m_i must be even; (iv) if $\Gamma^{(1/k)}$ is also integral, for some integer k, then k must divide each m_i, and, if $\Gamma^{(1/k)}$ is even, then m_i/k must be even; (v) if $(m_1,m_2,\ldots,m_n)$ works, so does $(k_1^2m_1,k_2^2m_2,\ldots,k_n^2m_n)$ for any integers k_i (hence the integers m_i are not unique); (vi) if some prime p divides m_1, say, but does not divide any other m_i, nor $|\Gamma|$, then we may take $m_1'=m_1/p^2$ instead of m_1.

The proof of this theorem can be found in the Appendix. It is also valid for the indefinite lattices, in which case the base lattice is of the form $(Z^{(m_1)}\oplus Z^{(m_2)}\oplus\cdots\oplus Z^{(m_L)};Z^{(m_1')}\oplus Z^{(m_2')}\oplus\cdots\oplus Z^{(m_R')})$.

Let us illustrate this theorem by listing the simple roots and the fundamental weights of simply laced simple Lie algebras in the orthogonal basis.

Example 2.1. The relevant information is given in Table II. The scaled factors m_i are given in the third column. Given a $Z^{(m_i)}$, the minimal nonzero component in the lattice is $\pm m_i/\sqrt{m_i}\equiv\pm\beta_i$, with $\beta_i\cdot\beta_j=m_i\delta_{ij}$. The second column of this table gives these β_i expressed in terms of the simple roots α_j, and the fourth column expresses the simple roots α_i directly in the orthogonal basis. The numbering of the simple roots in terms of the Dynkin diagrams is given in Fig. 1. The node beside which the letter i (without square brackets) is written is the node for the simple root α_i. The fifth column gives the orthogonal components of those fundamental weights λ_i defining the conjugacy classes $[i]$. Note that in this notation, the fundamental weight λ_i is that for the class $[i]$, and with the exception of A_r, this is usually not the dual vector of the simple root α_i.

These representations differ from the usual *orthonormal* basis of expressing the weight and root vectors of these algebras (see Table I and Ref. 10) in two ways. First, in the usual bases, quite often these vectors are expressed in a basis in a higher-dimensional space. Thus the vectors in A_r are expressed in $(r+1)$-dimensional space, and the vectors in E_6 and E_7 are both expressed in eight-dimensional space. In contrast, the orthogonal basis used in Theorem 2.5 lies in a vector space whose dimension is the rank of the Lie algebra. Second, the root and weight lattices cannot be expressed as direct sums of sublattices, but they can be obtained from gluing into a base lattice which is an r-fold direct sum for a rank-r algebra. The examples above show what these base lattices are. Consult Table II for more detailed correspondences between the different bases. (End of Example 2.1.)

The choice of Λ_{0R} is determined by the choice of the supercurrent, as we shall discuss in the next section. We will choose it to be the lattice Λ_{1R} defined in Theorem 3.1.

We have now discussed in a general way how the base lattice Λ_0 is to be chosen. There remains only the question of the choice of the glue group G. This is partially determined by the massless spectrum $\mathcal{S}$ in the following way. Among the massless particles, the gauge vector boson belongs to the base lattice *if* it is properly chosen to contain the gauge group. This is because its momentum $p=(p_L;p_R)$ is such that p_L transforms like the adjoint representation of the gauge group, and hence is a root of the gauge root lattice $\mathcal{G}$, and p_R for this vector boson must transform like the vector weight of SO(2). The corresponding shifted momentum $q=p-v_0$ therefore belongs to the base lattice Λ_0. While the gauge boson is thus contained in the base lattice, the other zero mass particles, including the quarks, the leptons, the Higgs boson, and possibly also the gaugino and the other supersymmetric partners, are given by the spectrum $\mathcal{S}$. If $p=(p_L;p_R)\in\mathcal{S}$, then $q=p-v_0$ constitute the glues of the glue group G. Note, however, that only certain components of p are determined by $\mathcal{S}$. The components of the vector p in the $\mathcal{G}$ directions are determined by their group properties, and its component along SO(2) is determined by its helicity. The other components of p are determined only by the massless condition $p_L^2=0,2$ and $p_R^2=1$, and the constraints that the glues must form an additive glue group with integral dot products. These constraints are surprisingly strong as evidenced by the arguments given for the proof of Theorem 5.11 at the end of the Appendix. Note also that the massless states in $\mathcal{S}$ may not yield a large enough glue group to make $\Lambda_0[G]$ self-dual. In that case, additional glues, hopefully massive, would have to be supplied to complete the job.

III. SUPERCURRENT

A heterotic string must possess a conformal and a superconformal current in order to eliminate all the negative-norm states. Conformal invariance has already

TABLE II. Orthogonal decomposition of the root lattices. See Fig. 1 for the labeling of the number i for the simple roots α_i, and Theorem 2.5 as well as Example 2.1 in the text for discussions. Column 2 gives the decomposition of the shortest nonzero root β_i in the one-dimensional lattice $Z^{(m_i)}$ in terms of the simple roots α_j, and column 4 gives the explicit representation of α_i in terms of the glue classes $[k_i]$ of $Z^{(m_i)}$. Note that the glue class $[k_i]$ corresponds to a component value of $k_i/\sqrt{m_i}=k_i\beta_i/m_i$. Finally, the last column expresses the representative weight vectors λ_i of class $[i]$ in the orthogonal basis. Note that the numberings are such that λ_i is the dual of α_i, for all i, only in A_r.

Algebra	Orthogonal basis β_i	m_i	Roots α_i	Weights λ_i
A_r	$\beta_{2k+1}=\alpha_{2k+1}$	$m_{2k+1}=2$	$\alpha_{2k+1}=\beta_{2k+1}$	$[0^{i-2},(i-1)/2,1,$ $i,0,i,0,\ldots],$ i odd
	$\beta_{2k}=\alpha_1+2\alpha_2$ $+\cdots+2k\alpha_{2k}+k\alpha_{2k+1},$ if $2k<r$	$m_{2k}=$ $2k(k+1),$ if $2k<r$	$\alpha_{2k}=(0^{2k-3},1-k,$ $-1,k+1,-1,0^{r-2k-1}),$ if $2k<r$	$[0^{i-1},i,0,$ $i,0,\ldots],$ i even
	$\beta_{2k}=\alpha_1+2\alpha_2+\cdots+r\alpha_r,$ if $2k=r$	$m_{2k}=$ $r(r+1),$ if $2k=r$	$\alpha_{2k}=(0^{2k-3},1-k,-1,2k+1),$ if $2k=r$	
D_r	$\beta_1=\alpha_{r-1}-\alpha_r$ $\beta_2=\alpha_{r-1}+\alpha_r$ $\beta_i=2\alpha_{r-i+1}$ $+\cdots+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r,$ $i>2$	$m_i=4$	$\alpha_i=(0^{r-i-1},-2,2,0^{i-1}),$ $i<r-1$ $\alpha_{r-1}=(2,2,0^{r-2})$ $\alpha_r=(-2,2,0^{r-2})$	$[1]=[1^r]$ $[2]=$ $[0^{r-1},2]$ $[3]=$ $[-1,1^{r-1}]$
E_6	$\beta_{1,2}=\alpha_2\mp\alpha_3$ $\beta_i=\alpha_2+\alpha_3+2\alpha_4$ $+\cdots+2\alpha_{i+1},\quad 2<i<6$ $\beta_6=4\alpha_1+3\alpha_2+5\alpha_3$ $+6\alpha_4+4\alpha_5+2\alpha_6$	$m_i=4,$ $i<6$ $m_6=12$	$\alpha_1=(1,(-1)^4,3)$ $\alpha_2=(2,2,0^4)$ $\alpha_i=(0^{i-3},-2,2,0^{7-i}),$ $i>2$	$[1]=$ $[0^5,4]$ $[2]=$ $[0^4,2,2]$
E_7	$\beta_{1,2}=\alpha_2\mp\alpha_3$ $\beta_i=\alpha_2+\alpha_3+2\alpha_4$ $+\cdots+2\alpha_{i+1},\quad 2<i<7$ $\beta_7=2\alpha_1+2\alpha_2+3\alpha_3$ $+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	$m_i=4,$ $i<7$ $m_7=2$	$\alpha_1=(1,(-1)^5,1)$ $\alpha_2=(2,2,0^5)$ $\alpha_i=(0^{i-3},-2,2,0^{8-i}),$ $i>2$	$[1]=$ $[1^6,2]$
E_8	$\beta_{1,2}=\alpha_2\mp\alpha_3$ $\beta_i=\alpha_2+\alpha_3+2\alpha_4$ $+\cdots+2\alpha_{i+1},\quad 2<i<8$ $\beta_8=4\alpha_1+5\alpha_2+7\alpha_3$ $+10\alpha_4+8\alpha_5+6\alpha_6+4\alpha_7$ $+2\alpha_8$	$m_i=4$	$\alpha_1=(1,(-1)^6,1)$ $\alpha_2=(2,2,0^6)$ $\alpha_i=(0^{i-3},-2,2,0^{9-i}),$ $i>2$	

