Anomaly in even dimensions with an arbitrary signature and the finite-temperature effect

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We study the anomaly in even dimensions with an arbitrary signature and the finite-temperature effect. We find that there is explicit signature dependence and that there is no temperature dependence for Abelian and non-Abelian anomalies in even dimensions with an arbitrary signature.

The anomaly plays a very important role in gauge theories. Extensive studies have been done to understand its mathematical origin and physical implications.[1-4] In the context of fundamental theories such as superstring theories, anomalies in higher dimensions have also been studied. On this basis we will study anomalies in even dimensions with an arbitrary signature. Our aim here is to see if any nontrivial effect will result in comparison with calculations of anomalies in Euclidean space or with one time dimension for which most of the calculations of anomalies in higher dimensions were done.[5] In higher dimensions it is not obvious that there is only one timelike dimension. It is, therefore, tempting to study anomalies in higher dimensions with an arbitrary signature. There have been many studies in higher-dimensional theories with more than one time dimension[6-8] and in some cases several time dimensions are desirable, e.g., with more than one time dimension it is possible to construct theories with zero cosmological constant.[7] It has been shown that an arbitrary change of signature in supersymmetric theories is not allowed and only certain time dimensions make it possible for supersymmetry to exist.[8] It is also not clear whether the naive Wick rotation can be applied to bring a Euclidean space to a space-time with more than one time dimension. A specific calculation may help to understand the structure of a space-time with arbitrary signature. The finite-temperature effect on the anomalies in higher dimensions with arbitrary signatures is also interesting to study. In four dimensions it has been shown that there is no temperature dependence for anomalies.[9] This is anticipated because anomalies are characterized by the topology of the gauge fields in a given space-time structure. The finite-temperature effects on the fields are continuously deformed from the zero-temperature case and therefore will not change the characteristics of topology of the fields. We find that this is also true in higher dimensions. In the following we will first carry out the calculation of anomalies in even dimension with an arbitrary signature and then study the temperature effect.

In D=even dimension with arbitrary signature, the anticommutation relation of the γ matrices is given by

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=g_{\mu\nu} , \tag{1}$$

where $g_{\mu\nu}=\mathrm{Diag}(1,\ldots,1,-1,\ldots,-1)$ in which t (time dimensions) of the diagonal elements are 1 and $D-t=s$ (space dimensions) of them are negative and off-diagonal elements are all zero. The minimal representation of γ is a $2^{D/2}$-dimensional matrix. The transpose and the complex conjugate of γ satisfy the same anticommutation relations. Because of the irreducibility of the D-dimensional Clifford algebra, there are always unitary transformations such that[10]

$$P\gamma_\mu P^{-1}=\gamma_\mu^\dagger,\quad P_1\gamma_\mu P_1^{-1}=-\gamma_\mu^\dagger , \tag{2}$$

and

$$C\gamma_\mu^T C^{-1}=-\gamma_\mu,\quad C_1\gamma_\mu^T C_1^{-1}=\gamma_\mu . \tag{3}$$

It is always possible to choose a convention such that $\gamma_t^\dagger=\gamma_t$, $\gamma_s^\dagger=-\gamma_s$ where t and s indicate the time and space dimensions, respectively. In the following we will work in this convention.

The Lagrangian for a free Dirac particle is

$$L=\int dx^D(\bar\psi i\partial\!\!\!/\psi-m_0\bar\psi\psi) , \tag{4}$$

where $\partial\!\!\!/=\gamma_\mu\partial^\mu$ and $\bar\psi=\psi^\dagger B^\dagger$. B is defined in such a way that L is Hermitian, so that we have

$$B\gamma_\mu^\dagger B=\gamma_\mu,\quad B=i(i)^{t(t+1)/2}\gamma_1\cdots\gamma_t \tag{5}$$

and that m_0 has a phase

$$m_0=(i)^{t+1}m , \tag{6}$$

where m is real and can be positive or negative.

