Geometrical quantization of bosonic string with Wess-Zumino term on genus-g Riemann surface

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(Received 23 June 1989)

By combining the method of geometrical quantization and Krichever and Novikov algebra, we study the holomorphic structure of a bosonic string with Wess-Zumino term on a genus-g Riemann surface Σ and arrive at the conclusion that the curvature on a Kähler manifold is proportional to the central extension of Krichever and Novikov algebra on Σ and vanishes at the critical dimension $d=26-(kd_G)/(C_V+k)$.

I. INTRODUCTION

Recently Bowick and Rajeev discovered that anomalies can be formulated as the curvature of some holomorphic bundles over gauge-group manifolds or proper coset spaces related to them.[1] Then Pilch and Warner derived the correct anomalies by only considering vacuum bundles on a Kähler manifold.[2] On the other hand, Kricherver and Novikov introduced a new general formalism[3] which proves to be a very important tool in the study of conformal field theories on Riemann surfaces of arbitrary genus. They consider the algebra (to be referred as KN algebra) of meromorphic vector fields which are holomorphic outside two distinguished points and introduce an explicit countable basis. By using this method, some authors have studied the bosonic string and superstring on a genus-g Riemann surface[4-7] and proved that the critical dimensions for any genus remain 26 and 10 for the bosonic string and superstring, respectively.

In this paper we wish to combine the above two research directions into a new approach which will serve to study the string field theory on a genus-g Riemann surface by an operator formalism. It might be quite interesting and useful from the geometric point of view. Here we think of the motion space of the bosonic string being $M^d\times G$, where M^d is a d-dimensional Minkowski space and G a d_G-dimensional group manifold. By using the method of geometrical quantization to study the Sugawara construction on a Riemann surface Σ of arbitrary genus, we shall prove that the curvature of a Kähler manifold on genus-g Riemann surface Σ is proportional to the central extension of the KN algebra on Σ.

The organization of this paper is as follows. Section II is a brief review of the KN algebra and the scheme of geometrical quantization. However, at the same time we introduce the bosonic string with the Wess-Zumino (WZ) term, which will be discussed later. As a warm up, we consider the holomorphic structure of pure bosonic string on a Riemann surface in Sec. III. But the main ingredient in this section is the method of calculating the curvature of a vacuum bundle in the high-genus case. Then in Sec. IV we are able to perform the geometrical quantization procedure for a bosonic string model with the WZ term, also on a genus-g Riemann surface. Section V contains a short conclusion.

II. REVIEW OF KN ALGEBRA AND GEOMETRICAL QUANTIZATION

We begin with a brief review of KN algebra. Consider a genus-g Riemann surface Σ with two distinguished points P_+,P_- and local coordinates Z_+,Z_- around them, such that $Z_\pm(P_\pm)=0$. On Σ one can introduce bases of meromorphic vector fields e_n, functions A_n, one-differentials ω_n, and half-differentials h_α, which are all holomorphic except at $P_\pm$. Here n is integer or half-integer according to whether g is even or odd, while α is integer (Ramond sector) or half-integer (Neveu-Schwarz sector). The behavior near $P_\pm$ is given by

$$A_n(Z_\pm)=a_n^\pm Z_\pm^{\pm n-g/2}[1+O(Z_\pm)]\ , \tag{1a}$$

$$\omega_n(Z_\pm)=b_n^\pm Z_\pm^{\mp n+g/2-1}[1+O(Z_\pm)](dZ_\pm)\ , \tag{1b}$$

$$e_n(Z_\pm)=c_n^\pm Z_\pm^{\pm n-g_0+1}[1+O(Z_\pm)]\frac{\partial}{\partial Z_\pm},\quad g_0=3g/2\ , \tag{1c}$$

$$h_\alpha(Z_\pm)=d_\alpha^\pm Z_\pm^{\mp\alpha-1/2}[1+O(Z_\pm)](dZ_\pm)^{1/2}\ . \tag{1d}$$

For $|n|\le g/2$ the definitions (1a) and (1b) must be modified, because of the Weierstrass theorem. In formula (1d) a branch cut is understood when α is a half-integer. The above basis elements are determined up to an arbitrary constant due to the Riemann-Roch theorem. So we may choose $a_n^+=b_n^+=c_n^+=d_\alpha^+=1$, the other coefficients will then be uniquely determined. One needs duality relations

$$\frac{1}{2\pi i}\oint_{C_\tau} A_m(Q)\omega_n(Q)=\delta_{mn}\ ,$$
$$\frac{1}{2\pi i}\oint_{C_\tau} h_\alpha(Q)h_\beta^+(Q)=\delta_{\alpha\beta}\ , \tag{2}$$

where $h_\alpha^+(Q)=h_{-\alpha}(Q)$ and C_τ denotes a level curve of the univalent function $\tau(Q)$ (Ref. 2). The Lie brackets of the basis element e_n are

$$[e_n,e_m]=\sum_{s=-g_0}^{g_0} c_{nm}^s e_{n+m-s}\ . \tag{3}$$

The structure constants c_{nm}^s can be calculated from the constants appearing in the expansion of e_n.