ready been used in the light-cone gauge construction under discussion, but we must still make sure of the existence of a super(conformal) current constructed out of the right-moving variables.

If Φ denotes the first nine right-moving compactified bosonic operators, and X_i,ψ_i denote, respectively, the right-moving uncompactified coordinates and its world-sheet supersymmetric partners (when bosonized, the ψ_i corresponds to the tenth right-moving compactified bosonic coordinate), then the supercurrent, which has conformal dimension 3/2, must be of the form[6]

$$T_F(z)=\sum_{i=1}^{2}\psi_i(z)\partial X_i(z)+t_F(z)\ ,$$
$$t_F(z)=\sum_{p_1\in\Gamma_1}A(p_1)\cdot{:}\partial\Phi(z)\exp[ip_1\cdot\Phi(z)]{:}+\sum_{p_3\in\Gamma_3}B(p_3){:}\exp[ip_3\cdot\Phi(z)]{:}\ . \tag{3.1}$$

The vectors in the set Γ_1 must be of norm 1, with $p_1\cdot A(p_1)=0$, and the vectors in the set Γ_3 must be of norm 3. When parallel-transported around the closed string, $z\to ze^{2\pi i}$, the supercurrent $T_F(z)$ must be periodic or antiperiodic depending on whether $\psi_i(z)$ is periodic or antiperiodic. This requires the shifted momenta $q_1=p_1-v_0$ and $q_3=p_3-v_0$ to be in the lattice Λ, if we suitably expand the nine-component vectors p_i into 32-component vectors p_i by setting all the other components equal to zero. Note in particular that $p_i\cdot v_0=0$.

The sets Γ_1 and Γ_3, as well as the coefficients $A(p_1)$ and $B(p_3)$, must be chosen to satisfy the superconformal algebra. The general solution for this is not known. One particular solution would be to have all $B(p_3)=0$. This corresponds to the supercurrent construction in a ten-dimensional string and it would not lead to a chiral four-dimensional theory. A solution that may lead to chiral theories is obtained by choosing all $A(p_1)=0$, all $B(p_3)$

equal, and the set Γ_3 to consist of nine mutually orthogonal vectors p_3 of norm 3. The corresponding shifted momentum $q_3 = p_3 - v_0$ is in Γ and is of norm 4. This supercurrent is, for example, the one used in the $\mathbf{Z}_3$ orbifold, as will be shown in Example 3.1 below. From now on, for the sake of concreteness, we shall adopt this solution and consider its consequences. The method, however, is equally applicable if another solution of (3.1) is taken. In that case, Theorem 3.1 has to be modified accordingly.

This supercurrent solution may be used to constrain the right-hand part of the lattice Λ_R. The proof of the following theorem is given in the Appendix.

Theorem 3.1. (i) Λ_R contains the *supercurrent base (sub)lattice*

$$\Lambda_{1R} \equiv (Z^{(12)},Z^{(12)},Z^{(12)},Z^{(12)},Z^{(12)},Z^{(12)},Z^{(12)},Z^{(12)},Z^{(12)},Z^{(4)})$$
$$\equiv ((12),(12),(12),(12),(12),(12),(12),(12),(12),(4))$$

and the glue classes $[(q_3)_i] = [0,\ldots,0,6,0,\ldots,0,2]$, where the "6" occurs at the ith place and $1 \le i \le 9$; (ii) let $q = (q_L; q_R) = p - v_0$ represent a massless fermion, then in the basis of Λ_{1R}, each of the 10 components of p_R must be equal to $+1$ or -1; (iii) suppose a spacetime supersymmetric theory contains a massless fermion $q = (q_L; q_R)$ not in the adjoint representation of the gauge group. If $p_R = q_R + v_0$ is chosen conventionally to be $p_R = ((-1)^{10}) \equiv \eta$, then the gaugino glue vector $\delta = (1^6(-1)^4) \in \Lambda_R$ is uniquely determined up to permutations of its first nine coordinates. In particular, there can be no $N > 1$ spacetime supersymmetry theory that is chiral.

It is necessary to explain what is meant by a spacetime supersymmetric theory in this context. We mean by this that a gaugino glue of the form $(0;\delta)$ exists, with δ belonging to class [3] in the helicity group SO(2) (the tenth coordinate of Λ_{1R}), and with $\delta^2 = (\delta + v_0)^2 = 1$. We also mean by this that every massless vector boson must be in the adjoint representation of the gauge group. Beyond this, no supersymmetric operator algebra is assumed. With this definition, we note that neither $(q_L;0)$ nor $(0;q_R)$ in (iii) of Theorem 3.1 may be in the self-dual lattice Λ for the string.

To illustrate this theorem, we shall consider the supercurrent in the familiar example of the $\mathbf{Z}_3$ orbital.

Example 3.1: $\mathbf{Z}_3$ orbifold. The gluing solution of this has been discussed in Table III of Ref. 5. It is reproduced as Table III here. Added to the bottom of this table are the vectors $q_1 = p_1 - v_0$ and $q_3 = p_3 - v_0$ defining the supercurrent (3.1). It is easy to check that the nine p_3 form an orthogonal set of norm 3 each.

If instead we choose the right-hand base lattice Λ_{0R} to be Λ_{1R} of Theorem 3.1, then the glue vectors $A'-D'$, and the supercurrent vectors q_1 and q_3, will take on another form. These new expressions are given in Table IV. (End of Example 3.1.)

The determinant of Λ_{1R} in Theorem 3.1 is $|\Lambda_{1R}| = (12)^9 4$. Incorporating the right-hand glues $(q_3)_i$ and δ, the right-hand glued lattice $\Lambda_{2R} \equiv \Lambda_{1R}[(q_3)_1,\ldots,(q_3)_9,\delta]$ has a determinant equal to $[\Lambda_{2R}] = (12)^9 4/(2^9 6)^2 = 3^7 \ge |\Lambda_R|$. Combining this with Theorem 2.4, and the fact that any additional right-hand glue must contribute a factor dividing $|\Lambda_{2R}|$ by some power of 3^2 (Theorem 2.1), we obtain immediately Theorem 3.2 below. If the theory is not spacetime supersymmetric, then the gaugino glue δ should not be included, and the bound for $|\Lambda_R|$ is increased to $3^9 2^2$.