Similar operators such as $1\pm\gamma_5$ in the four-dimensional space-time can also be defined in D dimensions and we have the analogous operator

$$1\pm\gamma,\quad \gamma=(i)^{t+1}\gamma_{D+1} , \tag{7}$$

where

$$\gamma_{D+1}=(i)^{D+t}\gamma_1\cdots\gamma_t\gamma_{t+1}\cdots\gamma_D,\quad (\gamma_{D+1})^2=1 . \tag{8}$$

The choice of γ is made such that when replacing ψ by $\frac{1}{2}(1\pm\gamma)\psi$ in Eq. (4), L remains Hermitian. Notice that when t is even Eq. (7) is not a projection operator.

The Lorentz group in our case is SO(t,s). Under Lorentz transformation, $x'_\mu=\Lambda_\mu{}^\nu x_\nu$ and ψ transforms as $S(\Lambda)\psi$ with $S(\Lambda)=\exp(-i\sigma_{\mu\nu}\omega^{\mu\nu}/4)$ where $\sigma_{\mu\nu}=(i/2)(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$ and $\omega^{\mu\nu}$ is an antisymmetric matrix. Lagrangian L of Eq. (4) remains invariant under the Lorentz transformation and the transformation properties of bispinor products can be worked out easily.

We can define parity, charge conjugation, and time reversal in space with an arbitrary signature in a manner analogous to Minkowski space. Parity transformation changes all signs of space dimensions and for spinors, $\psi_P(t,x)=P\psi(t,-x)$ and $\gamma_\mu^\dagger=P\gamma_\mu P^{-1}$ where P is defined in the first equation in Eq. (2). Under charge conjugation C, $\psi^c=CB^*\psi^*$ where C satisfies $C\gamma_\mu^T C^{-1}=-\gamma_\mu$ and time-reversal operation T will change all signs of time dimensions and for spinors $\psi_T(t,x)=T\psi(-t,x)$ where T satisfies $T^\dagger\gamma_\mu^* T=\gamma_\mu^\dagger$. We find C is given by Eq. (3) and $T=C_1$. With these definitions of C, P, and T, L remains invariant under CPT. With proper definition of the transformation properties of bosons under C, P, and T, it can be shown that there is a CPT theorem analogous to the CPT theorem in Minkowski space.

The transformation $e^{i\alpha\gamma}$ is analogous to the chiral transformation $e^{i\alpha\gamma_5}$ in four-dimensional space-time. Under this transformation L is invariant in the limit $m=0$.

We now study the anomaly due to the transformation $e^{i\alpha\gamma}$ in a theory in which the fermions are coupled to a gauge field A^μ. The Lagrangian is given by

$$L=\int dx^D[\bar\psi \not{D}\psi-(i)^{t+1}m\bar\psi\psi] , \tag{9}$$

where $\not{D}=\gamma_\mu(i\partial^\mu+gA^\mu)$, $A^\mu=A^{a\mu}T^a$, and T^a the group generator which is normalized as $\mathrm{Tr}T^aT^b=\frac{1}{2}\delta_{ab}$. The path integral of the fermions is given by

$$W=\int D\bar\psi D\psi \exp\left[i\int dx^D[\bar\psi\not{D}\psi-(i)^{t+1}m\bar\psi\psi]\right]$$
$$=\exp[J(m)] , \tag{10}$$

where $J(m)=\mathrm{Tr}\ln[i\not{D}-i(i)^{t+1}m]$.

To make the trace meaningful, one needs to regulate the ultraviolet divergence. We choose the Pauli-Villars regularization to subtract a term $J(M)$ identical in form to the original one but with a Ψ particle which obeys boson statistics and has an infinite mass M. The regularized J is

$$J=J(m)-J(M) . \tag{11}$$

Under an $e^{i\alpha\gamma}$ transformation,

$$J\to J(\alpha)=J(m,\alpha)-J(M,\alpha) ,$$

where $J(m,\alpha)=\mathrm{Tr}\ln[i\not{D}-i(i)^{t+1}me^{2i\alpha\gamma}]$.

