Shapiro-Virasoro amplitude and string field theory

Long Hua
Physics Department, New York University, New York, New York 10003

Michio Kaku
Institute for Advanced Study, Princeton, New Jersey 08540
and Physics Department, City College of the City University of New York, New York 10031
(Received 17 July 1989; revised manuscript received 19 January 1990)

One of the embarrassments of covariant string field theory has been the glaring failure to derive the Shapiro-Virasoro amplitude. Modular invariance appears explicitly violated: either the fundamental region is overcounted an infinite number of times, or it is undercounted because of a missing region. We try to approach this problem from a fresh point of view. Conventional wisdom holds that, in string field theory, the Veneziano amplitude can only be derived either in light-cone string field theory or Witten's string field theory. We show that this firmly held belief is actually wrong, that the Veneziano amplitude can actually be derived using vertices of *arbitrary* lengths. This is a highly nontrivial calculation. Using third elliptic integrals we show that a series of "miracles" occurs which allow us to cancel scores of unwanted terms in the measure, leaving us with the correct Koba-Nielsen variable and measure. We give three independent proofs of our result. When we generalize our results to closed-string scattering with arbitrary lengths, we find a new surprise, that we can successfully derive the Shapiro-Virasoro amplitude as long as a crucial four-string interaction term is added. We check by explicit computer calculation that we reproduce the correct region of integration for the four-closed-string amplitude. Crucial to the theory is the existence of the missing four-string tetrahedron graph, which precisely fills the missing integration region. We comment on the implications of this for geometric string field theory.

I. INTRODUCTION

Years ago, it was shown that the field theory of closed strings[1] in the light-cone gauge successfully reproduced the Shapiro-Virasoro amplitude. Both the measure and the region of integration were successfully reproduced.

Recently, Witten[2] proposed a successful covariant theory of open strings. Giddings[3,4] then showed that Witten's theory correctly reproduced the Veneziano amplitude. He showed this in two different ways.

(1) Using third elliptic integrals he constructed the explicit map from the upper-half plane to the four-open-string scattering world sheet.[3] He then was able to show that the measure of integration, expressed as a Jacobian from the variable τ to the Koba-Nielsen variable x, could precisely reproduce the usual Veneziano amplitude.

(2) Using the theory of Beltrami differentials,[4] he was able to generalize this proof to all orders in perturbation theory.

Attempts to generalize this to closed strings, however, have met with disappointment, because the symmetric closed-string configuration necessarily undercounts the fundamental region.[4] Similarly, the standard method of tracing over tachyon vertices overcounts the fundamental region an infinite number of times. Elaborate investigations exhausting a large class of three-string vertices have shown that the fundamental region is crucially undercounted.[5,6]

We feel that a fresh new approach to the problem is warranted, and that we must question some of our cherished beliefs. One of these beliefs is that string field theory can derive the Veneziano amplitude only via either the light-cone field theory[1] or Witten's string field theory.[2] In this paper, we show that this belief is actually wrong, that we can derive the Veneziano amplitude using strings of *arbitrary* lengths. This is a highly nontrivial result. In fact, in the beginning, we were skeptical ourselves whether this was really true. Therefore, to convince ourselves of the correctness of this result, we proved it in three different ways. In the first method, we use the theory of third elliptic integrals to derive the explicit conformal maps and the explicit measure which allows us to reproduce the Veneziano amplitude. In the second method, we use the method of Beltrami differentials to show that the amplitude is actually invariant under changes of the string lengths, contrary to accepted belief. In the third method, we use the "method of reflections" to show explicitly how to go smoothly from the light-cone four-string amplitude to the Witten four-string amplitude using an "interpolating vertex" which allows us to interpolate between these two theories.

However, the main surprise comes when we apply our results to the closed-string case. We find that, when calculating the closed-string scattering amplitude with strings of arbitrary lengths, we can rederive the Shapiro-Virasoro amplitude if we add an explicit four-string interaction, the tetrahedron graph.

Thus, the calculation of the Shapiro-Virasoro amplitude, using our techniques, shows that string lengths can be varied at will, as long as we keep track of the four-string interaction.

The original motivation of this paper was to see how geometric string field theory (which is manifestly reparametrization independent and hence independent of string lengths), when gauge fixed, could rederive the Veneziano and Shapiro-Virasoro amplitude. The reader who is interested in seeing how gauging-fixing geometric string field theory yields scattering amplitudes with varying string lengths is advised to consult Refs. 7–9, which are beyond the scope of this paper. Although this paper was motivated by geometric string field theory, the paper can stand by itself. We have, therefore, deliberately written this paper so that it can be read independent of geometric string field theory.

II. CALCULATION OF THE MEASURE

Let us begin with the following conformal map,[7,8] which takes us from the complex plane to a multiply connected sheet representing the three-string vertex:

$$\rho(z)=\alpha_1\ln(z-1)+\alpha_2\ln z + \sum_{r=1}^{3}\beta_r\ln[(az^2+bz+c)^{1/2}+(a_rz+b_r)]\ , \tag{2.1}$$

where

$$a_1=\frac{\alpha_1^2-\alpha_2^2+\alpha_3^2}{2\alpha_1},\quad a_2=\frac{-a_1^2+\alpha_2^2+\alpha_3^2}{2\alpha_2},\quad a_3=-\alpha_3 \tag{2.2}$$

$$b_1=\frac{\alpha_1^2+\alpha_2^2-\alpha_3^2}{2\alpha_1},\quad b_2=-\alpha_2,\quad b_3=\frac{\alpha_1^2-\alpha_2^2-\alpha_3^2}{-2\alpha_3}\ , \tag{2.3}$$

$$a=\alpha_3^2,\quad b=\alpha_1^2-\alpha_2^2-\alpha_3^2,\quad c=\alpha_2^2,\quad \beta_r=-\alpha_r\ , \tag{2.4}$$

