PHYSICAL REVIEW D VOLUME 41, NUMBER 2 15 JANUARY 1990

Covariant smoothed particle hydrodynamics on a curved background

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(Received 24 July 1989)

We show that the smoothed-particle-hydrodynamics discretization technique is compatible with the principles of general relativity if the contact interactions are modeled by spatial smoothing functions in the local frame comoving with the fluid. We then develop a smoothed-particle-hydrodynamics algorithm to model a non-self-gravitating isentropic fluid on a curved background. The equations of the fluid are discretized using a kernel whose spatial support is of a constant proper width in the local comoving frame. The one-dimensional relativistic shock problem is used for testing this algorithm.

I. CONTACT FORCES AND THE LOCAL COMOVING FRAME

Further insight into the dynamics of matter and fields in strong gravitational potentials requires, in part, truly three-dimensional relativistically covariant computer algorithms to reveal the complex motions and instabilities inherent to environments of this kind. Computer simulations that impose symmetries often times freeze out physical instabilities, and nonrelativistic codes both accumulate errors on the order of $v^2/2c^2$ per time step and exclude essential driving mechanisms (e.g., the gravitomagnetic force).

An obvious astrophysical problem which calls for a fully three-dimensional treatment is a "generic" case of a thick accretion disk around a spinning black hole: ordinarily the rotation axis of the disk and of the hole will not coincide. Therefore, effects of the frame dragging (Bardeen-Petterson process[1]) are important. Moreover, thick accretion tori exhibit nonaxisymmetric and hence inherently three-dimensional instabilities.[2] The recently developed smoothed-particle-hydrodynamics (SPH) discretization technique[3] provides a valuable tool for the analogous problems in the nonrelativistic regime,[4] and it may be the only tractable approach to problems of this nature.[5] However, every SPH code developed so far is valid only in the Newtonian limit. It is for this reason that we have developed a relativistically covariant formulation of SPH in this paper.

It was not clear that a consistent relativistic treatment of fluid dynamics by the SPH technique was even possible. Our initial concerns stemmed from the fact that SPH modeled pressure gradients in a three-dimensional noncovariant fashion. However, we demonstrate that such a formalism is indeed possible. *We conclude that the relativistic hydrodynamical contact interactions are mediated by kernels whose supports reside in the local frame comoving with the fluid.* We test our thesis by demonstrating via computer simulation that our algorithm reproduces shock propagation[6] (to within 3% of the expected shock velocity) in a relativistic Riemann-shock-tube experiment[7] over an extreme range of effective temperatures (Fig. 3).

A three-space model of the hydrodynamic contact forces by distributions does not lead to any contradictions with relativity. It does not assume in any way a superluminal speed of propagation of the interactions. The contact forces are due to the same kinetic processes that are responsible for the local thermodynamic equilibrium in the fluid. Our demand that the fluid instantaneously achieves local thermodynamic equilibrium is an assumption of relativistic kinetics and does not create any problems in relativity.[8,9]

However, it is important to keep in mind that hydrodynamics in general (and in particular the smoothed particle hydrodynamics) can only be used for describing phenomena whose (1) dynamical scale is much larger than the mean free path of the particle, and whose (2) speed of propagation is slow enough to preserve local thermodynamic equilibrium.

In relativistic SPH the choice of the three-space on which the support of the kernels is placed should be made carefully to avoid running into contradictions with (1) relativity or (2) with basic hydrodynamic assumptions. Along this line of reasoning, placing the support for the kernels in a laboratory frame is not consistent, because (1) it is relativistically noncovariant, and (2) it leads to a contradiction between the freedom of choice of the observer and the conditions necessary for using a hydrodynamic description.[10]

How do we represent the contact forces in SPH while preserving general covariance? The only possible solution to this problem is to adopt the proper space of the local comoving frame (LCF) of the fluid as the platform upon which we define the smoothed particle contact forces. In particular, the three-space kernels must be defined in the LCF so that their half-width is of a constant proper length. The LCF always exists and is well defined.

Such a choice makes our version of SPH truly Lagrangian, relativistic covariant, and solves all the problems noted above.

II. A SIMPLE EXAMPLE

We cannot think of a more expedient way to demonstrate our LCF approach to SPH than to examine it in $1+1$ dimension. For $(1+1)$-dimensional problems the number of the four-velocity components is reduced to two. The complete list of parameters, six altogether, then includes (1) two components of the fluid four-velocity $\mathbf{u}$, (2) baryon number density n, (3) mass density ρ, (4) temperature T, and (5) pressure p. The equations relating the parameters are (1) the momentum equation, (2) the baryon conservation condition, (3) the energy-conservation condition, (4) two state equations, and (5) the four-velocity normalization condition. Here, we replace the baryon conservation equation and the energy equation (entropy-conservation condition) by their first integrals. The baryon number and entropy conservation conditions are explicitly satisfied along the world tubes of the fluid in spacetime.