We have

$$\int dx^D[\partial^\mu\bar\psi\gamma_\mu\gamma\psi-i2(i)^{t+1}m\bar\psi\gamma\psi]=A , \tag{12}$$

where the anomaly A is given by

$$A=-2\,\mathrm{Tr}\frac{(i)^{t+1}M}{\not{D}-(i)^{t+1}M}\gamma$$
$$=-2\,\mathrm{Tr}\frac{(-1)^{t+1}M^2}{\not{D}^2-(-1)^{t+1}M^2}\gamma . \tag{13}$$

To evaluate A we choose a plane-wave basis. We thus have

$$A=\int dx^D\int dk^D(-2)\frac{1}{(2\pi)^D}e^{-ikx}\mathrm{Tr}\frac{(-1)^{t+1}M^2}{\not{D}^2-(-1)^{t+1}M^2}\gamma e^{ikx} . \tag{14}$$

In the limit $M=\infty$, the nonvanishing part of A is given by

$$A=\int dx^D(-2)\mathrm{Tr}\gamma(\gamma_\mu\gamma_\nu g\partial^\mu A^\nu)^{D/2}\int dk^D\frac{1}{(2\pi)^D}\frac{(-1)^{t+1}M^2}{[k^2-(-1)^{t+1}M^2]^{D/2+1}} . \tag{15}$$

Using the identities

$$\mathrm{Tr}\gamma\gamma_{\mu_1}\gamma_{\nu_1}\cdots\gamma_{\mu_{D/2}}\gamma_{\nu_{D/2}}=i^{D/2+1}(-1)^t(-2)^{D/2}\epsilon_{\mu_1\nu_1\cdots\mu_{D/2}\nu_{D/2}} \tag{16}$$

and

$$I^D=\int dk^D\frac{1}{(2\pi)^D}\frac{(-1)^{t+1}M^2}{[k^2-(-1)^{t+1}M^2]^{D/2+1}}=\frac{\pi^{D/2}}{(2\pi)^D}\frac{(-1)^{D/2+1}i^t}{(D/2)!} , \tag{17}$$

where ϵ is the totally antisymmetric tensor with $\epsilon_{12\cdots D}=1$, the final expression for the anomaly is

$$A=\int dx^D 2(-1)^t(i)^{t+D/2+1}\frac{1}{(4\pi)^{D/2}(D/2)!}\epsilon_{\mu_1\nu_1\cdots\mu_{D/2}\nu_{D/2}}F^{\mu_1\nu_1}\cdots F^{\mu_{D/2}\nu_{D/2}} , \tag{18}$$

where $F^{\mu\nu}=g(\partial^\mu A^{a\nu}-\partial^\nu A^{a\mu}+gif^{abc}A^{b\mu}A^{c\nu})T^a$. We see that the Abelian anomaly explicitly depends on the signature.

For the non-Abelian anomaly, the evaluation is more complicated. Consider the chiral Lagrangian

$$L=\int dx^D(\bar\psi\not{D}\psi-m_0\bar\psi\psi) , \tag{19}$$

where $\not{D}=\gamma_\mu[i\partial^\mu+gA^\mu\frac{1}{2}(1-\gamma)]$.

This theory admits the existence of two distinct currents, the covariant and the consistent currents.[4] Formally these two currents are identical:

$$L_\mu^a=\frac{1}{g}\frac{\delta L}{\delta A^{a\mu}}=\int dx^D\bar\psi\gamma_\mu\tfrac{1}{2}(1-\gamma)T^a\psi$$
$$=\tfrac{1}{2}(J_\mu^a-J_{D+1\mu}^a) . \tag{20}$$

