Wilson loops at finite temperature

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(Received 27 November 1989)

We study the behavior of Wilson lines at finite temperature in the simple situation in which there are only one or two Wilson lines present, and the topology of the spacetime is $R^2 \times S^1 \times S^1$. We find that the pattern of symmetry breaking is not affected by thermal fluctuations in this case and, in particular, the symmetry is not restored above a characteristic temperature.

I. INTRODUCTION

The Hosotani mechanism for symmetry breaking through vacuum expectation values of Wilson loops in gauge theories when the spacetime manifold is nonsimply connected is probably the most important one in the framework of string theories (cf. Ref. 1), at least when interpreting them from the field-theoretic point of view.

In Hosotani's paper,[2] in particular, the case of a SU(N) gauge theory in the manifold $M_3 \times S^1$ is studied in detail (especially in the case $N=2$) and it is concluded, independently of the number of fermions, that the minimum of the effective potential for fermionic twist $|\delta| < \pi/2$ corresponds to a nontrivial vacuum expectation value of the gauge field in the S^1 direction.

Manifolds of the type $M_4 \times S^3/\mathbb{Z}_2$, with different gauge groups have been studied by Evans and Ovrut[3] and by Birmingham, Kantowski, and Milton[4] at zero temperature, with conclusions similar to Hosotani's.

It has been actually claimed by Lee, Holman, and Kolb[5] that there is no energy barrier between different vacua when the internal manifold has infinite fundamental group (although such a barrier was found for the case of a finite π_1, such as $S^3/\mathbb{Z}_2$).

One-loop quantum fluctuations will generically lift the degeneracy between different vacua. Shiraishi[6] however has studied the effect of thermal fluctuations in the case in which the internal manifold has the topology $S^n/\mathbb{Z}_2$, and he claims that they do not restore the symmetry.

The purpose of this Brief Report is to study Hosotani's model at finite temperature, that is, an SU(2) gauge theory, with N_F species of fermions, and with an spacetime manifold $\mathbb{R}^d \times S^1(L) \times S^1(\beta)$. Some related work has been done by Hetrick and Ho.[7]

II. ONE WILSON LINE

The partition function of a gauge theory with gauge group $G = \mathrm{SU}(2)$, defined on a manifold of the type $\mathbb{R}^d \times (S^1)^2$, with n flavors of fermions in the fundamental representation, can be written as

$$Z = (\det - D^2)^{d/2} (\det M)^{1-d/2} , \tag{2.1}$$

where $D_\rho^{ab} = D_\rho \delta^{ab} - g f^{abc} B_\rho^c$ the gauge-covariant derivative, and $M = -D_\rho D^\rho$. We have divided the gauge field into a background B and quantum fluctuations Q; we furthermore assume that the background has nontrivial components only along one of the S^1, that is

$$\begin{aligned} &B_\mu = 0, \quad \mu = 1, \ldots, d+1 , \\ &B_{d+2} = \omega \sigma_3 . \end{aligned} \tag{2.2}$$

The boundary conditions on the fermions are

$$\begin{aligned} &\psi(x_{d+1} + L) = e^{i\epsilon} \psi(x) , \\ &\psi(x_{d+2} + \beta) = e^{i\delta} \psi(x) = -\psi(x) . \end{aligned} \tag{2.3}$$

The computation of the determinants is best done by using the ζ-function technique; the eigenvalues of M are

$$\begin{aligned} &\lambda^0_{n_1 n_2}(k) = k^2 + \left[\frac{2\pi}{\beta}\right]^2 n_1^2 + \left[\frac{2\pi}{L}\right]^2 n_2^2 , \\ &\lambda^\pm_{n_1 n_2}(k) = k^2 + \left[\frac{2\pi}{L}\right]^2 n_2^2 + \left[\frac{2\pi}{\beta}\right]^2 \left[n_1 \pm \frac{g\omega\beta}{2\pi}\right]^2 \end{aligned} \tag{2.4}$$

and the eigenvalues of D^2 can be written as

$$\lambda^\pm_{n_1 n_2}(k) = k^2 + \left[\frac{2\pi}{L}\right]^2 \left[n_2 + \frac{\epsilon}{2\pi}\right]^2 + \left[\frac{2\pi}{\beta}\right]^2 \left[n_1 + \frac{\pi \pm g\omega\beta/2}{2\pi}\right]^2 , \tag{2.5}$$

where $k \in \mathbb{R}$, and $n_1, n_2 \in Z$. It is quite easy to check that the parameters (β, L, ϵ) contain essentially all the "twisted tori" which could be constructed with boundary conditions on physical fields:

$$\begin{aligned} &\phi(\sigma_1 + 1, \sigma_2) = \phi(\sigma_1 + \tau_1, \sigma_2 + \tau_2) = \phi(\sigma_1, \sigma_2) , \\ &\sigma_1 \equiv x_{d+1}, \quad \sigma_2 \equiv x_{d+2} . \end{aligned} \tag{2.6}$$