Next, we consider the energy-momentum tensor of bosonic string with Wess-Zumino term[7]

$$T=-\partial X^\mu \partial X^\mu - \tfrac{1}{2}J^a J^a\ , \tag{4}$$

where $X^\mu(Q)$ is the bosonic field in space M^d and $J^a(Q)$ the conservation current in group manifold G. In order to perform the geometrical quantization later, we manage to construct $J^a(Q)$ as some fermionic currents[6] and then will be able to realize the Kac-Moody algebra in terms of them. For this purpose, let ψ^i be a multiplet of spin-$\frac{1}{2}$ fields homomorphic outside $P_\pm$, transforming according to a real representation of G and let T^a denote the antisymmetric matrices representing the generators of G. Then one has at the quantum level

$$J^a(Q)=\tfrac{1}{2}{:}\psi(Q)T^a\psi(Q){:}=\tfrac{1}{2}{:}\psi^i(Q)(T^a)_{ij}\psi^j(Q){:}\ , \tag{5a}$$

with T^a satisfying

$$T^aT^b=\tfrac{1}{2}f^{abc}T^c\ , \tag{5b}$$

$$\mathrm{Tr}(T^aT^b)=-K\delta^{ab}\ , \tag{5c}$$

$$C_V\delta^{cd}=f^{abc}f^{abd}\ , \tag{5d}$$

where f^{abc} is the structure constant of group G.

Now the objects appearing in (4) can be expanded in terms of the basis elements introduced in (1):

$$X^\mu(Q)=\sum_n X_n^\mu A_n(Q),\quad X_n^\mu=\frac{1}{2\pi i}\oint_{C_\tau} X^\mu(Q)\omega_n(Q)\ , \tag{6a}$$

$$\psi^i(Q)=\sum_\alpha \psi_\alpha^i h_\alpha(Q),\quad \psi_\alpha^i=\frac{1}{2\pi i}\oint_{C_\tau}\psi^i(Q)h_\alpha^\dagger(Q)\ , \tag{6b}$$

$$P^\mu(Q)=\sum_n P_n^\mu\omega_n(Q),\quad P_n^\mu=\frac{1}{2\pi i}\oint_{C_\tau}P^\mu(Q)A_n(Q)\ , \tag{6c}$$

$$J^a(Q)=\sum_n J_n^a\omega_n(Q),\quad J_n^a=\frac{1}{2\pi i}\oint_{C_\tau}J^a(Q)A_n(Q)\ , \tag{6d}$$

$$dX^\mu(Q)+P^\mu(Q)=\sqrt{2}\sum_n\alpha_n^\mu\omega_n(Q)\ ,$$
$$\sqrt{2}\alpha_n^\mu=\frac{1}{2\pi i}\oint_{C_\tau}[dX^\mu(Q)+P^\mu(Q)]A_n(Q)\ , \tag{6e}$$

where $P^\mu(Q)$ is the conjugate momentum of $X^\mu(Q)$. Let us introduce the Poisson brackets

$$[X^\mu(Q),P^\nu(Q')]=i^{-1}\delta^{\mu\nu}\Delta_\tau(Q,Q')\ , \tag{7a}$$

$$[\psi^i(Q),\psi^j(Q')]_+=i^{-1}\delta^{ij}\delta_\tau(Q,Q')\ , \tag{7b}$$

$$[J^a(Q),J^b(Q')]=i^{-1}f^{1bc}J^c(Q)\Delta_\tau(Q,Q') - \frac{i^{-1}}{2}K\,d\Delta_\tau(Q,Q')\delta^{ab}\ , \tag{7c}$$

where

$$\Delta_\tau(Q,Q')=\sum_n A_n(Q)\omega_n(Q')\ , \tag{7d}$$

$$\delta_\tau(Q,Q')=\sum_\alpha h_\alpha(Q)h_\alpha^\dagger(Q'),\quad Q,Q'\in C_\tau\ , \tag{7e}$$