Theorem 3.2. $|\Lambda_L| = |\Lambda_R| = 3^7$, 3^5, 3^3, or 3 if the theory is spacetime supersymmetric. Otherwise, the allowed values for $|\Lambda_L| = |\Lambda_R|$ are all of the above, plus each of the above multiplied by 2^2, plus 3^9 and $3^9 2^2$.

This theorem will be used in Sec. V to find the allowed gauge groups.

Theorem 2.5 allows us to choose the left-hand base lattice Λ_{0L} in an orthogonal basis. It, however, cannot specify what the scale factors m_i are. Now with the supercurrent input from a four-dimensional string, these can be specified partially:

Theorem 3.3. We may take the m_i's of Λ_{0L} in Theorem 2.5 to be $m_1 = 3k^2$, and $m_j = k^2$ for $2 \le j \le 22$, where k is some (as yet unknown) integer.

We have stated in Sec. II that it is *not* true that every self-dual lattice can be obtained from a standard one by shifting. However, when the supercurrent input is incorporated, the story is different.

Theorem 3.4. Any two strings with this supercurrent can be obtained from each other by shifting. In particular, all of them can be obtained by shifting from the $\mathbf{Z}_3$ orbifold of Example 3.1.

IV. CHIRAL THEORIES

According to (iii) of Theorem 3.1, $N > 1$ spacetime supersymmetry is never present with our choice of the supercurrent. This eliminates an important source of nonchirality. Theorems 4.1 and 4.2 below establish conditions for a theory to be chiral. The proofs of these theorems are found in the Appendix.

A word of caution is necessary about what we mean by a chiral theory. By a *nonchiral* theory, we mean one in which every left-handed massless fermion with fixed quantum numbers in $\mathcal{G}$ *and* $\mathcal{H}$ has a right-handed partner. Let us define "left-handed" helicity to be in the Conway-Sloane class [3] (for p) of the SO(2) (the tenth coordinate on the right-hand side), and "right-handed" helicity to be the ones in class [1]. See Refs. 5 and 7 for more information. So a left-handed fermion $q = (q_L; q_R)$ is a chiral partner of a right-handed fermion $q' = (q_L'; q_R')$ iff $q_L = q_L'$, the tenth coordinate of q_R belongs to class [3], and the tenth coordinate of q_R' belong to class [1]. Note that there are no requirements for the other components

TABLE III. $\mathbf{Z}_3$ orbifold with the original base lattice. The base lattice Λ_0 is given in the first row. The massless glues $A-E$ are given in the subsequent rows. The massive glue F is also necessary to glue Λ_0 to a self-dual lattice representing the $\mathbf{Z}_3$ orbifold. The vectors q_1 and $(q_3)_i$ are the ones from which the supercurrent is constructed. See Eq. (3.1) and the discussions below it.

Λ_0	E_8	E_6	A_2	$A_2^{(9)}$	$A_2^{(9)}$	$A_2^{(9)}$	$A_2^{(9)}$	$A_2^{(9)}$	$A_2^{(9)}$	A_2	$Z^{(12)}$	$Z^{(4)}$
A	0	0	0	0	0	0	0	0	0	2	[I]	
B	0	0	0		[i]		0	0	0	1	[II]	
C	0	1	1	0	0	0	0	0	0	1	[II]	
D	0	1	0		[ii]			[ii]		0	[II]	
E	0	0	2		[iii]			[ii]		0	[II]	
F	0	1	0	020	080	080	020	080	080	0	[II]	
	0	1	0	080	020	080	080	020	080	0	[II]	
q_1	0	0	0	0	0	0	0	0	0	0	3	1
$(q_3)_i$	0	0	0	0	0	0	e	0	0	α	1	-1
	0	0	0	0	0	0	0	e	0	β	1	-1
	0	0	0	0	0	0	0	0	e	γ	1	-1

States

$(A_2^{(9)}, A_2^{(9)}, A_2^{(9)})$ states
- $[\mathrm{i}]=(b,0,0),(0,b,0)$, or $(0,0,b)$
- $[\mathrm{ii}]=(c,c,c)$
- $[\mathrm{iii}]=(c,c,d),(c,d,c)$, or (d,c,c)

$A_2^{(9)}$ states
- $b=030,003$, or 066
- $c=080,008$, or 011
- $d=020,002$, or 077
- $e=0\bar{1}0,00\bar{1}$, or 011 $(\bar{1}\equiv -1)$

$(Z^{(12)}, Z^{(4)})$ states
- $[\mathrm{I}]=(0,0)$ or $(3,1)$
- $[\mathrm{II}]=(10,2)$ or $(1,3)$

A_2 states
- $\alpha=\lambda_1$
- $\beta=\lambda_1-\alpha_1$
- $\gamma=\lambda_1-\alpha_1-\alpha_2$

Glue groups and generators
- $(\mathbf{Z}_{12})[C]$
- $(\mathbf{Z}_3)^6[B]$
- $(\mathbf{Z}_9)^4[D]$
- $(\mathbf{Z}_3)^2[F]$

of q_R and q_R'. Note also that if a theory is nonchiral in our sense, it is nonchiral in a more usual sense as well.

Suppose $\Lambda=\Lambda_0[G]$ is a self-dual lattice for an $N=1$ spacetime supersymmetric theory in which at least one left-handed massless fermion is present. This means that G contains glues of the form $(0^{22};(q_3)_i)$, $(0^{22};\delta)$, and $(q_L;\eta)$, as defined in Theorem 3.1, with $q_L^2=0$ or 2. Let $g_a=(g_{aL};g_{aR})$ be the remaining independent glues in G, and let f_j be the jth $(1\le j\le 10)$ component of g_{aR}.

Theorem 4.1. The supersymmetric theory described above is nonchiral iff either for every glue g_a, $f_7+f_8+f_9=0$ (mod 6) is true, or for every glue g_a, $f_\alpha+f_\beta+f_\gamma=0$ (mod 6; α, β, and γ are any three distinct integers between 1 and 6) is true.

Theorem 4.1 tells us what glues to avoid if we want a supersymmetric chiral theory. The condition stated there deals only with the right movers. However, because of self-duality, the right movers are intimately related to the left movers and sometimes the information given for the left movers is already sufficient for us to rule out a chiral theory. Since gauge groups and gauge quantum numbers of the observed particles are specified by the left movers, it is much more convenient for applications if a criterion for nonchirality can be established by the left movers alone. The following theorem is a theorem of this kind. Spacetime supersymmetry is not required.

Theorem 4.2. Let $[q]=[(q_L;q_R)]$ be a glue class of a massless fermion not in the adjoint representation. If 3 does not divide the order n_L of the glue class $[q_L]$, then q has a massless chiral partner. In particular, if the theory is spacetime supersymmetric, then 3 divides n_L.

Note that the order n of a glue class $[g]$ is defined to be the smallest positive integer so that $n[g]=[0]$.

Theorems 4.1 and 4.2 illustrate the difficulty of constructing chiral theories. To illustrate Theorem 4.2, let us consider the following example.

Example 4.2: $\mathrm{SO}(10)\times\mathrm{U}(1)\times\mathrm{SO}(32)$ grand unified theory (GUT) models. Suppose we want to construct a $\mathrm{SO}(10)\times\mathrm{U}(1)$ GUT theory with shadow matter quantum number given by $\mathrm{SO}(32)=D_{16}$. For this we take the left-hand part Λ_{0L} of the base lattice $\Lambda_0=(\Lambda_{0L};\Lambda_{0R})$ to be $\Lambda_{0L}=D_5\oplus Z^{(l)}\oplus D_{16}^{+}$ (for some l), and a glue $q=(q_L;q_R)$

TABLE IV. Z_3 orbifold in the supercurrent base lattice. The notations are the same as those of Table III. The glues $A'-D'$ in this table are related to the glues $A-D$ in Table III.