where the string lengths $\alpha_1,\alpha_2,\alpha_3$ are totally arbitrary. It is straightforward, but subtle, to show that we can take different limits on this map and arrive at the old light-cone map as well as the map for Witten's three-string vertex (Fig. 1). For example, if we carefully take the limit that $\sum_{r=1}^{3}\alpha_r\to 0$ we find

$$\rho(z)\approx\alpha_1\ln(z-1)+\alpha_2\ln z\ , \tag{2.5}$$

which is the usual light-cone map for the three vertex (Fig. 2). In the limit that all three string lengths become equal, we find the limit

$$\rho(z)\approx\ln\frac{(z+1)(z-2)(-\frac{1}{2})+(z^2-z+1)^{3/2}}{z(1-z)}\ .$$

This, by itself, is rather surprising. Intuitively, we may have expected that taking the limit of the interpolating map (2.1) would yield singularities as we approached either the light-cone or Witten's map, preventing a unified description of both vertices in terms of a single interpolating vertex. However, a careful limiting process shows that there are no singularities whatsoever, that the limiting process is smooth.

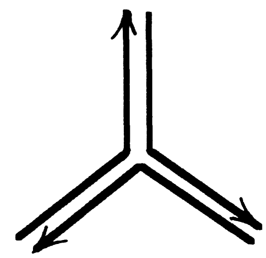

FIG. 1. In the midpoint gauge, three open strings in a vertex have equal parametrization length and join at the midpoint.

From the interpolating map, it is now a straightforward matter to construct the Neumann functions associated with this map, and then construct the vertex function $|V_{\alpha_1\alpha_2\alpha_3}\rangle$, which we call the interpolating vertex (Fig. 3).

The heart of this paper is whether or not we can rederive the usual Veneziano and Shapiro-Virasoro amplitude by taking the following contraction for the process $1+2\to 5\to 3+4$:

$$\int d\tau\langle V_{\alpha_1\alpha_2\alpha_5}|b_0e^{-\tau(L_0-1)}|V_{\alpha_5\alpha_3\alpha_4}\rangle\ , \tag{2.6}$$

where the external α_i are actually *arbitrary;* they are neither set equal to unity nor do they sum to zero. (The price we pay for this generality, we will see, is that the region of integration is not invariant.) The interpolating vertex $|V_{\alpha_1\alpha_2\alpha_3}\rangle$ satisfies the property

$$\prod_{i=1}^{3}\prod_{0<\sigma_i<a_{i,i-1}}\delta(X_i(\sigma_i)-X_{i-1}(\pi\alpha_i-\sigma_i))|V_{\alpha_1\alpha_2\alpha_3}\rangle=0\ , \tag{2.7}$$

where a_{ij} is the parametrization overlap between the ith and jth strings.

We have shown that this vertex function smoothly interpolates between the usual vertex functions defined in the midpoint and end-point configurations:

$$\begin{aligned}\lim_{\sum\alpha_i=0}|V_{\alpha_1\alpha_2\alpha_3}\rangle_{\text{interpolating}}&=|V_{\alpha_1\alpha_2\alpha_3}\rangle_{\text{end point}}\ ,\\ \lim_{|\alpha_i|=\alpha}|V_{\alpha_1\alpha_2\alpha_3}\rangle_{\text{interpolating}}&=|V_{\alpha\alpha\alpha}\rangle_{\text{midpoint}}\ .\end{aligned} \tag{2.8}$$

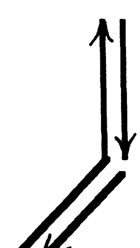

FIG. 2. In the end-point gauge, three open strings in a vertex have parametrization lengths which sum to zero, and join at their end points.

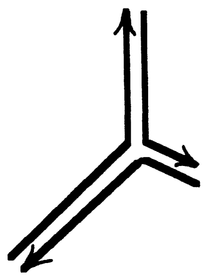

FIG. 3. In the interpolating gauge, three open strings in a vertex have arbitrary parametrization lengths. The interpolating vertex contains both the midpoint and the end-point vertices as special cases.

For closed-string interactions, we have almost an identical formalism, except the propagator now becomes

$$D=b_0\bar{b}_0(L_0+\bar{L}_0-2)^{-1}\int_0^{2\pi}d\theta\, e^{i\theta(L_0-\bar{L}_0)} \tag{2.9}$$

and the vertex becomes a function of two sets of independent operators. (We will treat mainly open strings in this section, but the generalization to closed strings is straightforward and is given in the next section. The only nontrivial feature of the generalization to closed strings is the region of integration, where a new tetrahedron graph emerges.)

To calculate the tachyon scattering amplitude, we now make a conformal transformation of variables to the upper-half plane (entire complex plane), where we find[3]

$$A_4=\int_0^\infty A_G\left\langle \exp\left[i\sum_i P_jX(j)\right]\right\rangle d\tau\ , \tag{2.10}$$

where A_G is the ghost contribution. (In geometric string field theory, the ghost sector has a nice group-theoretical interpretation: it is just the tangent space of the theory.)

Crucial to the calculation of the measure for the Veneziano (Shapiro-Virasoro) amplitude is the conformal map which takes us from the upper-half plane (entire complex plane) to the world sheet for the scattering of four open (closed) strings.

The conformal map is generated by the Schwartz-Christoffel transformation[4]

$$\frac{d\rho}{dz}=N\frac{[(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)]^{1/2}}{(z-\gamma_1)(z-\gamma_2)(z-\gamma_3)(z-\gamma_4)}\ , \tag{2.11}$$

where

$$\begin{aligned} &\gamma_i=(0,x,1,\infty)\ ,\\ &z_1=(ia_1+b_1),\quad \bar{z}_1=(-ia_1+b_1)\ ,\\ &z_2=(ia_2+b_2),\quad \bar{z}_2=(-ia_2+b_2)\ , \end{aligned} \tag{2.12}$$

where the overbar represents complex conjugation for open strings, while for closed strings the overbar does not (a_i and b_i are themselves complex variables for closed strings).