play the role of the δ function over C_τ for smooth tensors of weight 0 and $\frac{1}{2}$, respectively. Then Eqs. (6) can be recast in the form of their mode-expansion coefficients:

$$[X_m^\mu,P_n^\nu]=i^{-1}\delta^{\mu\nu}\delta_{mn}\ , \tag{8a}$$

$$[\alpha_m^\mu,\alpha_n^\nu]=i^{-1}\delta^{\mu\nu}\gamma_{mn}\ , \tag{8b}$$

$$[\psi_\alpha^i,\psi_\beta^j]_+=i^{-1}\delta^{ij}\delta_{\alpha+\beta,0}\ , \tag{8c}$$

$$[J_m^a,J_n^b]=i^{-1}f^{abc}\alpha_{mn}^s J_s^c-\frac{i^{-1}}{2}K\delta^{ab}\gamma_{mn}\ , \tag{8d}$$

where

$$\alpha_{mn}^s=\frac{1}{2\pi i}\oint_{C_\tau}\omega_s(Q)A_m(Q)A_n(Q)\ ,$$
$$\gamma_{mn}=\frac{1}{2\pi i}\oint_{C_\tau}A_m(Q)dA_n(Q)\ . \tag{8e}$$

Therefore, one obtains

$$L_n=\frac{1}{2\pi i}\oint_{C_\tau}e_n(Q)T(Q) = -\tfrac{1}{2}\sum_{j,k}l_{jk}^n\alpha_j^\mu\alpha_k^\mu-\tfrac{1}{2}\sum_{j,k}l_{jk}^nJ_j^aJ_k^a\ , \tag{9a}$$

with

$$l_{jk}^n=\frac{1}{2\pi i}\oint_{C_\tau}e_n(Q)\omega_j(Q)\omega_k(Q)\ . \tag{9b}$$

Hence the Poisson bracket for L_n reads

$$[L_m,L_n]=i^{-1}\sum_{s=-g_0}^{g_0}c_{mn}^sL_{m+n-s}\ . \tag{10a}$$

This relation reduces to the usual one in the genus-zero case. Then at the quantum level, because of the commutation relation of current operator J^a, this requires that L_n must be modified as

$$\hat{L}_n=-\tfrac{1}{2}\sum_{j,k}l_{jk}^n{:}\alpha_j^\mu\alpha_k^\mu{:}-\frac{1}{2(C_V+k)}\sum_{j,k}l_{jk}^n{:}J_j^aJ_k^a{:}$$
$$=-\tfrac{1}{2}\sum_{j,k}l_{jk}^n{:}\alpha_j^\mu\alpha_k^\mu{:}$$
$$-\frac{1}{8(C_V+k)}\sum l_{pq}^nE_{\alpha\beta}^pE_{\gamma\delta}^q(T^a)_{ij}(T^a)_{kl}{:}\psi_\alpha^i\psi_\beta^j\psi_\gamma^k\psi_\delta^l{:}\ , \tag{10b}$$

where

$$E^{n}_{\alpha\beta}=\frac{1}{2\pi i}\oint_{C_\tau} h_\alpha(Q)h_\beta(Q)A_n(Q) .$$

We now turn to discuss the geometrical quantization briefly. First one considers a complex Kähler manifold $\Gamma=\{Z_A,\bar{Z}_A\}$ and defines the Poisson brackets as

$$[Z_A,\bar{Z}_B]=i^{-1}\delta_{AB} . \tag{11}$$

The two-form in a Kähler manifold Γ should be

$$\omega=i\sum_A dZ_A\wedge d\bar{Z}_A=i\partial\bar{\partial}K , \tag{12}$$

where $K=\sum_A Z_A\bar{Z}_A$ is the Kähler potential. As a next step of prequantization, one constructs a Hilbert space H spanned by $\bar{\Psi}$ with an inner-product rule

$$\langle\Psi_1,\Psi_2\rangle=\int_\Gamma\bar{\Psi}_1\Psi_2\det\omega . \tag{13}$$

The third step of quantization amounts to a Kähler polarization procedure, i.e., to pose a constraint on Ψ:

$$\Psi=e^{-K/2}\Phi,\quad \partial\Phi=0 . \tag{14}$$

The condition (14) implies that Φ is the holomorphic function on Γ. These functions Ψ span the prequantum Hilbert space H_p and contain the quantum wave functions.