Using the results of the Appendix, we can write the free energy as

$$F \equiv -\frac{1}{\beta L} \ln Z = -\frac{1}{\beta L} (\zeta_B' - 2 n_f \zeta_F') , \tag{2.7}$$

where

$$\zeta'_B \equiv S(L,\beta,0,0)+S(L,\beta,0,g\omega\beta)+S(L,\beta,0-g\omega\beta)\,,$$
$$\zeta'_F \equiv S(L,\beta,\epsilon,\pi+g\omega\beta/2)+S(L,\beta,\epsilon,\pi-g\omega\beta/2) \tag{2.8}$$

and the function S is defined in (A10). The behavior of the function S is easily seen to imply that the free energy is minimized when

$$g\omega\beta=4\pi\mathbb{Z}, \quad \epsilon=\pi\,, \tag{2.9}$$

independently of the number of fermionic flavors. Were the boundary condition on the thermal S^1, δ be allowed to vary, we would have obtained this same condition for $\pi/2<|\delta|\leq\pi$, but for $|\delta|<\pi/2$, the minimum would be found at

$$g\omega\beta=2\pi(2\mathbb{Z}+1)\,. \tag{2.10}$$

III. TWO WILSON LINES

Let us now consider the more general case in which the background gauge field has nontrivial components along both S^1,s,: that is,

$$B_\mu=0, \quad \mu=1,\ldots,d\,,$$
$$B_{d+1}=\mathbf{v}\cdot\boldsymbol{\sigma}, \quad B_{d+2}=\omega\sigma_3\,. \tag{3.1}$$

It is not difficult to check that the eigenvalues of the operator M in this case are

$$\lambda^{00}_{n_1n_2}(k)\equiv k^2+\left[\frac{2\pi}{\beta}\right]^2 n_1^2+\left[\frac{2\pi}{L}\right]^2 n_2^2\,,$$
$$\lambda^{\pm\pm}_{n_1n_2}(k)=k^2+\left[\frac{2\pi}{\beta}\right]^2\left[n_1\pm\frac{g\omega\beta}{2\pi}\right]^2+\left[\frac{2\pi}{L}\right]^2\left[n_2\pm\frac{gvL}{2\pi}\right]^2, \tag{3.2}$$

where v is the modulus of the vector $\mathbf{v}$:$v^2\equiv\sum_{i=1}^3 v_i^2$ and $n\equiv(n_1n_2)\in Z^2$.

This means that the bosonic zeta function can be written in terms of the function S of the Appendix as

$$\zeta'_B(0)=S(L,\beta,0,0)+2S(L,\beta,gvL,g\omega\beta)\,. \tag{3.3}$$

On the other hand, the eigenvalues of the Dirac operator squared are

$$\lambda^{\pm}_{n_1n_2}(k)=k^2+\left[\frac{2\pi}{\beta}\right]^2\left[n_1+\frac{\delta\pm g\omega\beta/2}{2\pi}\right]^2+\left[\frac{2\pi}{L}\right]^2\left[n_2+\frac{\epsilon\pm gvL/2}{2\pi}\right]^2. \tag{3.4}$$

This means that the fermionic part of the determinant (for $\delta=\pi$ so that Fermi-Dirac statistics is implemented in one of the S^1,s)