Notice that there is a Riemann cut between z_1 and $\bar{z}_1$ and another Riemann cut between z_2 and $\bar{z}_2$. (See Figs. 4 and 5.) These cuts, in turn, make the definition (2.11) well defined. Notice that the conformal map transforms γ_i in the z plane into one of the points at infinity in the ρ plane. Thus, we can set the external lengths of the strings by

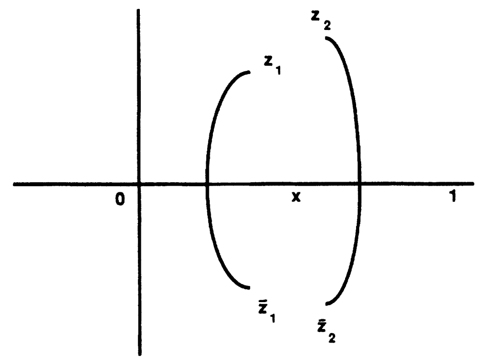

FIG. 4. Riemann surface for four-open-string scattering in the z plane. There are Riemann cuts which extend symmetrically from the upper- to the lower-half plane.

$$\frac{d\rho}{dz}\approx\frac{\alpha_i}{z-\gamma_i}\ , \tag{2.13}$$

where z is near γ_i. This, in turn, gives us four constraints:

$$\alpha_i=N\frac{\sqrt{(\gamma_i-b_1)^2+a_1^2}\,\sqrt{(\gamma_i-b_2)^2+a_2^2}}{\prod_{\substack{j=1\\ j\neq i}}^{4}(\gamma_i-\gamma_j)}\ , \tag{2.14}$$

where

$$\alpha_1+\alpha_2-2\delta_1=|\alpha_3|+|\alpha_4|-2\delta_2\ . \tag{2.15}$$

Notice that we have the midpoint or end-point configuration in the following limits:

$$\begin{aligned} &\text{midpoint:}\quad |\alpha_i|=1,\quad \delta_i=\tfrac{1}{2}\ ,\\ &\text{end point:}\quad \sum_{i=1}^{4}\alpha_i=0,\quad \delta_i=0\ . \end{aligned} \tag{2.16}$$

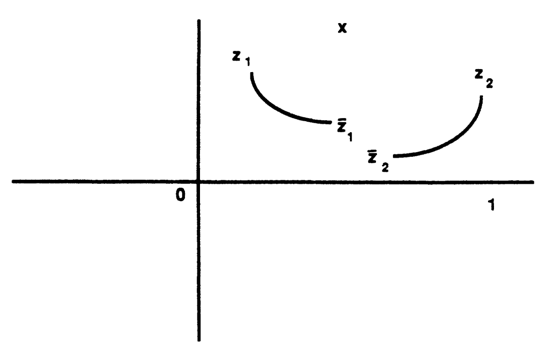

FIG. 5. Riemann surface for four-closed-string scattering in the z plane. Notice that the Riemann cuts are totally arbitrary.

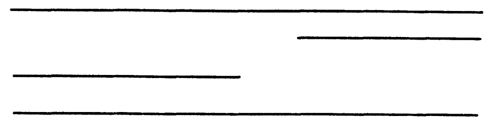

FIG. 7. Riemann surface for the four-open-string scattering in the end-point gauge.

Notice that by changing the value of δ_i, we can smoothly interpolate between the end-point and midpoint configurations. δ_i in effect measures the deviation away from being a planar graph. When δ_i equals zero, we have the usual light-cone configuration when the external lengths sum to zero. When δ_i equals $\frac{1}{2}$, then we have the midpoint configuration. Much more complicated is the interpolating configuration, where δ_i is arbitrary (see Figs. 6–8).

From the map, we can derive the expression for τ and δ_i:

$$\begin{aligned} \tau &= \mathrm{Re}[\rho(z_2)-\rho(z_1)] \ , \\ \pi\delta_i &= \mathrm{Im}[\rho(z_i)-\rho(0)] \ . \end{aligned} \tag{2.17}$$

The counting of the unknowns and constraints is important here, especially for the computer calculation. For open strings, we have five unknowns and five constraints:

5 unknowns: $(N,z_1,\bar{z}_1,z_1,\bar{z}_2)$,

5 constraints for (α_i,δ_1) .

[We do not have to fix the value of δ_2 because this is already fixed by (2.15).] For closed strings, the overbar no longer denotes complex conjugation, so the z's now contain eight independent unknowns, and N also becomes complex. Furthermore, the constraints for the α_i and δ_1 are now complex, so we have ten equations for them. (Because these string lengths must be real, the imaginary part of their constraints must be set to zero.) In total, we have ten unknowns and ten constraints:

10 unknowns: $(N,z_1,\bar{z}_1,z_2,\bar{z}_2)$,

10 constraints for (α_i,δ_1) .

It is essential to notice that the constraints, because they involve cuts and are highly coupled, must be solved implicitly. This means that there are probably no analytic solutions, except in exceptional situations with a high degree of symmetry. For the arbitrary case, we are forced to use a computer.