III. CALCULATING THE CURVATURE OF VACUUM BUNDLE ON GENUS-g RIEMANN SURFACE

We only consider the bosonic string and take $d_G=0$. Following Ref. 8, we assume that there is a symmetry generated by some canonical transformations which keep the two-form ω intact

$$\hat{Z}'_A=A_{AB}\hat{Z}_B+B_{AB}\hat{\bar{Z}}_B , \tag{15}$$

$$\hat{\bar{Z}}'_A=\bar{A}_{AB}\hat{\bar{Z}}_B+\bar{B}_{AB}\hat{Z}_B , \tag{16}$$

$$A_{AD}\bar{A}_{AC}-B_{AC}\bar{B}_{AD}=\delta_{DC} , \tag{17a}$$

$$A_{AC}B_{BC}-B_{AC}A_{BC}=0 . \tag{17b}$$

Suppose now these transformations form a homogeneous space M, then one can define a vacuum bundle on M: at a point $(A,B)\in M$, the fiber space is the subspace of H_n annihilated by $\hat{Z}'_A$. The condition $\hat{Z}'_A|\Omega'\rangle=0$ together with (15)–(17) lead to the expression of a new vacuum $|\Omega'\rangle$ in terms of the old one $|\Omega\rangle$:

$$|\Omega'\rangle=\exp(-\tfrac{1}{2}\hat{\bar{Z}}_AK_{AB}\hat{\bar{Z}}_B)|\Omega\rangle , \tag{18}$$

where

$$K_{AB}=(A^{-1})_{AC}B_{CB} \tag{19}$$

is a holomorphic coordinate of M, while $|\Omega'\rangle$ is a homomorphic basis of the vacuum bundle. The Hermitian metric is induced by the inner product of these vacuum states:

$$h\equiv\langle\Omega'|\Omega'\rangle=\det{}^{-1/2}(1-\bar{K}K) , \tag{20}$$

which can be proved by substituting (18) into (15) and performing the Gaussian integration. Then the curvature of vacuum bundle is calculated via the formula[9]

$$F=\bar{\partial}\partial\ln h=\tfrac{1}{2}\mathrm{tr}\bar{\partial}\partial(\bar{K}K) . \tag{21}$$

Here the derivative is taken with respect to the holomorphic coordinate at the origin of M, so the fact that K being small has been used.

In order to use the above formulas after geometrical quantization for calculating the curvature of vacuum bundle on Kähler manifold, we manage to write γ_{mn} defined in (8) into factorized form,

$$\gamma_{mn}=\xi_{m-j}\xi_{jn}-\xi_{n-j}\xi_{jm}\quad (j>0) , \tag{22}$$

and define the complex coordinates as

$$\begin{aligned}&Z^\mu_l=\xi^{-1}_{n-l}\alpha^\mu_n,\quad \bar{Z}^\mu_l=\xi^{-1}_{ln}\alpha^\mu_n ,\\ &\alpha^\mu_n=\xi_{n-l}Z^\mu_l+\xi_{ln}\bar{Z}^\mu_l ,\end{aligned} \tag{23}$$

where

$$\bar{\xi}_{nl}=\xi_{-l-n},\quad \xi^{-1}_{jn}\xi_{ln}=\delta_{jl},\quad \xi^{-1}_{jm}\xi_{m-l}=0\quad (j,l>0) . \tag{24}$$

In the case of $g=0$, $\xi_{ml}=\sqrt{m}\,\delta_{m+l,0}$. Notice that the above prescriptions enable us to get the following commutation relation for quantized operators $\hat{Z}^\mu_l$ and $\hat{\bar{Z}}^\mu_l$:

$$[\hat{Z}^\mu_l,\hat{\bar{Z}}^\nu_j]=\delta^{\mu\nu}\delta_{lj},\quad [\hat{Z}^\mu_l,\hat{Z}^\nu_j]=[\hat{\bar{Z}}^\mu_l,\hat{\bar{Z}}^\nu_j]=0 . \tag{25}$$

According to (12), we now have the two-form

$$\omega=i\sum dZ^\mu_n\wedge d\bar{Z}^\mu_n . \tag{26}$$

Under the infinitesimal transformation generated by $iV_n\hat{L}_n(\bar{V}_n=V_{-n})$,

$$\begin{aligned}\hat{Z}^{\mu\prime}_j&=\hat{Z}^\mu_j+[\hat{Z}^\mu_j,iV_n\hat{L}_n]\\ &=\hat{Z}^\mu_j+iV_m\varphi^m_{nk}\xi^{-1}_{n-j}\xi_{k-l}\hat{Z}^\mu_l\\ &\quad+iV_m\varphi^m_{nk}\xi^{-1}_{n-j}\xi_{lk}\hat{\bar{Z}}^\mu_l ,\end{aligned} \tag{27}$$