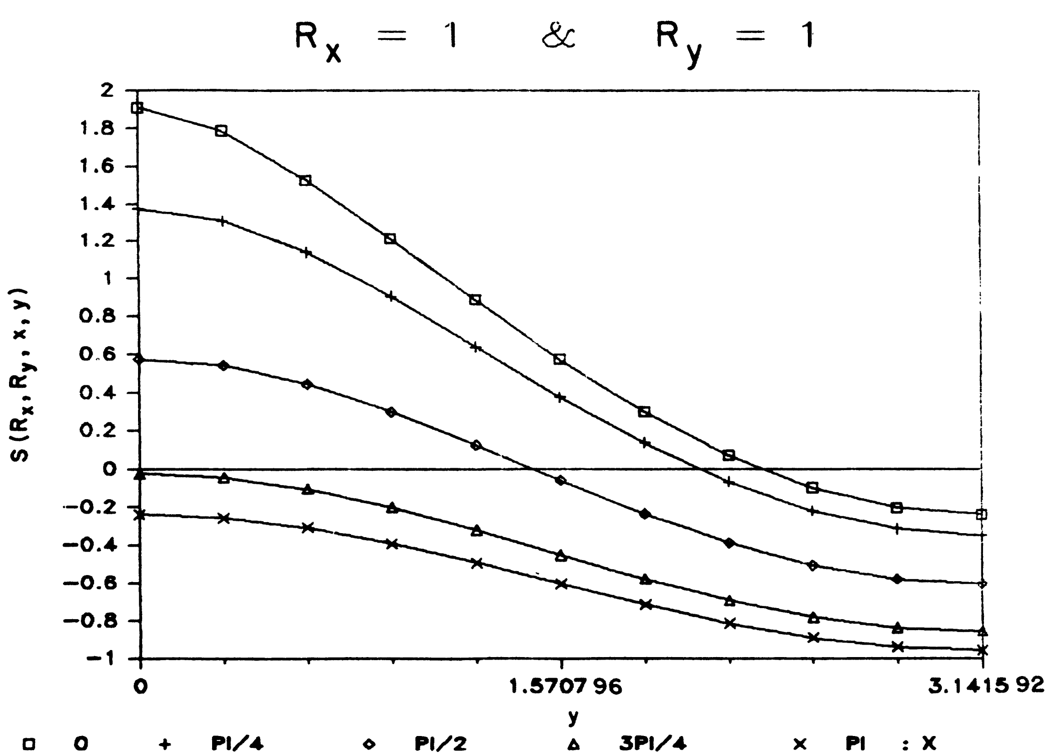

FIG. 1. The auxiliary function $S(R_x,R_y;x,y)$ is represented (for $R_x=R_y=1$) as a function of the variable y, and this is done for several values of the variable x. □ denotes $x=\pi/4$, ◇ denotes $x=\pi/2$, △ denotes $x=3\pi/4$, and × denotes $x=\pi$.

$$\zeta_F'(0)=S(L,\beta,\epsilon+gvL/2,\pi+g\omega\beta/2)+S(L,\beta,\epsilon-gvL/2,\pi+g\omega\beta/2) \tag{3.5}$$

and the free energy is

$$F=\frac{1}{\beta L}(2n_f\zeta_F'-\zeta_B') \, .$$

Using the properties of the S function, it is easy to realize that the stationary points of the free energy are independent of the number of flavors present.

If we fix, moreover, the convenient variable $x\equiv g\omega\beta/2$, we have, in terms of $y\equiv gvL/2$, the following set of minima: for $0\le y<\pi/2$, the minimum is at $\epsilon=\pi$, that is, for antiperiodic fermions in the $d+1$ coordinate; for $\pi/2<y\le\pi$, this minimum corresponds to $\epsilon=0$ (periodic fermions); and for the boundary case $y=\pi/2$ both $\epsilon=0$ and $\epsilon=\pi/2$ correspond to minima.

On the other hand (still keeping x fixed), for $0\le\epsilon<\pi/2$, there is a minimum in terms of the other variable, for $y=\pi$. This minimum shifts to $y=0$ for $\pi/2<\epsilon\le\pi$, and for the boundary point $\epsilon=\pi/2$ we have again two minima at $y=0$ and $y=\pi$. All these properties are conveniently summarized in Fig. 1.

IV. DISCUSSION

We have obtained, at least in the simple case in which only up to two Wilson lines are allowed, that the position of the minimum is not changed with respect to the zero-temperature result; that is, that thermal fluctuations do not spoil the symmetry-breaking pattern.

In a sense, this was only to be expected, since symmetry breaking in our case is a topological effect, which should not be affected by small quantum fluctuations. Explicit studies of the somewhat related physics of two-dimensional vortices in the X-Y model, however, show that explicit calculations can unveil some surprising results (such as a Kosterlitz-Thouless-type phase transition.)

We need to study more general configurations, however, before definite conclusions can be drawn on the behavior of Wilson lines at finite temperature. Work is in progress on this (and related) matters.

ACKNOWLEDGMENTS

This work has been partially supported by Comisión Interministerial de Ciencia y Tecnología (CICYT) (Spain).

APPENDIX: STUDY OF THE ζ FUNCTION

We are interested in the function

$$\zeta_{[a,x]}(s)\equiv\int\frac{d^nk}{(2\pi)^n}\sum_{n_1,n_2\in\mathbb{Z}}[k^2+a_1^2(n_1+x_1)^2+a_2^2(n_2+x_2)^2]^{-s} \, . \tag{A1}$$

We can first perform the integration, by using

$$\int\frac{d^nk}{(2\pi)^n}(k^2+\alpha^2)^{-s}=\frac{\pi^{n/2}}{(2\pi)^n}\frac{\Gamma(s-n/2)}{\Gamma(s)}(\alpha^2)^{-s+n/2} \tag{A2}$$

getting in this way the result

$$\zeta_{[a,x]}(s)\Gamma(s)=\frac{\pi^{n/2}\Gamma(s-n/2)}{(2\pi)^n}\times\sum_{n_1,n_2\in\mathbb{Z}}[a_1^2(n_1+x_1)^2+a_2^2(n_2+x_2)^2]^{-s+n/2} \, . \tag{A3}$$