In spite of the complexity of the conformal map, we will show in this section that it is possible to derive an exact analytic expression for the Jacobian from τ to the Koba-Nielsen variable x, where

$$x=\frac{(\gamma_2-\gamma_1)(\gamma_3-\gamma_4)}{(\gamma_2-\gamma_4)(\gamma_3-\gamma_1)} \ . \tag{2.18}$$

To begin, we perform the integration in (2.11). To do this, we must rewrite the expression in a more manageable form. It takes a bit of work to prove

$$\frac{(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)}{\prod_{i=1}^{4}(z-\gamma_i)}=1+\sum_{i=1}^{4}\frac{A_i}{z-\gamma_i} \ , \tag{2.19}$$

where

$$A_i=\frac{[(\gamma_i-b_1)^2+a_1^2][(\gamma_i-b_2)^2+a_2^2]}{\prod_{\substack{j=1 \\ j\neq i}}^{4}(\gamma_i-\gamma_j)} \ . \tag{2.20}$$

(The simplest way of checking our expression for A_i is to take the limit as $z\to\gamma_i$. Most terms in the product vanish, leaving us only the expression for A_i. By cyclic symmetry, we then have all the A_i.)

Thus, the integral we wish to calculate is given by

$$\rho(z)=\int\frac{N\,dz}{\sqrt{(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)}}+\sum_{i=1}^{4}\int\frac{NA_i\,dz}{(z-\gamma_i)\sqrt{(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)}} \ . \tag{2.21}$$

With some work, we can reexpress everything in terms of elliptic integrals of the first, second, and third kinds, which was first done in Ref. 7. To standardize our notation, we will use extensively the identities contained within Ref. 12. For reference, we will also give the equa-

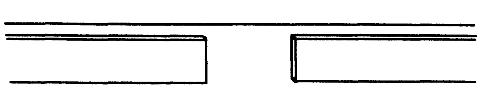

FIG. 6. Riemann surface for the four-open-string scattering amplitude in the midpoint gauge.

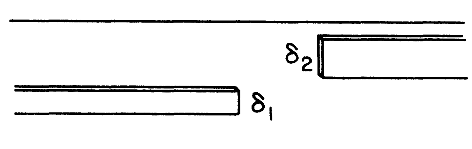

FIG. 8. Riemann surface for the four-open-string scattering in the interpolating gauge. This amplitude smoothly interpolates between the amplitudes for midpoint scattering and end-point scattering.

tion number used in Ref. 10.

First, let us define our notation. We define a first elliptic integral as

$$F(\phi,k)=\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} =\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} =\int_0^{u_1} du =u_1 =\mathrm{sn}^{-1}(y,k) , \quad (2.22)$$

where

$$y=\sin\phi, \quad \phi=amu_1 . \quad (2.23)$$

We define the second elliptic integral as

$$E(\phi,k)=\int_0^y \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt =\int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta =\int_0^{u_1} \mathrm{dn}^2u\,du . \quad (2.24)$$

Third elliptic integrals are defined as

$$\Pi(\phi,\alpha^2,k)=\int_0^y \frac{dt}{(1-\alpha^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}} =\int_0^\phi \frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} =\int_0^{u_1} \frac{du}{1-\alpha^2\mathrm{sn}^2u} . \quad (2.25)$$

Complete first elliptic integrals are defined as

$$K(k)=K=\int_0^{1/2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=F(\tfrac{1}{2}\pi,k) . \quad (2.26)$$

Complete second elliptic integrals are given as

$$E(\tfrac{1}{2}\pi,k)=E=\int_0^{1/2\pi}\sqrt{1-k^2\sin^2\theta}\,d\theta . \quad (2.27)$$

Heuman's lambda function is defined as

$$\Lambda_0(\phi,k)=\frac{2}{\pi}[EF(\phi,k')+KE(\phi,k')-KF(\phi,k')] , \quad (2.28)$$

where

$$k'^2=1-k^2 . \quad (2.29)$$

Finally, the Jacobi zeta function is defined as

$$Z(\phi,k)=E(\phi,k)-\frac{E}{K}F(\phi,k) . \quad (2.30)$$

With these definitions, we can now begin performing the integrations. We first note [Eq. (267.00)]

$$gF(\phi,k')=\int_{y_1}^y \frac{dz}{\sqrt{(z-z_1)(z-\bar z_1)(z-z_2)(z-\bar z_2)}} =g\,\mathrm{tn}^{-1}(\tan\phi,k') , \quad (2.31)$$

where

$$A^2=(b_1-b_2)^2+(a_1+a_2)^2 ,$$
$$B^2=(b_1-b_2)^2+(a_1-a_2)^2 ,$$
$$k'^2=\frac{4AB}{(A+B)^2} ,$$
$$g=\frac{2}{A+B} , \quad (2.32)$$
$$y_1=b_1-a_1g_1 ,$$
$$g_1^2=\frac{4a_1^2-(A-B)^2}{(A+B)^2-4a_1^2} ,$$
$$\phi=\arctan\left[\frac{y-b_1+a_1g_1}{a_1+g_1b_1-g_1y}\right] .$$

We also need the important integral [Eqs. (267.02) and (361.64)]

$$\int_{y_1}^y \frac{dz}{(z-\gamma_i)\sqrt{(z-z_1)(z-\bar z_1)(z-z_2)(z-\bar z_2)}} =\frac{g}{a_1+b_1g_1-g_1\gamma_i}\left[g_1F(\phi,k')+\frac{\omega_i-g_1}{1+\omega_i^2}[F(\phi,k')+\omega_i^2\Pi(\phi,1+\omega_i^2,k')+\omega_i(\omega_i^2+1)f_i]\right] , \quad (2.33)$$

where

$$\omega_i=\frac{a_1+b_1g_1-\gamma_ig_1}{b_1-a_1g_1-\gamma_i} \quad (2.34)$$

and

$$f_i=\tfrac{1}{2}(1+\omega_i^2)^{-1/2}(k^2+\omega_i^2)^{-1/2} \times\ln\frac{\sqrt{k^2+\omega_i^2}-\sqrt{1+\omega_i^2}\mathrm{dn}u}{\sqrt{k^2+\omega_i^2}+\sqrt{1+\omega_i^2}\mathrm{dn}u} , \quad (2.35)$$

where

$$\mathrm{dn}u=\sqrt{1-k'^2\sin^2\phi} . \quad (2.36)$$

(Because of the delicate nature of the analytic continuation of our results to complex values of the arguments, we give an independent proof of these important equations in Appendix A where we can check the validity of the analytic continuation.)