At the classical level this current transforms covariantly, under a gauge transformation:

$$A^{\mu} \rightarrow A'^{\mu} = A^{\mu} - D^{\mu}\alpha \ . \tag{21}$$

At the quantum level, however, in order to correctly specify the current, we have to invoke a regularization prescription to control the ultraviolet divergence. This regularization will determine if this current is covariant or consistent. Using a Pauli-Villars regularization and carrying out a similar calculation as for an Abelian anomaly, one obtains the covariant anomaly G^a:

$$G^a = D^{\mu} J^a_{D+1\mu} = 2\,\mathrm{Tr}\,T^a \gamma \frac{(-1)^{t+1} M^2}{\not{D}^2 - (-1)^{t+1} M^2} \ , \tag{22}$$

we have a similar form for G^a as the Abelian anomaly, i.e.,

$$G^a = -\int dx^D 2(-i)^t (i)^{D/2+t+1} \frac{1}{(4\pi)^{D/2}(D/2)!} \epsilon_{\mu_1\nu_1 \cdots \mu_{D/2}\nu_{D/2}} \mathrm{Tr}\,T^a F^{\mu_1\nu_1} \cdots F^{\mu_{D/2}\nu_{D/2}} \ . \tag{23}$$

This anomaly does not satisfy the consistent condition.[11] There is a difference X^a_μ between $J^a_{D+1\mu}$ and the consistent current $J^a_{c\mu}$,

$$J^a_{c\mu} = J^a_{D+1\mu} + X^a_\mu \ , \tag{24}$$

and X^a_μ satisfies the condition

$$\int dx^D A^{a\mu} X^a_\mu = 0 \ , \tag{25}$$

irrespective of whether or not the current that enters in the construction of $J^a_{c\mu}$ is conserved. X^a_μ has been constructed in higher dimensions.[3,4,12] We will use the representation given in Ref. 12:

$$\begin{aligned} J^a_{c\mu} = & -i\,\mathrm{Tr}\gamma_\mu \gamma T^a \frac{1}{\not{D} - (i)^{t+1} m} - \frac{1}{\not{D} - (i)^{t+1} M} \\ & - \frac{i}{g}\int_0^g dg' g' \mathrm{Tr}\gamma_\mu \gamma T^a \left[\frac{1}{\not{D} - (i)^{t+1} M} - \frac{1}{\not{D} + (i)^{t+1} M} \right] \\ & \times A \frac{1}{\not{D} - (i)^{t+1} M} \ . \end{aligned} \tag{26}$$

The first term corresponds to the covariant current and X^a_μ is given by

$$\begin{aligned} X^a_\mu = & -\frac{i}{g}\int_0^g dg' g' \mathrm{Tr}\gamma_\mu \gamma T^a \left[\frac{1}{\not{D} - (i)^{t+1} M} - \frac{1}{\not{D} + (i)^{t+1} M} \right] A \frac{1}{\not{D} - (i)^{t+1} M} \\ = & -\frac{i}{g}\int_0^g dg' g' \mathrm{Tr}\gamma_\mu \gamma T^a \frac{2(-1)^{t+1} M}{\not{D}^2 - (-1)^{t+1} M^2} A \frac{M}{\not{D}^2 - (-1)^{t+1} M^2} \ . \\ = & \int dk^D \left[-\frac{i}{g} \right] (i)^{D/2+1} (-1)^t (-2)^{D/2} \epsilon_{\mu\sigma\alpha_1\beta_1 \cdots \alpha_{(D/2-1)}\beta_{(D/2-1)}} \\ & \times 4 \int_0^g dg' g' \mathrm{Tr} \sum_{m=0}^{D/2-1} [T^a (F^{\alpha_i\beta_i})^m A^\sigma (F^{\alpha_j\beta_j})^{D/2-m-1}] I^D \ . \end{aligned} \tag{27}$$