Introducing now the identity

$$\lambda^{-s}\Gamma(s)=\int_0^\infty d\tau\,\tau^{s-1}e^{-\lambda\tau} \tag{A4}$$

we get

$$\zeta_{[a,x]}(s)\Gamma(s)=\frac{\pi^{n/2}}{(2\pi)^n}\int_0^\infty d\tau\,\tau^{s-n/2-1}e^{-(a_1^2x_1^2+a_2^2x_2^2)\tau}\theta_3\left[\frac{ia_1^2\tau}{\pi}x_1,\frac{ia_1^2\tau}{\pi}\right]\theta_3\left[\frac{ia_2^2\tau x_2}{\pi},\frac{ia_2^2\tau}{\pi}\right] , \tag{A5}$$

where the Jacobi elliptic theta function is defined through

$$\theta_3(z,\tau)\equiv\sum_{n\in\mathbb{Z}}e^{i2\pi nz}e^{in^2\pi\tau} \, . \tag{A6}$$

We decompose the interval of integration into $\int_0^\delta+\int_\delta^\infty$, and transform the first of those by using the modular transformation properties of Jacobi's function, i.e.,

$$\theta_3(iz/\omega,i/\omega)=\omega^{1/2}e^{\pi z^2/\omega}\theta_3(z,i\omega) \tag{A7}$$

and we get an equivalent expression which is well behaved at $s=0$ (cf. Ref. 8), namely,

$$\zeta_{[a,x]}(s)\Gamma(s)=\frac{\pi^{n/2}}{(2\pi)^n}\left\{\frac{\pi}{a_1a_2}\frac{\delta^{s-n/2-1}}{s-n/2-1}+\frac{\pi}{a_1a_2}\int_{\delta-1}^\infty d\tau\,\tau^{-s+n/2}\left[\theta_3\left[x_1,\frac{i\pi\tau}{a_1^2}\right]\theta_3\left[x_2,\frac{i\pi\tau}{a_2^2}\right]-1\right]\right.$$
$$\left.+\int_\delta^\infty d\tau\,\tau^{s-n/2-1}e^{-(a_1^2x_1^2+a_2^2x_2^2)\tau}\theta_3\left[\frac{ia_1^2\tau x_1}{\pi},\frac{ia_2^2\tau}{\pi}\right]\theta_3\left[\frac{ia_2^2\tau x_2}{\pi},\frac{ia_2^2\tau}{\pi}\right]\right\} . \tag{A8}$$

(All this is true for arbitrary δ.) Working out the derivative, we finally get

$$\zeta'_{[a,x]}=\frac{4\pi(2+n/2)}{2\pi^{1+3n/2}}\frac{1}{a_1a_2}\left[a_1^{2+n}\sum_{n_1=1}^{\infty}\frac{\cos 2\pi x_1 n_1}{n_1^{2+n}}+a_2^{2+n}\sum_{n_1=1}^{\infty}\frac{\cos 2\pi x_2 n_1}{n_1^{2+n}}+2\sum_{n_1 n_2=1}^{\infty}\frac{\cos 2\pi n_1 x_1 \cos 2\pi n_2 x_2}{(n_1^2/a_1^2+n_2^2/a_2^2)^{1+n/2}}\right]. \tag{A9}$$

In our case ($n=2$),

$$S(L,\beta,x,y)\equiv\zeta^1_{[2\pi/L,2\pi/\beta,x/2\pi,y/2\pi]}(0)=\frac{2L\beta}{\pi}\left[\frac{1}{L^4}\hat{S}(x)+\frac{1}{\beta^4}\hat{S}(y)+F(L,\beta,x,y)\right], \tag{A10}$$

where the function $\hat{S}(x)$ is

$$\hat{S}(x)\equiv\sum_{n=1}^{\infty}\frac{\cos nx}{n^4}=\zeta(4)-\frac{1}{48}x^2(x-2\pi)^2,\quad 0\le x<2\pi \tag{A11}$$

and F is defined through

$$F(L,\beta,x,y)\equiv 2\sum_{n_1,n_2}\frac{\cos n_1 x\cos n_2 y}{(L^2n_1^2+\beta^2n_2^2)^2}. \tag{A12}$$

The function $S(L,\beta,x,y)$ for fixed L and β is a monotonically decreasing function of x and y for $0\le x,y\le\pi$; this means that it has absolute maxima at $(x,y)=(m_1,m_2)2\pi$ and absolute minima at $(x,y)=(2m_1+1,2m_2+1)\pi$.

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