We are in a position to integrate our original expression (2.11). After integration, we find

$$\rho(z)=gNF(\phi,k')\left[1+\sum_{i=1}^{4}\frac{A_i}{a_1+b_1g_1-g_1\gamma_i}\times\left[g_1+\frac{\omega_i-g_1}{1+\omega_i^2}\right]\right]+\sum_{i=1}^{4}\frac{gNA_i}{a_1+b_1g_1-g_1\gamma_i}\frac{\omega_i-g_1}{1+\omega_i^2}\times[\omega_i^2\Pi(\phi,1+\omega_i^2,k')+\omega_i(\omega_i^2+1)f_i]\ . \tag{2.37}$$

At this point, we see for the first time how nontrivial this calculation really is. Notice that the map is exceedingly nonlinear, and that δ_1 and τ are defined implicitly by this integral. For this complicated expression to reduce to the Veneziano formula would require a series of "miracles." Fortunately, this series of miracles actually takes place.

The first simplification is to notice that, after a series of manipulations, the coefficient of $F(\phi,k')$ actually vanishes. To see this, let us write

$$\frac{g_1+\dfrac{\omega_i-g_1}{1+\omega_i^2}}{a_1+b_1g_1-g_1\gamma_i}=\frac{b_1-\gamma_i}{(b_1-\gamma_i)^2+a_1^2}=\mathrm{Re}\frac{1}{b_1+ia_1-\gamma_i}\ , \tag{2.38}$$

where we have used the fact that

$$1+\omega_i^2=\frac{[(b_1-\gamma_i)^2+a_1^2](1+g_1^2)}{(b_1-a_1g_1-\gamma_i)^2}\ .$$

This, in turn, allows us to write down the coefficient of $F(\phi,k')$ using (2.19):

$$g\ \mathrm{Re}\left[1+\sum_{i=1}^{4}\frac{A_i}{z-\gamma_i}\right]_{z=z_1}=g\ \mathrm{Re}\left[\frac{(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)}{\prod_{i=1}^{4}(z-\gamma_i)}\right]_{z=z_1}=0\ . \tag{2.39}$$

Notice that $\rho(z)$ is now written entirely in terms of third elliptic integrals and the function f_i.

Our next step is to impose the constraint coming from δ_1. We have

$$\pi\delta_1=\mathrm{Im}[\rho(z_1)-\rho(y_1)]\ , \tag{2.40}$$

where we have set $y=z_1$ in (2.33) and y_1 is a real number. This, in turn, fixes

$$\tan\phi=i\ . \tag{2.41}$$

Because ϕ is now complex, we must analytically continue the third elliptic integral, using [Eqs. (161.02) and (411.01)]

$$\Pi(i\tilde{\phi},\alpha^2,k)=\frac{i}{1-\alpha^2}[F(\tilde{\beta},k')-\alpha^2\Pi(\tilde{\beta},1-\alpha^2,k')]\ , \tag{2.42}$$

where $\tan\tilde{\beta}=\sinh\tilde{\phi}$ and

$$\Pi(\alpha^2,k)=\frac{K}{1-\alpha^2}+\tfrac{1}{2}\pi\alpha^2\frac{\Lambda_0(\beta,k)-1}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}\ . \tag{2.43}$$

Using these identities, we can write

$$\Pi(\phi,1+\omega_i^2,k')=-\tfrac{1}{2}\pi i\frac{\sqrt{1+\omega_i^2}}{\sqrt{k^2+\omega_i^2}}\frac{\Lambda_0(\beta_i,k)-1}{\omega_i}\ , \tag{2.44}$$

where

$$\sin^2\beta_i=\frac{1}{1+\omega_i^2} \tag{2.45}$$

and $\phi=i\tilde{\phi}$.

Let us now write down an expression for δ_1. In the limit that $\tan\phi=i$, we find that $\mathrm{dn}u$ goes to infinity. This, in turn, allows us to show

$$f_i\approx\ln\frac{-\mathrm{dn}u}{\mathrm{dn}u}\approx+i\pi\ . \tag{2.46}$$

Let us now put everything together and write down

$$\delta_1=-\tfrac{1}{2}gN\frac{(\omega_i-g_1)\omega_i}{a_1+b_1g_1-g_1\gamma_i}\sum_{i=1}^{4}\frac{A_i\Lambda_0(\beta_i,k)}{\sqrt{(1+\omega_i^2)(k^2+\omega_i^2)}}\ . \tag{2.47}$$

Fortunately, at this point another "miracle" occurs and we have

$$\frac{g(\omega_i-g_1)\omega_i}{a_1+b_1g_1-g_1\gamma_i}\frac{\sqrt{(\gamma_i-b_1)^2+a_1^2}\sqrt{(\gamma_i-b_2)^2+a_2^2}}{\sqrt{(1+\omega_i^2)(k^2+\omega_i^2)}}=1\ . \tag{2.48}$$

(We demonstrate this in Appendix B.)

As a result, we now have the simple equation

$$\delta_1=-\frac{1}{2}\sum_{i=1}^{4}\alpha_i[\Lambda_0(\beta_i,k)]\ . \tag{2.49}$$

Our next task is to calculate τ itself and the Jacobian which takes us to the Koba-Nielsen variable x. We define τ as

$$\tau=\mathrm{Re}[\rho(z_2)-\rho(z_1)]\ . \tag{2.50}$$

Because we are now taking $y=z_1,z_2$, we have

$$\tan\phi_1=i,\quad \phi_1=i\infty\ ,\qquad \tan\phi_2=\frac{i}{k},\quad \phi_2=\arcsin\frac{1}{k'}\ . \tag{2.51}$$

(We prove this in Appendix C.)