Λ_1	E_8	E_6	A_2	$A_2^{(9)}$	$A_2^{(9)}$	$A_2^{(9)}$	(12)	(12)	(12)	(12)	(12)	(12)	(12)	(12)	(12)	(4)
q_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(q_3)_1$	0	0	0	0	0	0	6	0	0	0	0	0	0	0	0	2
$(q_3)_2$	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	2
$(q_3)_3$	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	2
$(q_3)_4$	0	0	0	0	0	0	0	0	0	6	0	0	0	0	0	2
$(q_3)_5$	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0	2
$(q_3)_6$	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	2
$(q_3)_7$	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	2
$(q_3)_8$	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	2
$(q_3)_9$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	2
A'	0	0	0	0	0	0	2	2	2	0	0	0	2	2	2	0
	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	0
B'	0	0	0	b'	0	0	1	1	1	1	1	1	1	1	1	3
	0	0	0	0	b'	0	1	1	1	1	1	1	1	1	1	3
	0	0	0	0	0	b'	1	1	1	1	1	1	1	1	1	3
C'	0	1	1	0	0	0	1	1	1	1	1	1	1	1	1	3
D'	0	1	0	080	080	080	$\bar{1}$	1	1	$\bar{1}$	1	1	1	$\bar{1}$	$\bar{1}$	1
	0	1	0	008	080	080	1	$\bar{1}$	1	$\bar{1}$	1	1	1	$\bar{1}$	$\bar{1}$	1
	0	1	0	080	008	080	$\bar{1}$	1	1	1	$\bar{1}$	1	1	$\bar{1}$	$\bar{1}$	1
	0	1	0	080	080	008	$\bar{1}$	1	1	$\bar{1}$	1	1	$\bar{1}$	1	$\bar{1}$	1

States:

$A_2^{(9)}$ states:
$b'=030$ or 003 $\mathbf{Z}^{(12)}\equiv(12)$ states:
$\bar{1}=-1$

Glue groups and generators:
$(\mathbf{Z}_2)^9[q_3]$
$(\mathbf{Z}_3)^2[A']$
$(\mathbf{Z}_3)^6[B']$
$(\mathbf{Z}_6)^2[C']$
$(\mathbf{Z}_3)^4[D']$

to contain massless quarks and leptons in the 16-dimensional representation of SO(10). Since D_{16}^+ is meant to be the shadow group, this glue q for the observed fermions should have no components in it. Now the 16-dimensional representation of SO(10) belongs to class [1] of D_5. To make it massless we need $q_L^2=2$, and hence a component equal to $\pm\sqrt{3/4}$ along U(1). This latter belongs to class $[3m]$ of $Z^{(12m^2)}$. Thus $l=12m^2$ and the order of the glue is $4m$. According to Theorem 4.2, the GUT fermions have a chance to be chiral only when 3 divides the order of the glue class, meaning in this case that 3 must divide $4m$, and hence m. The smallest scaled-up lattice in this case is $Z^{(108)}$, corresponding to $m=3$.

Thus the resulting left-hand base lattice Λ_{0L} is $D_5 \oplus Z^{(12m^2)} \oplus D_{16}^+$. Its determinant is $|\Lambda_{0L}| = 4\times 12m^2 = 3\times 4^2 m^2| \geq |\Lambda_L|$. Thus $|\Lambda_L|$, which must be equal to $|\Lambda_{0L}|$ divided by the square of a number (Theorem 2.1), may indeed have the allowed values listed in Theorem 3.2. Nevertheless, it can be proved (see Theorem 5.11) that a chiral theory based on the gauge group $SO(10)\times U(1)\times SO(32)$, or $SO(10)\times U(1)\times E_8 \times E_8$, can never be constructed. (End of Example 4.1.)

V. GENERAL THEOREMS

Using the knowledge obtained in the previous sections, a number of general results on four-dimensional strings can be established. The proofs of the theorems are given in the Appendix.

It should be borne in mind that all we are starting from is a self-dual lattice containing the sublattice Λ_{1R}. When we say supersymmetry all that we mean is the existence of the gaugino glue vector δ defined in Theorem 3.1 (iii), and the absence of a massless vector boson in any other than the adjoint representation of the gauge group.

Beyond these we do not carry an operator supersymmetric algebra, or any other operator framework. Thus although some of the theorems stated below (e.g., Theorems 5.1 and 5.2) are familiar,[1,2,11] they are usually proven with the help of the supersymmetric algebra, which is absent in the present approach. In our framework these "familiar" theorems are not immediately obvious, and it is gratifying that with much less input in the gluing string framework, they are still valid.

Theorem 5.1. Tachyons are absent in spacetime supersymmetric theories.

Theorem 5.2. To each massless scalar boson in a supersymmetric theory, there is one and only one massless fermion with the same gauge quantum numbers. To each massless gauge vector boson in such a theory, there are two massless fermions (gaugino), one left-handed and another one right-handed.

The following theorem shows the absence of low-mass and high-spin fundamental particles. This theorem holds in much greater generality; it is included here because the proof contained in the Appendix makes use of this self-dual lattice formalism alone. By gravitons we mean the particles associated with the metric tensor $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, and the dilaton ϕ. By gravitinos we mean their supersymmetric partners.

Theorem 5.3. Other than the gravitinos and the gravitons, there can be no massless particles of spin greater than one.

The following theorem about the Higgs boson, long suspected to be true,[2] makes it impossible to construct conventional grand unified theories in this framework. Supersymmetry is not required for the following proof. A similar result has been obtained in Ref. 12. It is also known that this theorem may be false for other string models.[13]

Theorem 5.4. No massless scalar particles in a chiral theory can lie in the adjoint representation of the gauge group.

Next, we will establish what are the allowed gauge groups. With the help of Theorem 3.2, one can establish the following theorem.

Theorem 5.5. Let $\Lambda_0=(\Lambda_{0L};\Lambda_{0R})$ be a base lattice, and Λ_L,Λ_R be the maximal Euclidean sublattices, of a four-dimensional string. Then $A_2\oplus\Lambda_{0L}$ and $A_2\oplus\Lambda_L$ can both be glued to one of the 23 inequivalent Niemeier lattices. Moreover, all the glues for $A_2\oplus\Lambda_L$ are of order 3.

We remark parenthetically that there are 24 inequivalent self-dual even Euclidean lattices of 24 dimensions. One of them has a minimum nonzero norm to be 4 (the Leech lattice), and the 23 others have a minimum nonzero norm to be (the Niemeier lattices).

Recall from Theorem 3.2 that $|\Lambda_{0L}|$ may take on values 3^7, 3^5, 3^3, or 3 for a spacetime supersymmetric theory. The following two theorems deal with the relationship between these values and the chirality of the theory.

Theorem 5.6. A $|\Lambda_L|=3^7$ spacetime supersymmetric theory is necessarily chiral, in the sense defined in Sec. IV.

Theorem 5.7. A $|\Lambda_L|=3$ theory is necessarily nonchiral.

When $|\Lambda_L|=3^3$ or 3^5, the theory can be either chiral or nonchiral.