In this paper, we use the state equations for an isentropic classical, ideal, one-component gas derived by Synge.[8] However, all our procedures can be carried out for any relativistic state equations.

To define the proper space of the LCF at an event we must first define the world tube. As in any discrete mathematics representation of a continuous system there is no unique way to define this world tube. However, we adopt an equipartition technique in the frame of the LCF. We approximate the space in question by a piecewise flat space, so that it is orthogonal to the four-velocity of each smoothed particle in the cell, and flat over the world tube associated with each smoothed particle. The walls of the world tubes can be placed uniquely only if an additional condition on the distances from the walls to particles is imposed. The distances have to be evaluated in the proper space of the LCF (otherwise the covariance will be violated) which is undefined without the proper placing of the walls. The two problems of (1) recovering the proper space of the LCF and of (2) placing the walls of world tubes around the particles are, consequently, coupled—"bootstrapped." They are described by the system of four coupled algebraic equations ($\overline{LB}=\overline{BC}$—equipartition condition, $\mathbf{u}_l\cdot\vec{LB}=0$, $\mathbf{u}_m\cdot\vec{BC}=0$, $\vec{CM}=\text{const}\times\mathbf{u}_m$), the solution of which gives the answer to the both problems (Fig. 1).

Following the standard SPH technique of discretizing the hydrodynamics equations, we (1) adopt the LCF, (2) express the fluid state in terms of the baryon density n and reciprocal temperature $\xi=c^2/kT$ (T is the temperature, k is Boltzmann's constant), and (3) use the $m=4$ basis spline w_4 of half-width $2h$ and normalized for planar geometries.[10]

The discretized SPH equations then become[11]

$$n_k L(m\xi_k)=\text{const}\ , \tag{1}$$

$$c^{-2}\frac{d\mathbf{u}_l}{d\tau}=-\frac{1}{mn_l G(m\xi_l)}\sum_k \frac{\alpha_k}{\xi_k}\nabla_l w_{lk}\ , \tag{2}$$

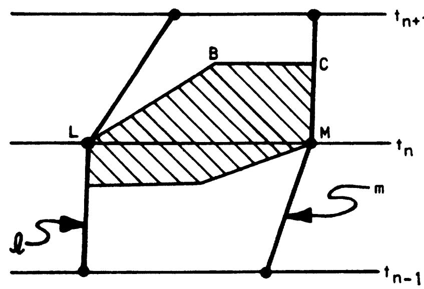

FIG. 1. The local comoving frame (LCF) of the fluid associated with particles l and m as represented in the laboratory frame (three horizontal lines). The acceleration imparted to particle l at event L by particle m is mediated in SPH by a contact force (kernel modeled) acting along the LCF (line segment $\overline{LBC}$). Points B and C are determined by the equipartition conditions. Point B is the flux tube boundary between l and m. Similarly, the acceleration imparted to particle m at point M by particle l is due to the contact force interaction along the bottom edge of the shaded region. This procedure is implemented for each pair of adjacent particles. The boundary of the shaded region can be thought of as a pictorial representation of the covariant kernel in SPH. Larger relative velocities give rise to kernels that are more spread out in time (the shaded region becomes larger).

$$c^{-2}p_k=\frac{n_k}{\xi_k}\ , \tag{3}$$

$$\rho_k+c^{-2}p_k=mn_k G(m\xi_k)\ , \tag{4}$$

$$n_k V_k=\alpha_k=\text{const}\ , \tag{5}$$

$$\mathbf{u}_l\cdot\mathbf{u}_l=-c^2\ . \tag{6}$$

Here k and l label the particles, α_k is the baryon number of particle k, m is the baryon rest mas, and $G(m\xi)$ and $L(m\xi)$ are functions used by Synge[8] to define the equation of state. In particular, they are defined as

$$\begin{aligned} G(m\xi)&=\frac{K_3(m\xi)}{K_2(m\xi)}\ ,\\ L(m\xi)&=\frac{m\xi}{K_2(m\xi)}\exp\left[-\frac{m\xi K_3(m\xi)}{K_2(m\xi)}\right]\ , \end{aligned} \tag{7}$$

where $K_n(m\xi)$ is the nth hyperbolic Bessel function.