where

$$\varphi^m_{nk}=\frac{1}{2\pi i}\oint\omega_n(Q)e_m(Q)dA_k(Q) .$$

Hence we obtain the coefficients A_{AB} and B_{AB} in (15) and (16) and thereby the holomorphic and antiholomorphic coordinates K and $\bar{K}$ [see (19)] (we omit the superscript μ for simplicity in notation):

$$\begin{aligned}&K_{jl}=iV_m\varphi^m_{nk}\xi^{-1}_{n-j}\xi_{lk} ,\\ &\bar{K}_{jl}=iV_m\varphi^m_{nk}\xi^{-1}_{jn}\xi_{k-l}\quad (j,l>0) .\end{aligned} \tag{28}$$

Therefore, one easily gets the curvature of vector vacuum bundle on the Kähler manifold:

$$F=\frac{d}{2}\sum\bar{\partial}\partial(V_mV_{m'})\chi_{mm'} , \tag{29}$$

where

$$\chi_{mm'}=-\varphi^m_{nk}\varphi^{m'}_{n'k'}(\xi^{-1}_{jn}\xi_{jk'})(\xi_{k-j'}\xi^{-1}_{n'-j'})\quad (j',j>0) , \tag{30}$$

and d is the dimension of target space.

In the case of $g=0$, one has the conditions $n=k'<0$, $n'=k>0$, so $\chi_{mm'}=\frac{1}{6}m(m^2-1)\delta_{m+m',0}$. But for the case of $g=0$, one still has the conditions $n'>-\Lambda(g)$, $k>-\Lambda(g)$, $n<\Lambda(g)$, $k'<\Lambda(g)$, where $\Lambda(g)$ is some positive constant depending on g and vanishes as $g=0$. These conditions stem from the fact that $\gamma_{mn}\neq 0$, $|n+m|\leq g$. However, $\chi_{mm'}$ can also be written as

$$\chi_{mm'}=-\sum_{\substack{n<\Lambda\\k>\Lambda}}\varphi^m_{nk}\varphi^{m'}_{kn}+\sum_{n<\Lambda}\sum_{j>0}\sum_{k,k'=-\Lambda}^{\Lambda}\varphi^m_{nk}\varphi^{m'}_{k'n}(\xi_{kj}\xi^{-1}_{k'j})$$
$$+\sum_{k>-\Lambda}\sum_{j>0}\sum_{n,n'=-\Lambda}^{\Lambda}\varphi^m_{nk}\varphi^{m'}_{kn'}(\xi^{-1}_{-jn}\xi_{-jn'})$$
$$-\sum_{j,j'>0}\sum_{-\Lambda}^{\Lambda}\varphi^m_{nk}\varphi^{m'}_{k'n'}(\xi^{-1}_{-jn}\xi_{-jn'})(\xi_{kj'}\xi^{-1}_{k'j'})\ . \tag{31}$$

Obviously , the last three terms in (31) contribute nothing in the case of $g=0$. All four terms in (31) are "local" with the first one just being the two-cocycle in Ref. 5.

Finally, we turn to the ghost fields and choose the relevant phase space as $\Gamma=\{b_n,c_n\}$. The vacuum of ghosts is defined by the requirement

$$\begin{aligned}&\hat{c}_n|\Omega_N\rangle=0\quad n\geq -N\ ,\\&\hat{b}_n|\Omega_N\rangle=0\quad n\geq N+1\ .\end{aligned}\tag{32}$$

The formulas used for the vacuum bundle need to be modified for discussing the ghost case. We introduce in this case the canonical transformation and a new vacuum as[8]

$$\begin{aligned}&\hat{c}\,'_n=A'_{nl}\hat{c}_l+B'_{nl}\hat{\bar{c}}_l\ ,\\&|\Omega'_N\rangle=\exp(-\hat{\bar{b}}_lK'_{ln}\hat{\bar{c}}_n)|\Omega_N\rangle\ ,\end{aligned}\tag{33}$$

where $K'_{ln}=(A'^{-1})_{lm}B'_{mn}$. Then the curvature of ghost vacuum bundle turns out to be

$$F_{\rm gh}={\rm tr}\bar{\partial}\partial(K'^{(N)}\bar{K}'^{(-N-1)})\ . \tag{34}$$