Again, several simplifications occur. First, we can eliminate the f_i contribution to τ because they are either purely imaginary constants or zero:

$$f_i(z_1)\approx\ln\frac{-\mathrm{dn}\,u}{\mathrm{dn}\,u}\approx i\pi\ ,\qquad f_i(z_2)\approx\ln 1\ . \tag{2.52}$$

Next, we calculate τ, we must compute the difference between two third elliptic integrals. Fortunately, we have the equation [Eq. (116.03)]

$$\Pi(\theta,\alpha^2,k)\pm\Pi(\beta,\alpha^2,k)=\Pi(\phi,\alpha^2,k)\pm\left[\frac{\alpha^2}{(\alpha^2-1)(\alpha^2-k^2)}\right]^{1/2}\operatorname{arctanh}\left[\frac{\sin\theta\sin\beta\sin\phi\sqrt{\alpha^2(\alpha^2-1)(\alpha^2-k^2)}}{1-\alpha^2\sin^2\phi+\alpha^2\sin\theta\sin\beta\cos\phi\sqrt{1-k^2\sin^2\phi}}\right], \tag{2.53}$$

where

$$\phi=2\arctan\left[\frac{\sin\theta\sqrt{1-k^2\sin^2\beta}\pm\sin\beta\sqrt{1-k^2\sin^2\theta}}{\cos\theta+\cos\beta}\right]. \tag{2.54}$$

Now let us substitute the values that are relevant to our calculation of τ in (2.51) into (2.53). We find

$$\Pi(\phi_2,1+\omega_i^2,k')-\Pi(\phi_1,1+\omega_i^2,k')=\Pi(\phi,1+\omega_i^2,k')\ . \tag{2.55}$$

Notice that we were able to eliminate the length term involving the tanh. This is because

$$\phi=2\arctan\frac{\sin\phi_2\sqrt{1-k'^2\sin^2\phi_1}-\sin\phi_1\sqrt{1-k'^2\sin^2\phi_2}}{\cos\phi_2+\cos\phi_1}=\tfrac{1}{2}\pi\ . \tag{2.56}$$

This, in turn, means that the tanh term can be eliminated because arctanh $\infty=i\frac{1}{2}\pi$ is imaginary.

Our expression for τ now becomes

$$\tau=g\sum_{i=1}^{4}NA_i\frac{\omega_i^2(\omega_i-g_1)}{(a_1+b_1g_1-g_1\omega_i)(1+\omega_i^2)}\Pi(\tfrac{1}{2}\pi,1+\omega_i^2,k')\ . \tag{2.57}$$

Last, we note that [Eq. (415.01)]

$$\Pi(\alpha^2,k)=-\frac{\alpha KZ(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\ , \tag{2.58}$$

where $A=\arcsin\alpha^{-1}$. Thus, our final expression for τ can be written in a surprisingly simple fashion as

$$\tau=-K(k')\sum_{i=1}^{4}\alpha_iZ(\beta_i,k')\ . \tag{2.59}$$

Our next step is to differentiate τ with respect to x in order to arrive at a closed expression for the measure. We need, however, a series of identities concerning how to differentiate the various elliptic integrals. Equations (710.00)[10] and (710.06)[10] give us

$$\frac{d}{dk'}[K(k')Z(\beta_i,k')]=\frac{k'E(k')}{k^2}\frac{\sin\beta_i\cos\beta_i}{r(\beta_i,k')}\ . \tag{2.60}$$

Equation (710.11)[10] gives us

$$\frac{d}{dk}\Lambda_0(\beta_i,k)=\frac{2}{\pi k}[E(k)-K(k)]\frac{\sin\beta_i\cos\beta_i}{r(\beta_i,k')}\ . \tag{2.61}$$

Equation (730.04)[10] gives us

$$\frac{d}{d\beta_i}[K(k')Z(\beta_i,k')]=\frac{r^2(\beta_i,k')K(k')-E(k')}{r(\beta_i,k')}\ . \tag{2.62}$$

Equation (730.04)[10] gives us

$$\frac{d}{d\beta_i}\Lambda_0(\beta_i,k)=\frac{2}{\pi r(\beta_i,k')}[E(k)-k'^2\sin^2\beta_iK(k)]\ , \tag{2.63}$$

where

$$r(\theta,k')=\sqrt{1-k'^2\sin^2\theta}\ . \tag{2.64}$$

We now use a very convenient formula, which will prove useful to our calculation of the Jacobian:

$$\sum_{i=1}^{4}\alpha_i\frac{\sin\beta_i\cos\beta_i}{r(\beta_i,k')}=0\ . \tag{2.65}$$

(We will prove this in Appendix D.)

With these formulas, we can now differentiate τ and δ_1 with respect to β_i:

$$\begin{aligned} d\tau&=-\sum_{i=1}^{4}\alpha_i\frac{r^2(\beta_i,k')K(k')-E(k')}{r(\beta_i,k')}d\beta_i\ ,\\ d\delta_1&=\sum_{i=1}^{4}\alpha_i\frac{E(k)-k'^2\sin^2\beta_iK(k)}{r(\beta_i,k')}d\beta_i=0\ . \end{aligned} \tag{2.66}$$

We also use [Eq. (110.10)]

$$K(k')E(k)+K(k)E(k')-K(k)K(k')=\tfrac{1}{2}\pi\ . \tag{2.67}$$

Now let us multiply the expression for $d\delta_1$ by $K(k')/K(k)$ and add this to the expression for $\delta\tau$. We then use (2.67) to find

$$d\tau=\tfrac{1}{2}\pi\frac{1}{K(k)}\sum_{i=1}^{4}\frac{\alpha_id\beta_i}{r(\beta_i,k')}\ . \tag{2.68}$$