The next three theorems give a classification of the allowed gauge groups in the presence of shadow matter. Recall from Sec. I that a shadow group is one in which no glue is allowed to have nonzero components both inside and outside of the confines of the corresponding root lattice. For that reason a shadow-group lattice has to be self-dual in order for the whole string lattice to be so, and this confines the allowed shadow-group lattices to $\mathrm{E}_8=D_8^+$, $\mathrm{E}_8\times\mathrm{E}_8$, and D_{16}^+. With this factorization of the shadow group, the possible Niemeier lattices that can be built out of $A_2\oplus\Lambda_L$ according to Theorem 5.5 are greatly reduced. Then it can be shown that

Theorem 5.8. When shadow matter is present, we may take the m_i's in Theorem 2.5 to be products of 2's and 3's only.

Theorem 5.9. If the shadow-group lattice is E_8, then Λ_L is a sublattice of either $E_6\oplus E_8\oplus E_8$, or $(D_{13}\oplus Z^{(12)})[1,3]\oplus E_8$, and hence the gauge group $\overline{\mathcal{G}}$ is a subgroup of either $E_6\times E_8$ or $D_{13}\times\mathrm{U}(1)$.

Theorem 5.10. If the shadow group is of rank 16, then Λ_L is a sublattice of $E_6\oplus$ (shadow-group lattice), and hence the gauge group $\overline{\mathcal{G}}$ is a subgroup of E_6.

Define a *first-order ungluing* of a Euclidean lattice Λ to be any lattice Λ' for which there exists a glue g of order 3 satisfying $\Lambda=\Lambda'[g]$. Define a *second-order ungluing* of Λ to be a first-order ungluing of any first-order ungluing of Λ, etc. For example, the first-order ungluings of E_8 are A_8, $E_6\oplus A_2$, $D_7\oplus Z^{(36)}[1,9]$, and $E_7\oplus Z^{(18)}[1,9]$. E_7 itself has five first-order ungluings.

Thus the above theorems imply Λ_L must be a first-, second-, or third-order ungluing of E_6, $E_6\oplus E_8$, or $D_{13}\oplus Z^{(12)}[1,3]$, if the resulting string is to be chiral and has shadow matter. Conversely, any such ungluing determines Λ_L and hence the gauge group of a supersymmetric string with the supercurrent of Sec. III.

Similarly, with or without shadow matter, it can be shown using Ref. 14 that Λ_R is a first-, second-, or third-order ungluing of $D_9\oplus Z^{(12)}[1,3]$, $E_6\oplus Z_4$, $Z_9\oplus Z^{(3)}$, or $E_8\oplus Z\oplus Z^{(3)}$. In particular, because there is only a finite number of integral Euclidean lattices with a given determinant and dimension, there is a finite number of possible strings whose supercurrent is as in Sec. III. Moreover, in theory at least, these could be systematically and completely enumerated.

The standard-model group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ has rank 4. It is, therefore, conceivable to have a rank-16 shadow matter group in the theory. If this were possible, the standard model would effectively be contained in a rank $22-16=6$ grand-unified group. Unfortunately, the following theorem shows that this is impossible, at least not when the theory is *grand unifiable* and contains the supercurrent of Sec. III. By a grand-unifiable standard model, we mean one in which all the observed fermions (quarks and leptons) are constructed out of glues $q=(q_L;q_R)$ with all $q_R=\eta$. We call that grand unifiable because if this were not the case, it is hard to imagine how the fermions can be unified into a larger gauge group.

Theorem 5.11. A grand-unifiable standard model, with an arbitrary number of generations, with or without the usual choice of Higgs boson, will be nonchiral if it contains 16 dimensions of shadow matter and the supercurrent of Sec. III. A corollary to this result is that no chiral GUT string, containing the standard model in a natural way, may have 16 dimensions of shadow matter.

VI. CONCLUSION

String theory is the unique candidate for a grand unified theory with finite quantum gravity incorporated. It also distinguishes itself from the usual grand unified theory based on four-dimensional quantum field theory in that the gauge group and the spectrum of particles are no longer arbitrary in a string theory. In this and a previous paper,[5] we have undertaken a systematic study of the properties of heterotic strings of the lattice type. We have found out, in Secs. III–V, a number of restrictions resulting from the supercurrent we impose on the theory in Sec. III. We have also found that chirality places a very strong restriction on the theory. Essentially, a good string theory containing the standard model is very difficult to construct using the lattice string approach. Either the gauge group is far too large, or the theory is nonchiral. See the theorems in Sec. V. This had been found to be the case by practical constructions,[2,3] but to our knowledge, systematic studies of the type carried out in this paper have not been attempted before.

The conclusions of this paper are reached for lattice strings with the supercurrent chosen in Sec. III. Some of these conclusions (e.g., Theorems 5.1–5.3) hold in much greater generality, whereas others (e.g., Theorem 5.4) do not. The theorems for the shadow matters (Theorems 5.9 and 5.10) are probably the most far-reaching physical results of this paper, though this paper also contains a number of technical and mathematical developments in the theory of lattices (e.g., Theorems 2.4, 2.5, and 5.5–5.7). The results for the shadow matters, as stated, are valid only for lattice strings because the gauge group has to have rank 22. We do not even know whether the results are valid for other supercurrents. In this sense, this paper does not solve the all-important question of whether a string containing the standard model can be constructed. It only gives some answers to the specific cases of lattice strings with the specified supercurrent. Nevertheless, it provides a systematic analysis, to the extent that theorems can be rigorously proven, at least for this limited class of strings. Much more work is necessary to analyze the cases of other supercurrents, and to determine whether the present methods are helpful in constructing more general conformal field theories along the lines of Ref. 15. Work is underway in those directions.

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APPENDIX: PROOFS OF THE PREVIOUS RESULTS

In the following pages we will prove the remarks mentioned in the main body of this paper.

Let Λ be a self-dual indefinite lattice of signature $(n_L;n_R)$. Then any $x\in\Lambda$ may be written as $x=(x_L;x_R)$, where $x\cdot y=x_L\cdot y_L-x_R\cdot y_R$. Define the projection lattice $\Lambda'_L\equiv\{x_L \mid \exists x_R \text{ such that } (x_L;x_R)\in\Lambda\}$; define Λ'_R similarly.

As mentioned earlier, for both mathematical and physical reasons we assume throughout this paper that Λ contains a sublattice $(\Lambda_{0L};\Lambda_{0R})$, where Λ_{0L} and Λ_{0R} are, respectively, Euclidean lattices of dimensions n_L and n_R. This occurs precisely when Λ'_L and Λ'_R are rational lattices, i.e., when all their dot products are rational. In this case there exist *maximal* lattices Λ_L and Λ_R, such that: $(\Lambda_{0L};\Lambda_{0R})$ is a sublattice of Λ iff Λ_{0L} and Λ_{0R} are sublattices of Λ_L and Λ_R, of dimensions n_L and n_R, respectively. The following result follows quickly from the self-duality of Λ.

Lemma A. $\Lambda_L=(\Lambda'_L)^*$ and $\Lambda_R=(\Lambda'_R)^*$.

Theorem 2.4. $|\Lambda_L|=|\Lambda_R|$.

Proof. Let $\Lambda=(\Lambda_L;\Lambda_R)[G]$. Let g_i be the independent generators of G, of order n_i. Then by Theorem 2.2, $|\Lambda_L||\Lambda_R|=\prod n_i^2$. Since Λ_R is maximal, $(g_i)_L$ is also of order n_i (with respect to Λ_L). Thus,

$$\frac{1}{|\Lambda_L|}=|\Lambda'_L|=|\Lambda_L[(g_i)_L]|=\frac{|\Lambda_L|}{\prod n_i^2}=\frac{1}{|\Lambda_R|}\ ,$$

by the lemma. Q.E.D.

Theorem 2.4 will be used in the proof of Theorem 5.5 given below.