To benchmark our approach by computer we examine the shock velocity of a one-dimensional Riemann shock tube over a wide range of temperatures ($1/m\xi=0$ to $1/m\xi=2$) (see Fig. 2). It was not our intent to optimize the computer implementation of Eqs. (1)–(6). In particular, we (1) employ a first-order accurate forward differencing scheme, (2) do not introduce any artificial viscosity, and (3) do not implement the variable kernel technique within the LCF. Nevertheless, our computer simulation agrees with the analytic solution of this problem by Synge[8] as depicted in Fig. 3.

III. FUTURE EXTENSIONS

Smoothed particle hydrodynamics can be extended from the level described here to a calculational tool for

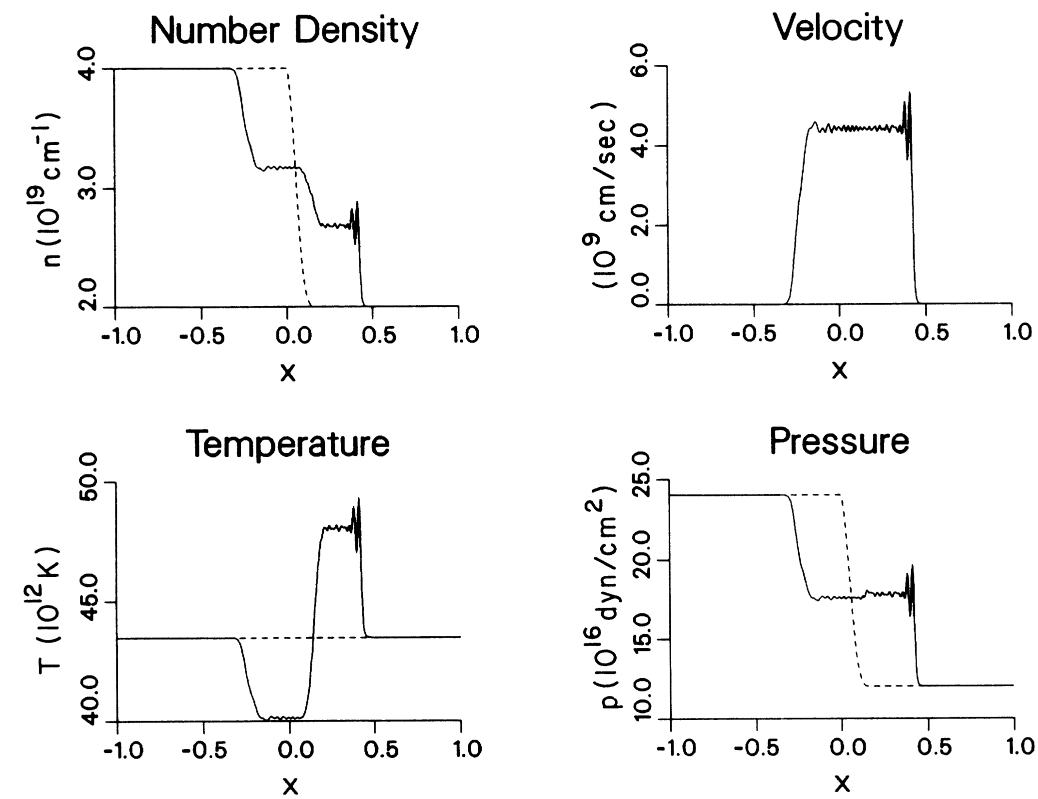

FIG. 2. A graphical representation of the one-dimensional Riemannian shock solution for a neutron gas at ultrarelativistic temperatures ($1/m\xi=2 \Longrightarrow T\sim 4.3\times 10^{13}$ K). Initially the neutron gas was at rest with a jump discontinuity in number density (n) at $x=0$ cm and was spread over 32 particles (dashed lines indicate initial conditions). The number density jumps from 2×10^{19} cm^{-1} at $x=0.16$ cm to 4×10^{19} cm^{-1} at $x=0$ cm. The gas was simulated by 400 particles over a 2-cm span with kernel width $h=7.5\times 10^{-3}$ cm and evolved for 1.99×10^{-11} sec (398 time steps at 5×10^{-14} sec/step). All quantities are in cgs units.