The consistent anomaly is given by

$$\begin{aligned} \tilde{G}^a = & D^\mu J^a_{c\mu} = D^\mu J^a_{D+1\mu} + D^\mu X^a_\mu \\ = & -(i)^{D/2+1}(-1)^t(-2)^{D/2} \int dx^D \Bigg[4\frac{1}{g} \epsilon_{\mu\sigma\alpha_1\beta_1 \cdots \alpha_{(D/2-1)}\beta_{(D/2-1)}} \\ & \times \int_0^g dg' g'(g'-g) \mathrm{Tr} \left[[T^a, A^\mu] \sum_{m=0}^{D/2-1} (F^{\alpha_i\beta_i})^m A^\sigma (F^{\alpha_j\beta_j})^{D/2-m-1} \right] \\ & -2\int_0^g dg' \epsilon_{\mu_1\nu_1 \cdots \mu_{D/2}\nu_{D/2}} \mathrm{Tr}\,T^a F^{\mu_1\nu_1} \cdots F^{\mu_{D/2}\nu_{D/2}} \Bigg] I^D \ . \end{aligned} \tag{28}$$

Let us now investigate the finite-temperature effect on the anomalies G^a and $\tilde{G}^a$. We identify one of the time dimensions, e.g., t_1, to correspond to the ordinary Minkowski time and temperature is defined in the usual sense. We use the imaginary formalism of the field theory at finite temperature. In this formalism the integration of the momentum $\int dk^D$ is replaced by

$$i \sum_{n=-\infty}^{\infty} \int dk^D \frac{2\pi}{\beta} \delta \left[k_1 - i\frac{2\pi}{\beta}(n+\tfrac{1}{2}) \right]$$

(Ref. 13) where β is the inverse temperature. Notice that for all the anomalies we discussed before there is a common factor I^D in the momentum integral. The temperature dependence of I^D will determine the temperature effect on the anomalies. At finite temperature we have $I^D \rightarrow I^D(\beta)$,

$$I^D(\beta)=\sum_{n=-\infty}^{\infty}\int dk^D\frac{2\pi}{\beta}i\delta\left[k_1-i\frac{2\pi}{\beta}(n+\tfrac{1}{2})\right]\frac{1}{(2\pi)^D}\frac{(-1)^{t+1}M^2}{[k^2-(-1)^{t+1}M^2]^{D/2+1}}$$
$$=\sum_{n=-\infty}^{\infty}\int dk^{D-1}\frac{2\pi}{\beta}\frac{i}{(2\pi)^D}\frac{(-1)^{t+1}M^2}{\left[-\left[\frac{2\pi}{\beta}(n+\tfrac{1}{2})\right]^2+k'^2-(-1)^{t+1}M^2\right]^{D/2+1}}\,, \tag{29}$$

where $k'^2=k_2^2+\cdots+k_t^2-k_{t+1}^2-\cdots-k_D^2$. We note that

$$|I^D(\beta)-I'|<\left|\int dk^{D-1}\frac{2\pi}{\beta}\frac{i}{(2\pi)^D}\frac{(-1)^{t+1}M^2}{\left[-\left[\frac{\pi}{\beta}\right]^2+k'^2-(-1)^{t+1}M^2\right]^{D/2+1}}\right|\,, \tag{30}$$

where

$$I'=\int_{-\infty}^{\infty}dx\int dk^{D-1}\frac{2\pi}{\beta}\frac{i}{(2\pi)^D}\frac{(-1)^{t+1}M^2}{\left[-\left[\frac{2\pi}{\beta}x\right]^2+k'^2-(-1)^{t+1}M^2\right]^{D/2+1}}\,.$$

If we integrate over x and k we obtain

$$I'=I^D\,. \tag{31}$$

The right-hand side of Eq. (30) is $O(1/M)$. In the limit $M=\infty$, we have

$$I^D(\beta)=\Gamma$$

and therefore

$$\lim_{M\to\infty}I^D=\lim_{M\to\infty}I^D(\beta)\,. \tag{32}$$

We have the same expression for the anomalies at finite temperature as the ones at zero temperature. We thus confirm that the anomaly does not depend on temperature.

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