For calculating it one needs the infinitesimal transformation of ghost fields induced by $iV_n\hat{L}_n$,

$$\begin{aligned}\hat{c}\,'_m&=\hat{c}_m+[\hat{c}_m,iV_n\hat{L}_n]\\&=\hat{c}_m+i\sum_{s=-g_0}^{g_0}V_nc^s_{nj}\delta_{m,n+j-s}\theta_+(j+N)\hat{c}_j\\&\quad -i\sum_{s=-g_0}^{g_0}V_nc^s_{nj}\delta_{m,n+j-s}\theta_-(j+N+1)\hat{c}_j\ ,\end{aligned}\tag{35}$$

where

$$\theta_+(j)=\begin{cases}1, & j\geq 0,\\0, & j<0,\end{cases}\quad \theta_-(j)=\begin{cases}1, & j\leq 0,\\0, & j>0.\end{cases}$$

Hence,

$$\begin{aligned}F_{\rm gh}&=\sum[\bar{\partial}\partial(V_jV_k)]\tilde{\chi}_{jk}\\&=\sum_{r,s=-g_0}^{g_0}\sum_{j+k=r+s}c^r_{jn}c^s_{k,j+n-r}\\&\qquad\times\theta_-(n+N+1)\theta_+(j+n-r+N)\\&\qquad\times[\bar{\partial}\partial(V_jV_k)]\ .\end{aligned}\tag{36}$$

It can be immediately seen that $\tilde{\chi}$ has the following properties.

(i) $\tilde{\chi}_{ij}=-\tilde{\chi}_{ji}$.

(ii) It is independent of the coordinate system.

(iii) It does satisfy the Jacobi identity.[4]

(iv) It is "local" in the sense that

$$\tilde{\chi}_{ij}=0\quad\text{for } |i+j|>3g\ , \tag{37}$$

as follows from an elementary computation of the zeros and poles in P_+. By the uniqueness theorem of the central extension of the KN algebra,[4] $\tilde{\chi}_{ij}$ must be proportional to the two-cocycle expressed in (30) or (31). In $g=0$ case, one has

$$\tilde{\chi}_{ij}=-26\chi_{ij}\ . \tag{38}$$

Then, up to trivial cocycles, the relation

$$\tilde{\chi}_{ij}=-26\chi_{ij} \tag{39}$$

follows for any genus g. We conjecture that this relation remains valid as it holds for $g=0$. So the vanishing of the total curvature $R=F+F_{\rm gh}$ requires

$$d=26 \tag{40}$$

to guarantee the reparametrization symmetry of bosonic string at the quantum level.

IV. THE GEOMETRICAL QUANTIZATION OF BOSONIC STRING WITH THE WZ TERM

We are not in a position to discuss the bosonic string model with the WZ term. For a classical observable f whose Hamiltonian vector field X_f $[X_f=-\omega^{-1}(df)]$ satisfying

$$\mathcal{L}_VX_f=[V,X_f]\in P\quad\text{for any } V\in P\ , \tag{41}$$

where V is a vector while P is the subspace of Γ after polarization, one can construct a quantum operator $\hat{O}(f)$ acting on ψ (Ref. 10):

$$\hat{O}(f)\psi=-i\nabla_{X_f}\psi+f\psi\ , \tag{42}$$

where the covariant derivative $\nabla_{X_f}=\omega^{AB}\partial_{Af}\nabla_B$.

After substitution of (23) into (9)

$$\begin{aligned}L_n=&-\tfrac{1}{2}\sum l^n_{jk}(\xi_{j-l}\xi_{k-m}Z^\mu_lZ^\mu_m+\xi_{lj}\xi_{mk}\bar{Z}^\mu_l\bar{Z}^\mu_m\\&+\xi_{j-l}\xi_{mk}Z^\mu_l\bar{Z}^\mu_m+\xi_{lj}\xi_{k-m}\bar{Z}^\mu_lZ^\mu_m)\\&-\tfrac{1}{2}\sum l^n_{jk}J^a_jJ^a_k\end{aligned}\tag{43}$$

we proceed to carry out the geometrical quantization procedure. As discussed in Ref. 10, this amounts to transforming the coordinates into operators as follows:

$$\left.\begin{matrix} Z_l^\mu \\ \bar{Z}_l^\mu \end{matrix}\right\} \to \begin{cases} \hat{Z}_l^\mu = Z_l^\mu , \\ \hat{\bar{Z}}_l^\mu = -\dfrac{\partial}{\partial Z_l^\mu} , \end{cases} \tag{44a}$$

$$\psi_\alpha^i \to \hat{\psi}_\alpha^i = \begin{cases} \psi_\alpha^i \ (\alpha > 0), \\ \dfrac{\partial}{\partial \psi_\alpha^i} \ (\alpha < 0). \end{cases} \tag{44b}$$