Our final step is to compute the Jacobian itself. We first wish to convert $d\beta_i$ into $d\gamma_i$, and then $d\gamma_i$ into dx. It is easy to show from (2.34) and (2.45) that

$$d\beta_i=-\frac{a_1(1+g_1^2)d\gamma_i}{(1+\omega_i^2)(b_1-a_1g_2-\gamma_i)^2}\ . \tag{2.69}$$

It is not hard to show from (2.69) and from Appendix B that

$$\sum_{i=1}^{4}\frac{\alpha_i}{r(\beta_i,k')}=-\frac{N}{g}\frac{d\gamma_i}{\prod_{\substack{j=1\\ j\neq i}}(\gamma_i-\gamma_j)}\ . \tag{2.70}$$

This, in turn, allows us to write

$$d\tau = \frac{\pi N}{2gK(k)} \sum_{i=1}^{4} \frac{d\gamma_i}{\prod_{\substack{j=1\\ j\neq i}} (\gamma_i - \gamma_j)} . \tag{2.71}$$

Let us now calculate dx. Because of (2.18), it is easy to show

$$dx = x(1-x)\frac{(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)(\gamma_1-\gamma_4)} d\gamma_1 \tag{2.72}$$

if we choose γ_1 to be the independent variable. Thus, we can write

$$\frac{d\tau}{dx} = -\frac{\pi N}{2K(k)gx(1-x)(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)} . \tag{2.73}$$

In summary, although the original conformal map with external legs was complicated by the presence of the external constraints, we find that nontrivial identities allow us to cancel most of these terms, leaving us with surprisingly simple expressions for δ_1 in (2.49), τ in (2.59), and $d\tau/dx$ in (2.73).

Now, we calculate the amplitudes themselves.

III. CALCULATION OF THE VENEZIANO AND SHAPIRO-VIRASORO AMPLITUDES

We must combine this Jacobian factor with the ghost contribution coming from the contraction of the b_0 with the vertices. We know that b_0 transforms as a second-rank tensor under the conformal group, i.e., b_{zz}. Thus, we can calculate how it transforms under a conformal transformation.

In the world sheet of the four-string interaction, the ghost factor is a line integral which bisects the intermediate strip:

$$b_0 = \int d\sigma\, b_{zz}(\sigma) . \tag{3.1}$$

We can now make a conformal transformation from the world sheet of four interacting strings to the upper-half plane. Let us bosonize the ghost:

$$b_{ww} = e^{-i\phi_+(w)} . \tag{3.2}$$

Then the ghost factor transforms as

$$A_G = \int_C \frac{dz}{2\pi i}\frac{dz}{dw} \exp\left[-\sum_{j<k}\langle \phi_j\phi_k\rangle\right] \times \exp\left[\sum_j \langle \phi_j \phi_+(z)\rangle\right] , \tag{3.3}$$

where the curve C in the w plane is a vertical line which bisected the propagator, while the transformed curve C in the z plane is a curved line which encircles the Riemann cut. Because the function we are integrating is analytic, we can rearrange the curve so that it wraps around the Riemann cut, i.e., becomes twice the integral from some real number y_1 to z_1. We find

$$\begin{aligned} \langle \phi_j\phi_k\rangle &= -\tfrac{1}{2}(\ln|z_j-z_k| + \ln|z_j-\bar{z}_k|) , \\ \langle \phi_j\phi_+(z)\rangle &= -\tfrac{1}{2}[\ln(z_j-z)+\ln(z_j-\bar{z})] . \end{aligned} \tag{3.4}$$

Inserting the values of z_i, we find

$$\begin{aligned} A_G &= \int_C \frac{dz}{2\pi i}\frac{dz}{dw}\frac{\prod_{i<j}(\gamma_i-\gamma_j)}{\prod_{j=1}^{4}(z-\gamma_j)} \\ &= 2\int_{y_1}^{z_1}\frac{dz}{2\pi i N}\frac{x(1-x)(\gamma_1-\gamma_3)^3(\gamma_2-\gamma_4)^3}{\sqrt{(z-b_1)^2+a_1^2}\sqrt{(z-b_2)^2+a_2^2}} \\ &= 2\frac{g}{\pi N}x(1-x)K(k)(\gamma_1-\gamma_3)^3(\gamma_2-\gamma_4)^3 , \end{aligned} \tag{3.5}$$

where we have used the fact that the integral along the cut reproduces a first complete elliptic integral from (2.31). We now have the final result

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{2g}{\pi N}K(k)x(x-1)(\gamma_1-\gamma_3)(\gamma_2-\gamma_4) \\ &= A_G(\gamma_1-\gamma_3)^{-2}(\gamma_2-\gamma_4)^{-2} , \end{aligned} \tag{3.6}$$

where g is given by (2.32). Thus, we have a precise cancellation between the measure and the ghost contribution in the interpolating configuration.

Let us now put everything together to derive the Veneziano amplitude. The amplitude we wish to calculate is given by

$$A_4 = \int d\tau\, A_G \exp\left[-\sum_{i<j} P_i\cdot P_j \langle X(i)X(j)\rangle\right] . \tag{3.7}$$

The X-dependent contribution can be calculated from

$$\langle X(i)X(j)\rangle = -\ln|z_i-z_j| - \ln|z_i-\bar{z}_j| . \tag{3.8}$$

Thus, the functional average over the X-dependent factors becomes

$$\begin{aligned} \prod_{i<j}(\gamma_i-\gamma_j)^{2P_i\cdot P_j} &= (\gamma_1-\gamma_3)^{-2}(\gamma_2-\gamma_4)^{-2}\left[\frac{(\gamma_1-\gamma_2)(\gamma_3-\gamma_4)}{(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)}\right]^{2P_1\cdot P_2}\left[\frac{(\gamma_2-\gamma_3)(\gamma_1-\gamma_4)}{(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)}\right]^{2P_2\cdot P_3} \\ &= x^{2P_1\cdot P_2}(1-x)^{2P_2\cdot P_3}(\gamma_1-\gamma_3)^{-2}(\gamma_2-\gamma_4)^{-2} . \end{aligned} \tag{3.9}$$