The following theorem has numerous applications, only some of which have been touched in this paper. In Ref. 8 it will be used to analyze the following (related) questions.

(i) When is it possible to shift between two given lattices? (This was only partially answered by Theorem 4.4 in Ref. 5.)

(ii) Given a non-self-dual base lattice, is it possible to glue it into a self-dual lattice? (That its determinant be square is a necessary but not sufficient condition.)

The value of this theorem lies in the ease with which it allows us to prove results for arbitrary integral lattices.

Theorem 2.5. Let Λ be any Euclidean integral lattice of dimension n. Then there exist positive integers $m_1,\ldots,m_n$ such that Λ can be obtained by gluing $\mathbf{Z}^{(m_1)}\oplus\cdots\oplus\mathbf{Z}^{(m_n)}$.

Proof. It is sufficient by Theorem 2.1 to prove that Λ contains a sublattice of the given form, or, in other words, to show that Λ contains n mutually orthogonal vectors.

Let β_1 be any nonzero vector in Λ, and define $m_1=\beta_1^2$. If $n=1$, we are done.

Now proceed inductively. Suppose there are $k<n$ linearly independent orthogonal vectors $\beta_1,\ldots,\beta_k$ in Λ, and define $m_i\equiv\beta_i^2$, for each i. Then some $y_k\in\Lambda$ can be found which is linearly independent of the β_i's. Now define

$$\beta_{k+1}\equiv[(m_2\cdot m_3\cdots m_k)(\beta_1\cdot y_k)]\beta_1+\cdots+[(m_1\cdot m_2\cdots m_{k-1})(\beta_k\cdot y_k)]\beta_k-(m_1\cdot m_2\cdots m_k)y_k\ ,$$

and $m_{k+1}\equiv\beta_{k+1}^2$. Then $\beta_i\cdot\beta_{k+1}=0$ for all $i\leq k$. Q.E.D.

The various parts of the theorem, as listed in Sec. II, are trivial to verify. For example, part (i) follows from Theorem 2.2, while parts (ii) and (iii) follow immediately from the fact that $\mathbf{Z}^{(m_1)}\oplus\cdots\oplus\mathbf{Z}^{(m_n)}$ is a sublattice of Λ. Of course, these m_i are by no means uniquely determined.

The proof of this theorem for indefinite Λ reduces to the definite case, since any "orthogonal decomposition" of $(\Lambda_L;\Lambda_R)$ is necessarily one for Λ. Note, though, that it holds in the indefinite case only because of our assumption that Λ_L and Λ_R exist.

For notational convenience we write $((m_1),\ldots,(m_k))$ for $Z^{(m_1)}\oplus\cdots\oplus Z^{(m_k)}$. Also, the abbreviation $(m)^k$ stands for $((m),\ldots,(m))$ (k times). For example, an orthogonal decomposition for D_n is (4).^{n}

Now consider the supercurrent given in Sec. III. Recall that it involved nine mutually orthogonal vectors $(p_3)_i$, each of norm 3, and each orthogonal to v_0. Moreover, the corresponding shifted momenta $(q_3)_i\equiv(p_3)_i-v_0$ are all in the lattice Λ_R.

Note that $2v_0$ is also in Λ_R—this follows from Eq. (2.1) and the self-duality of Λ. In fact, $2v_0$ generates the transverse spacetime group $\mathrm{SO}(2)=Z^{(4)}$. Hence each $2(p_3)_i$ is in Λ_R. The set of ten vectors $\{2(p_3)_1,\ldots,2(p_3)_9,2v_0\}$ are mutually orthogonal and hence span an *orthogonal base lattice* for Λ_R. Denote this sublattice by Λ_{1R}; it is just the lattice $((12)^9,(4))$ given in Theorem 3.1 (i).

We may take the lattice vectors $(q_3)_i$ to be glues. They must belong to the glue classes $[0,\ldots,0,\pm6,0,\ldots,0,-2]=[0,\ldots,0,6,0,\ldots,0,2]$. This completes the proof of Theorem 3.1 (i).

Let $q=(q_L;q_R)$ represent a massless fermion, and define $f_i\in\mathbf{Z}$, $(1\leq i\leq 10)$ by

$$q_R=\sum_{i=1}^{9}f_i\frac{2(p_3)_i}{12}+f_{10}\frac{2v_0}{4}\equiv(f_1,\ldots,f_{10})\ .$$

Because q is a fermion, f_{10} must be odd; because f_{10} is odd and $q\cdot(q_3)_i$ must be an integer for each i, we have that each f_i must be odd. Because q is massless, $(q_R+v_0)^2$ must equal 1. Thus $f_{10}=-1$ or -3 is forced. In either case, $\sum_{i=1}^{9}f_i^2$ must equal 9, implying $f_i=\pm1$, $(1\leq i\leq 9)$. This gives us Theorem 3.1 (ii).

Before completing the proof of Theorem 3.1, we will first address the question of chirality. Note that if our theory contains any massless fermions, we may take one to be $q=(q_L;\eta)$. Remember that our definition of nonchirality is a little weaker than the usual one (see Sec. IV).

Lemma B. Let $q=(q_L;\eta)$ be any massless fermion. Then the theory is nonchiral (in the sense of Sec. IV)$\Longrightarrow$either (2^{10}) or $(2^3,0^6,2)\in\Lambda_R$, up to a permutation of $(2^3,0^6)$. Also, $(2^3,0^6,2)\in\Lambda_R\Longrightarrow$nonchirality of the theory. If a gaugino is present, the theory becomes nonchiral iff $(2^3,0^6,2)\in\Lambda_R$, up to a permutation.

Proof. Note first that the theory is nonchiral$\Longrightarrow q$ has a massless chiral partner q'. Now consider $q''\equiv q-q'$; it is not difficult to verify q'' is of the form of either $(0^{22};2^{10})$ or $(0^{22};2^3,0^6,2)$.

Now suppose that a vector q'' of the form $(0^{22};2^3,0^6,2)$ is in Λ, and let q' be any left-handed massless fermion. Then $q''\cdot q'\in\mathbf{Z}$ implies $q'-q''$ is massless. Therefore the theory is nonchiral.

If a gaugino is present, by the following analysis it must be of the form $(0^{22};\delta)$. Subtracting twice the gaugino from $(0^{22};2^{10})$ and subtracting three of the glues given in (i) establishes the final statement of the lemma. Q.E.D.

Assume there is a fermion q in the theory with $q_R=\eta$, and assume in addition that $(0^{22};\eta)$ is not in the theory (otherwise η rather than δ would be the gaugino). Suppose $q'=(0^{22};q_R')$ is a massless fermion in the theory, where $q_R'=(f_1,\ldots,f_{10})$, and without loss of generality take $f_{10}=-1$. Then $\eta\cdot q_R'\in\mathbf{Z}$, so by Theorem 3.1 (ii) we know that either all ten of the f_i's are -1, or four are -1 and six are $+1$. By assumption we can discard the first possibility.

Assume there is a fermion q in the theory with $q_R=\eta$, and assume in addition that $(0^{22};\eta)$ is not in the theory (otherwise η rather than δ would be the gaugino). Suppose $q'=(0^{22};q_R')$ is a massless fermion in the theory, where $q_R'=(f_1,\ldots,f_{10})$, and without loss of generality take $f_{10}=-1$. Then $\eta\cdot q_R'\in\mathbf{Z}$, so by Theorem 3.1 (ii) we know that either all ten of the f_i's are -1, or four are -1 and six are $+1$. By assumption we can discard the first possibility.