Thus the appropriate commutation relations are realized:

$$[\hat{Z}_l^\mu, \hat{\bar{Z}}_j^\nu] = \delta^{\mu\nu}\delta_{lj}, \quad [\hat{\psi}_\alpha^i, \hat{\psi}_\beta^j] = \delta^{ij}\delta_{\alpha+\beta,0} . \tag{45}$$

Therefore, the quantum operator corresponding to L_n reads $(n>0)$

$$\hat{O}(L_n) = -\tfrac{1}{2}\sum l_{jk}^n \left[\xi_{j-l}\xi_{k-m} Z_l^\mu Z_m^\mu + \xi_{lj}\xi_{mk}\frac{\partial}{\partial Z_l^\mu}\frac{\partial}{\partial Z_m^\mu} - \xi_{j-l}\xi_{mk} Z_l^\mu \frac{\partial}{\partial Z_m^\mu} - \xi_{lj}\xi_{k-m} Z_m^\mu \frac{\partial}{\partial Z_l^\mu} \right]$$
$$- \frac{1}{8(C_V+k)} \sum l_{pq}^n E_{\alpha\beta}^p E_{\gamma\delta}^q (T^a)_{ij}(T^a)_{kl} : \hat{\psi}_\alpha^i \hat{\psi}_\beta^j \hat{\psi}_\gamma^k \hat{\psi}_\delta^l : . \tag{46a}$$

The case of $n=0$ is a special one

$$\hat{O}(L_0) = -\tfrac{1}{2}\sum l_{jk}^0 : \hat{\alpha}_j^\mu \hat{\alpha}_k^\mu : - \frac{1}{8(C_V+k)} \sum l_{pq}^0 E_{\alpha\beta}^p E_{\gamma\delta}^q$$
$$\times (T^a)_{ij}(T^a)_{kl} : \hat{\psi}_\alpha^i \hat{\psi}_\beta^j \hat{\psi}_\gamma^k \hat{\psi}_\delta^l : + ia(g) . \tag{46b}$$

Here after considering quantization we have renormalized the coefficient of each term in (46), and the constant term $ia(g)$ is added to account for the possible ambiguity in normal ordering. Actually, $a(g)$ is the ground-state energy. It can be proved that[10]

$$\hat{O}(L_{-n}) = -[\hat{O}(L_n)]^+ . \tag{47}$$

The action of diffeomorphism of Kähler manifold Γ on the loop space is tantamount to a change of the complex structure used to polarize the phase space during quantization, and thereby to a change in operators $\hat{O}(L_n)$. Such a change defines a covariant derivative: $\nabla_{L_n} = \mathcal{L}_{L_n} + \hat{O}(L_n)$ with $\mathcal{L}_{L_n}$ being the Lie derivative. Furthermore, the curvature is given by

$$F(L_m, L_n) = [\nabla_{L_m}, \nabla_{L_n}] - \nabla_{[L_m, L_n]}$$
$$= [\hat{O}(L_m), \hat{O}(L_n)] - \hat{O}([L_m, L_n])$$
$$= \chi_{mn} - \sum_{s=-g_0}^{g_0} a(g) c_{mn}^s \delta_{m+n-s,0} , \tag{48a}$$

where

$$\chi_{mn} = \frac{d}{2}\sum_{-\infty}^{\infty}\sum_{l,l'>0} \varphi_{jp}^m \varphi_{k'q}^n (\xi_{pl}^{-1}\xi_{k'l})(\xi_{j-l'}\xi_{q-l'})$$
$$+ \frac{K\, d_G}{2(C_V+K)} \sum_{\substack{\alpha,\beta>0 \\ \gamma,\delta>0}} F_{\alpha\beta\gamma\delta}^m F_{-\alpha-\beta-\gamma-\delta}^n - \{n \leftrightarrow m\} . \tag{48b}$$

with

$$F_{\alpha\beta\gamma\delta}^m = \frac{1}{2\pi i}\oint_{C_\tau} e_m(Q) h_\alpha(Q) h_\beta(Q) h_\gamma(Q) h_\delta(Q) . \tag{48c}$$