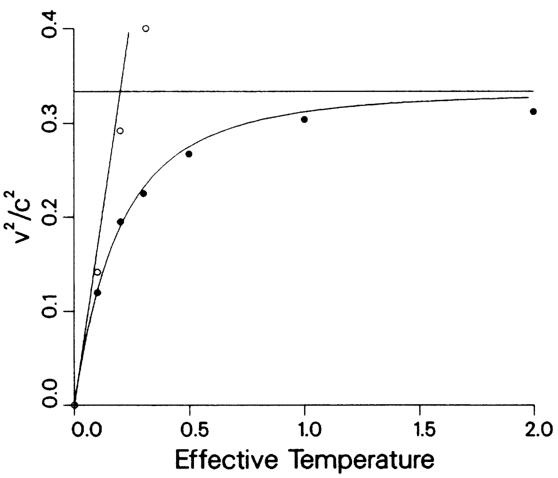

FIG. 3. A graphical representation of our algorithm's ability to reproduce the sound velocity squared (v^2/c^2) as a function of the effective temperature ($1/m\xi$). The analytic solution (curved line) is reproduced to better than 3% in velocity by our code (solid dots). The horizontal line indicates the sound saturation speed ($v=c/\sqrt{3}$) for any relativistic gas shock. The graph also represents the nonrelativistic solution (straight line with slope $\frac{5}{3}$) and the nonrelativistic SPH solution (circles). We obtained velocities in excess of 64c using the nonrelativistic equations. We never exceeded the limiting sound velocity ($c/\sqrt{3}$) using our relativistic code.

fully general three-dimensional relativistic astrophysics by introducing step-by-step the improvements needed; each improvement in the context where it can be most easily tried and tested.

(1) *Extension of the LCF from one to three dimensions.* Ideally, we seek to incorporate our algorithm into existing nonrelativistic SPH codes. Furthermore, we wish to do this by (1) augmenting an existing code with an LCF-finding subroutine, and (2) replacing the nonrelativistic equations with the corresponding covariant expressions. The only extension that we foresee making in our LCF subroutine is the added condition that the length of the LCF segment joining the particles be extremal (the algebra associated with this is straightforward). Here we reconstruct the LCF in a pairwise fashion. This pairwise reconstruction of the LCF is identical to the method employed in our one-dimensional code to second order in the relative velocities of the particles.

(2) *Curvature.* Extending our algorithm from flat spacetime to a curved spacetime requires no more than (1) the addition of a connection coefficient to the right-hand side of the momentum equation, and (2) the simple modification of the line element subroutine for $ds^2=\eta_{\mu\nu}dx^\mu dx^\nu$ to $ds^2=g_{\mu\nu}dx^\mu dx^\nu$, where $g_{\mu\nu}$ is the curved-spacetime metric. It is our intention to calculate a one-dimensional shock freeze on the horizon of a black hole.

Although the covariant SPH algorithm (CSPH) presented here is more computationally demanding than the nonrelativistic SPH algorithm, a three-dimensional code appears to (1) be computationally tractable, (2) be easily obtained by the slight modification of existing three-dimensional SPH codes,[5] and (3) preserve the essential advantages that the truly Lagrangian SPH technique has offered in the nonrelativistic regime. We therefore conclude that three-dimensional covariant hydrodynamics on a curved background via SPH is possible and forthcoming.

ACKNOWLEDGMENTS

We extend our thanks to W. Benz, W. Kluźniak, N. LaFave, R. Matzner, C. McCarter, R. Stellingwerf, and J. Wilson for helpful discussions and to C. Oliver for computer access. This work was supported by the Air Force Office of Scientific Research.

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[1]J. M. Bardeen and J. A. Petterson, Astrophys. J. (Lett.) **195**, L65 (1975).

[2]J. C. B. Papaloizou and J. E. Pringle, Mon. Not. R. Astron. Soc. **213**, 799 (1985); **208**, 31 (1984).

[3]J. J. Monaghan, Comput. Phys. Commun. **48**, 89 (1988).

[4]W. H. Zurek and W. Benz, Astrophys. J. **308**, 123 (1986)

[5]W. Benz, Comput. Phys. Commun. **48**, 97 (1986).

[6]J. J. Monaghan, J. Comput. Phys. **52**, 375 (1983).

[7]J. F. Hawley, L. L. Smarr, and J. R. Wilson, Astrophys. J. **277**, 296 (1984).

[8]J. L. Synge, *Relativistic Gas* (North-Holland, Amsterdam, 1957).

[9]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[10]J. J. Monaghan, Comput. Phys. Rep. **3**, 122 (1985); **3**, 80 (1985).

[11]We have taken advantage of our one-dimensional fluid by solving for the volume V analytically and using Eq. (5) to solve for n. We could have adopted the standard SPH technique to obtain n directly by the integral interpolant method—an approach valid in higher dimensions.