Assume there is an $N>1$ spacetime supersymmetry in the theory. Then there are N such gauginos. Let q' and q'' be any two independent ones. Then $q'\cdot q''=0$, so $q'+q''=(0^{22};2^3,0^6,-2)$, up to a permutation of the right-hand coordinates. Hence, by the above lemma, chirality would be violated.

This concludes the proof of Theorem 3.1. Incidently, the theory would also be nonchiral if both η and δ were gauginos, even if all other massless fermions were also in the adjoint representation of the gauge group.

Theorem 3.4 follows from the analysis in a future publication.[8] Lemma B and self-duality immediately give us Theorem 4.1.

Let $q=(q_L;q_R)$ be any massless fermion. Let n_L be the smallest positive integer such that $n_Lq_L\in\Lambda_L$ and suppose that 3 does not divide it. Then $(0^{22};4n_Lq_R)\in\Lambda$. Let $q_R=(f_1,\ldots,f_{10})$ and $[4n_Lq_R]=[(f_1',\ldots,f_{10}')]$ as usual. Then we may take $f_{10}'=0$ and [by Theorem 3.1 (ii)] $f_i'=\pm4(1\leq i\leq 9)$. Adding and subtracting the glues of Theorem 3.1 (i) gives us $q''=\pm(0^{22};2q_R)\in\Lambda$, with the sign depending on the value of n_L (mod 3). The massless chiral partner of q then is $q\mp q''$.

Note that the previous paragraph implies $2q_L\in\Lambda_L$, so $n_L=1$ or 2. If the theory is supersymmetric (i.e., a gaugino is present), then $|\Lambda_L|=|\Lambda_R|$ divides 3^7, and so is odd. Thus, no order can be even, so n_L must be 1. In other words, $(0^{22};q_R)$ is the gaugino.

This gives us Theorem 4.2. It is more convenient in practice to define n_L (as was actually done in the text of this paper) to be the smallest positive integer such that $n_L q_L \in \Lambda_{0L}$. This n_L will necessarily be a multiple of the n_L used in the above proof, so if Theorem 4.2 holds for the latter $n_L <$ as was shown above, then it must hold for the former.

Suppose $q=(q_L;q_R)$ is a tachyon. Then $p_R^2 =(q_R+v_0)^2$ must be less than 1. Let $q_R=(f_1,\ldots,f_{10})$ as usual. Then $f_{10}=-1$, -2, or -3.

If $f_{10}=-1$ or -3, because of the supercurrent glues in Theorem 3.1 (i) all f_i, $(1\le i\le 9)$, must be odd. Then $p_R^2\ge\sum_{i=1}^{9}(\pm1)^2/12+(\pm1)^2/4=1$.

Therefore f_{10} must equal -2. The supercurrent glues now imply that all f_i are even. Thus 0, 1, or 2 of them are ±2's, and the rest are 0's. In any of these three cases, dotting q with the gaugino yields the rational number $n/12+\frac{1}{2}=(n+6)/12$, where $n=0,\pm2,\pm4$. Thus the dot product can never be integral, so the tachyon cannot be present. This is Theorem 5.1.

Let $q=(q_L;q_R)$ be a massless scalar boson. Then $q_R=(f_i)=(\pm2,\pm2,\pm2,0^6,-2)$, up to a permutation of the first nine components. Note that $q\cdot\delta\in\mathbf{Z}$ implies that either $f_i\delta_i\ge0$ for all $i\le9$, or $f_i\delta_i\le0$ for all $i\le9$. In the former case the desired superpartner of q is $q-\delta$, and in the latter it is $q+\delta$. The case of a massless vector boson q' is even easier: its superpartners are clearly $q'\pm\delta$, since $q_R'=(0^{10})$. This gives us Theorem 5.2.

Theorem 5.3 follows from the fact that to be massless a particle q must satisfy $p_R^2=1$.

If $q=(q_L;f_i)$ were to represent a particle in the adjoint representation, then we may take $q_L=(0^{22})$. If it was a scalar, then all f_i would be even (by dotting with the supercurrent glues); to be massless f_{10} would have to be -2, three of the f_i $(1\le i\le9)$ would be ±2, and the remaining six would be 0. The dot product of q with the fermion $q'=(q_L';\eta)$ would have to be an integer, requiring that the three ±2's be of the same sign. Lemma B now tells us the resulting theory must be nonchiral. Hence chirality forbids an adjoint Higgs boson, and thus most GUT's. (Note that supersymmetry was not used in the proof.) This is Theorem 5.4.

Now we will turn to the proof of Theorem 5.5. Although it follows immediately from the analysis of Ref. 8, it is possible to prove it using more elementary means.

Let Λ_1 and Λ_2 be any two integral lattices, and define $\Lambda_1\sim\Lambda_2$ if there exist a third lattice Λ' and glue groups G_1 and G_2 such that $\Lambda'[G_1]\approx\Lambda_1$, and $\Lambda'[G_2]\approx\Lambda_2$ ("$\approx$," or "lattice equivalence," was defined at the beginning of this paper). For example, $E_8\sim Z_8$ (take $\Lambda'=D_8$)—in fact, as will be proved in Ref. 8, Λ is an n-dimensional Euclidean self-dualizable lattice iff $\Lambda\sim Z_n$. Also, $\Lambda_1\sim\Lambda_2$ iff Λ_1 can be rotated so that $\Lambda_1\cap\Lambda_2$ is of dimension equal to that of Λ_1 and Λ_2.

It is easy to prove "$\sim$" defines an equivalence relation. Also, $\Lambda_1\sim\Lambda_1'$ and $\Lambda_2\sim\Lambda_2'$ implies $\Lambda_1\oplus\Lambda_2\sim\Lambda_1'\oplus\Lambda_2'$. Any lattice is $\sim$ to its orthogonal decomposition $((m_1),\ldots,(m_n))$. Note that $((l_1^2m_1),\ldots,(l_n^2m_n))\sim((m_1),\ldots,(m_n))$, for any $l_i\in\mathbf{Z}$.

Theorem 2.3 translates into the statement:

Theorem C. Given any integral lattices Λ_1 and Λ_2, if $\Lambda_1\sim\Lambda_2$ then Λ_1 is self-dualizable iff Λ_2 is.

"Λ_1 is self-dualizable," of course, means that it can be glued to a self-dual lattice. As was mentioned in Ref. 5, this result was obtained by applying the shifting operation.

Since $\Lambda\sim(\Lambda_L;\Lambda_R)$, and Λ for a string is self-dual, the above theorem tells us that $(\Lambda_L;\Lambda_R)$, or $(\Lambda_{0L};\Lambda_{0R})$, is self-dualizable.

Lemma D. $\Lambda_R\sim((3),(1)^9)$.

Proof. $\Lambda_R\sim((12)^9,(1))\sim((3),(3)^4,(3)^4,(1))$. Now, it is straightforward to verify (and in any case will be explicitly shown in Ref. 8) that $(3)^4\sim(1)^4$ [in fact, $(l)^4\sim(1)^4$ for any l]. Q.E.D.

Incidentally, this implies Theorem 3.3, using results proved in Ref. 8.

Lemma E. $A_2\oplus\Lambda_L$ is self-dualizable.

Proof. Define $\Lambda_R''\equiv((3),(1)^9)$. By the above lemma, $(\Lambda_L;\Lambda_R'')$ is self-dualizable. Let g_i be the relevant independent glues. Then we may take $(g_i)_R=0$ for all but one g_i—call it g_0. Let $\Lambda_L''\equiv\Lambda_L[g_i,\ i\neq0]$. Then $(\Lambda_L'';\Lambda_R'')[g_0]$ is self-dual.