Using the relation $\xi_{jn}\xi_{kn} = \delta_{jk}$, one may simplify (48b) to obtain

$$\chi_{mn} = \frac{d}{2}\sum_{\substack{k>\Lambda \\ l<\Lambda}} \varphi_{lk}^m \varphi_{kl}^n - \frac{d}{2}\sum_{l>0}\sum_{k,k\le -\Lambda}^{\Lambda} \left[\sum_{p>-\Lambda} \varphi_{kp}^m \varphi_{pk'}^n (\xi_{k-l}^{-1}\xi_{k'-l}) + \sum_{p<\Lambda} \varphi_{pk}^m \varphi_{k'p}^n (\xi_{kl}^{-1}\xi_{k'l}) \right]$$
$$+ \frac{d}{2}\sum_{l,l'>0}\sum_{-\Lambda}^{\Lambda} \varphi_{pk}^m \varphi_{k'p'}^n (\xi_{p-l}^{-1}\xi_{p'-l})(\xi_{kl}^{-1}\xi_{k'l}) + \frac{k\, d_G}{2(C_V+K)} \sum_{\substack{\alpha,\beta>0 \\ \gamma,\delta>0}} F_{\alpha\beta\gamma\delta}^m F_{-\alpha-\beta-\gamma-\delta}^n - \{n \leftrightarrow m\} , \tag{49}$$

where $\Lambda(G)$ is a constant depending on g and vanishes at $g=0$. In Eq. (49) the first two terms and their exchange terms in n and m are just the central extension of the KN algebra of the bosonic string with WZ term. We note also that all terms in (49) are local.

Equation (48) tell us that the quantum version of $\hat{O}(L_n)$, does not form a representation of a Kähler manifold Γ. Only by introducing ghosts are we able to construct a unitary representation of Γ. The ghost states comprise the fiber of the line bundle over Γ, so the curvature of the line bundle is interpreted as the ghost contribution to this representation. Michelsson[11] has proposed a new derivation of the curvature of line bundle as follows. The infinitesimal action of diffeomorphism on the Hilbert space of ghost sector is precisely the adjoint action of the algebra of vector fields L_p's itself:

$$(L_p)_{nm}=[L_p,L_m]_n=\sum_{s=-g_0}^{g_0} c_{pm}^{s}\delta_{n,p+m-s} . \tag{50}$$

So the curvature can be calculated as

$$\begin{aligned}R(L_p,L_q)&=\mathrm{tr}[(L_p)_b(L_q)_c-(L_q)_b(L_p)_c]\\&=\sum_{m\le 0}\sum_{n>0}[(L_p)_{nm}(L_q)_{mn}-(L_q)_{nm}(L_p)_{mn}]\\&=\tilde{\chi}_{pq} ,\end{aligned} \tag{51}$$

where

$$\tilde{\chi}_{pq}=\sum_{r,s=-g_0}^{g_0}\sum_{p+q=\gamma+s}\sum_{m\le 0}[c_{pm}^{s}c_{q,p+m-s}^{\gamma}\theta_+(p+m-s)-c_{qm}^{s}c_{p,q+m-s}^{\gamma}\theta_+(q+m-s)] . \tag{52}$$

Here we have obtained the curvature of line bundle as the same result as the ghost contribution to the anomaly of the KN algebra on genus-g Riemann surface.[4]

Since all terms in Eqs. (48a) and (51) are local, we manage to combine the curvature of line bundle with that of the holomorphic vector bundle over the same Kählar manifold Γ on genus-g Riemann surface. Then the total curvature of product bundle of them reads

$$R_T(L_m,L_n)=F(L_m,L_n)+R(L_m,L_n) . \tag{53}$$

We do not have the explicit expression for ξ_{ml}, which are all zero in $g=0$ case. So we conjecture that both F and R are proportional to the two-cocycle term $\chi(e_m,e_n)$ (Ref. 4) in the central extension of the KN algebra for a bosonic string up to trivial cocycles, as it holds for the $g=0$ case:

$$R_T(L_m,L_n)=\left[d+\frac{K\,d_G}{C_V+K}-26\right]\chi(e_m,e_n) . \tag{54}$$

Then vanishing of R_T leads to the critical dimension

$$d=26-\frac{K\,d_G}{C_V+K} . \tag{55}$$

V. CONCLUSION

By using the geometrical quantization method of Bowick and Rajeev as well as the KN algebra on high-genus Riemann surface Σ, we have studied the holomorphic structure of bosonic string model with Wess-Zumino term on Σ and found that the total curvature of product bundle of line bundle and the holomorphic vector bundle of Kähler manifold Γ and Σ is proportional to the central extension of the KN algebra for this model. In critical dimension shown as (40) or (55), one obtains a reparametrization gauge-invariant theory at the quantum level. Though the results (40) and (55) are well known for some time, the arguments for their validity on a Riemann surface with high genus are important because one has to consider various Riemann surfaces in general when dealing with quantized string field theory. The above method can be extended to superstring theory and will be reported in a subsequent paper.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation in China under Grant No. KR12051 through the Nankai Institute of Mathematics.

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