Note that $(g_0)_R\in(\Lambda_R'')^*$. Therefore $(g_0)_L\in(\Lambda_L'')^*$. Also, $(g_0)_R^2\equiv\frac{1}{3}$ (mod 1), so $(g_0)_L^2\equiv\frac{1}{3}$ (mod 1).

Now consider $A_2\oplus\Lambda_L''$, and glue to it $g=[1,(g_0)_L]$. This glue has the same order as g_0, and the determinants $|A_2|$ and $|\Lambda_R''|$ are equal. Since the norm of g is integral, and $g\in A_2^*\oplus(\Lambda_L'')^*$, $(A_2\oplus\Lambda_L'')[g]$ is self-dual.

Since $\Lambda_L\sim\Lambda_L''$, the theorem follows. Q.E.D.

Note that up to here supersymmetry has not been used. It comes in the next paragraph.

From the spin-statistics connection Λ_L is even. Choose the g_i such that $A_2\oplus\Lambda_L[g_i]$ is self-dual. Replace each g_i with $2g_i$. Their orders, being odd (since $|A_2|\cdot|\Lambda_L|$ is odd, by Theorem 3.2), are unchanged. Therefore, $A_2\oplus\Lambda_L[2g_i]$ is also self-dual. Since both the base lattice and the glues have even norms, it is a 24-dimensional type-II lattice. It has roots, and so must be one of the 23 Niemeier lattices.

This gives us Theorem 5.5 [that the glues have order 3 follows from the fact that any glue g of $(\Lambda_L;\Lambda_R)$ must satisfy $3g_R\in\Lambda_R$].

Theorem 5.6 follows from Lemma B and the fact that if $|\Lambda_L|=3^7$, then $\Lambda_R=\Lambda_{1R}[(q_3)_1,\ldots,(q_3)_9,\delta]$—no combination of base lattice vectors and glue vectors can give us something of the form $(2^3,0^6,2)$.

Theorem 5.7 is more difficult. It is most quickly established by consulting a table, recently compiled by Conway and Sloane,[14] of all the integral lattices of small dimension and determinant. $|\Lambda_L|=3$ implies Λ_R is an integral lattice of dimension 10 and determinant 3. According to the aforementioned table, the only possibilities for Λ_R are (i) $D_9(12)[13]$, (ii) $E_7Z_2(6)[103]$, (iii) E_6Z_4, (iv) A_2Z_8, (v) A_2E_8, (vi) $Z_9(3)$, and (vii) $E_8Z(3)$.

From the previous analysis, we know $A_2\Lambda_R$ is self-dualizable. This eliminates (ii), (iv), and (v).

The spin-statistics connection tells us that there is a vector $2v_0\in\Lambda_R$ with norm 4, such that the dot product of $2v_0$ with any vector of even and/or odd norm is an even and/or odd integer. This constraint immediately eliminates (i) and (vi). Also, chirality demands that there be at most one gaugino [Theorem 3.1 (iii)], forbidding case (iii). Thus the only remaining possibility is (vii), where $2v_0$ is the sum of the norm-1 and norm-3 vectors. However, the supercurrent vectors $(q_3)_i$ have a dot product of 2 with $2v_0$ but a norm of only 4—a quick check confirms that not even one such vector can be found, given this choice for Λ_R. Hence we get Theorem 5.7. Supersymmetry is not needed in the proof.

Theorem 5.8 follows from the following two theorems and a result that will be proven in a later paper.[8]

Theorems 5.9 and 5.10 also use the Conway-Sloane table.[14] When eight dimensions of shadow matter are present, we are interested in a 14-dimensional lattice Λ' with determinant equal to $|A_2|=3$. Spin-statistics demands that Λ_L be an even lattice, and since the order of Λ'/Λ_L is odd, Λ' must also be even. Hence the only candidates are $E_6 \oplus E_8$ and $D_{13}(12)[13]$. $A_2 \oplus \Lambda'$, for both these choices of Λ', is self-dualizable, as it should be.

Similar reasoning produces Theorem 5.10.

Finally, we will address Theorem 5.11: a "grand-unifiable" standard model with 16 dimensions of shadow matter cannot be chiral, at least not with the supercurrent studied in this paper.

Choose as the right-hand base lattice Λ_{1R} as usual; for the left-hand base lattice choose $\Lambda_{sm} \equiv A_2 \oplus A_1 \oplus \Lambda_{16} \oplus (ab^2) \oplus (c) \oplus (d)$, where A_2 corresponds to $SU_c(3)$, A_1 to $SU_L(2)$, $\Lambda_{16}=E_8 \oplus E_8$ or D_{16}^+ is for the shadow matter, and (ab^2) corresponds to $U_y(1)$. (c) and (d) are hidden. a,b,c,d are as yet undetermined positive integers.

There are (at least) five massless glues, whose left-hand sides are given by

Particle	A_2	A_1	(ab^2)	(c)	(d)	Λ_{16}
left-handed quark q_L	[1]	[1]	b	c_1	d_1	0
left-handed electron e_L	[0]	[1]	$-3b$	c_2	d_2	0
right-handed electron e_R	[0]	[0]	$-6b$	c_3	d_3	0
right-handed up quark u_R	[1]	[0]	$4b$	c_4	d_4	0
right-handed down quark d_R	[1]	[0]	$-2b$	c_5	d_5	0

A priori, the only constraint on the right-hand side of these glues is Theorem 3.1 (ii). We may assume without loss of generality that the right-hand side of the *momentum* vector p for the left-handed quark q_L, is $((-1)^{10})$. By "grand-unifiable" we simply mean that the momentum for the right-hand side of e_L also be $((-1)^{10})$, and that for e_R, u_R, and d_R all be $-\eta$. Then the left-hand sides of the vectors $q=p-v_0$ for $q_L, \ldots, d_R$ must all have norm 2 (to be massless) and have integer dot products with each other (to be lattice vectors).

The proof of Theorem 5.11 is computational, and consists of several cases. An outline of the argument used is given below.

By an argument similar to that used in Theorem 3.1 (i), we may assume $d_1=0$ and $[1]^2+[1]^2+b^2/(ab^2)+c_1^2/c=2$. Hence we may take $c=6a(5a-6)r^2$ and $c_1=(5a-6)r$, where r is some integer. $c_2, \ldots, c_5$ can be determined by taking dot products of the respective vectors $(e_L)_L, \ldots, (d_R)_L$ with $(q_L)_L$ and noting that the result must be an integer between -3 and 3. d and d_2 can be determined by using $(e_L)_L^2=2$. d_3, d_4, d_5 can be obtained by dotting $(e_R)_L, (u_R)_L, (d_R)_L$ with $(e_L)_L$. The vast majority of the cases are eliminated by demanding $(e_R)_L^2=(u_R)_L^2=(d_R)_L^2=2$. This also restricts the value of a.

Further cases may be dismissed by requiring (see Lemma E) that $A_2 \oplus \Lambda_{sm}$ be self-dual.

Exactly four cases, each with $a=30$, survive all previous constraints. For example, one of them is

Particle	A_2	A_1	$(30b^2)$	$(20r^2)$	$(12s^2)$
q_L	[1]	[1]	b	$4r$	0
e_L	[0]	[1]	$-3b$	$3r$	$3s$
e_R	[0]	[0]	$-6b$	$-4r$	0
u_R	[1]	[0]	$4b$	$-4r$	0
d_R	[1]	[0]	$-2b$	$-3r$	$-3s$

These remaining possibilities can be eliminated by showing $|\Lambda_L|=3$, impossible for a physically acceptable theory (see Theorem 5.7).

It is much easier to directly handle GUT's [e.g., SO(10)] in the above manner, for there are far fewer cases in general.

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