Putting everything together, notice that the factor of $(\gamma_1-\gamma_3)^{-2}(\gamma_2-\gamma_4)^{-2}$ cancels with an equivalent term coming from $d\tau/dx\ A_G$, leaving us with

$$A_4 = \int dx\, x^{2P_1\cdot P_2}(1-x)^{2P_2\cdot P_3} . \tag{3.10}$$

Finally, it is possible to show that the s- and t-channel graphs add precisely to give the region [0,1] for the x integration, leaving us with the Veneziano amplitude. To show this, notice that the condition $\tau=\infty$ in (2.59) is easily satisfied by the singularity $K(1)=\infty$, so that $k'=1$. But this, in turn, can be satisfied in (2.32) by setting $A=B$, or by setting a_1 or a_2 to be equal to zero.

Now let us satisfy the constraints contained within (2.14) and (2.49). Because we set $\gamma_i=(0,x,1,\infty)$, we can easily show that the condition $x=1$ is satisfied if $a_1=0, b_1=1$ or if $a_2=0, b_2=1$.

Thus, the condition $\tau=\infty$ is satisfied for $x=1$.

Similarly, the condition $x=0$ can be satisfied by the constraints for $a_1=b_1=0$ or by $a_2=b_2=0$. This, in turn, corresponds to $k'=1$, so τ is once again divergent.

More subtle is the point in x space where the two turning points have the same real part, $\tau=0$, which we call x_0. In general, because we must satisfy both (2.49) and (2.59) with $\tau=0$, an exact analytical expression for x_0 does not exist.

Here let us take the very special configuration in which the $|\alpha_i|$ are all taken to be the same and $\delta_1=\delta_2=\delta$. In this case, $\tau=0$ just gives us $k=1$. Using the fact that $\Lambda_0(\beta_i,1)=2\beta_i/\pi$, we can write

$$\delta_1=-\frac{1}{2}\sum_{i=1}^{4}\alpha_i(2\beta_i/\pi) .$$

Then the solution for x_0 is

$$x_0=\sin^2(\tfrac{1}{2}\pi\delta) .$$

This expression has the correct properties. For the midpoint configuration, we have, for example, $\delta=\frac{1}{2}$, so that $x_0=\frac{1}{2}$, which is the result of Ref. 3. For the end-point configuration, in the limit, it is easy to check by conformal maps that the s-channel graph by itself occupies the entire region of integration, and that the t-channel graph vanishes. Setting $\delta=0$ for the end-point configuration we find that $x_0=0$, so that the t-channel graph vanishes, as expected. Our expression for x_0 therefore reproduces the known results in the midpoint and the end-point configurations.

Thus, our final result is that we must add the s- and t-channel contributions in the interpolating configuration as

$$A(t)+A(s)=\left[\int_0^{x_0}+\int_{x_0}^{1}\right]dx\, x^{2P_1\cdot P_2}(1-x)^{2P_2\cdot P_3} . \tag{3.11}$$

We can also show that, when investigating the t- and u-channel diagrams, there is a missing region in the integration which is not covered by the usual t- and u-channel graphs. This can be checked analytically when we go to the end-point configuration. (The missing region vanishes in the midpoint configuration.) This has also been verified in the interpolating configuration by computer. This missing region, of course, corresponds to the four-string interaction, first introduced in Ref. 1.

As an added check on our results, we wish to show that our expression for the measure agrees with the usual midpoint and end-point measures when we take the appropriate limits.

For the midpoint configuration for open strings,[3] we take the symmetric location of the external strings:

$$\gamma_i=\left[\frac{-1}{\alpha},-\alpha,\alpha,\frac{1}{\alpha}\right] , \tag{3.12}$$

where α is a real number. We also take z_1 and z_2 to be purely imaginary, such that $a_1=\gamma$ and is the inverse of a_2.

With this choice, we can show that

$$\begin{aligned} &b_1=b_2=0,\quad a_1=\gamma=\frac{1}{a_2} ,\\ &g_1=0,\quad \omega_i=-\frac{a_1}{\gamma_i},\quad \sin^2\beta_i=\frac{1}{1+a_1^2/\gamma_i^2} . \end{aligned} \tag{3.13}$$

Thus, the δ_1 constraint (2.49) becomes

$$\tfrac{1}{2}=\Lambda_0(\theta_1,k)-\Lambda_0(\theta_2,k) , \tag{3.14}$$

where

$$\sin^2\theta_1=\frac{\beta^2}{\beta^2+\gamma^2},\quad \sin^2\theta_2=\frac{\alpha^2}{\alpha^2+\gamma^2} , \tag{3.15}$$

where

$$\beta=\frac{1}{\alpha},\quad \theta_1=\beta_1=\beta_4,\quad \theta_2=\beta_2=\beta_3 . \tag{3.16}$$

Notice that this is precisely the expression found in the midpoint configuration.[3]

Similarly, for the τ variable in (2.59) we find

$$\tau=2K(k')[Z(\beta_2,k')-Z(\beta_1,k')] . \tag{3.17}$$

Again, this is in agreement with Ref. 3.

Now we calculate the midpoint limit of our results for $d\tau/dx$. In the midpoint limit, we have

$$\begin{aligned} &A=a_2+a_1,\quad B=a_2-a_1 ,\\ &k=\gamma^2,\quad g=\gamma ,\\ &N=\frac{2\alpha(1/\alpha^2-\alpha^2)}{\sqrt{\alpha^2+\gamma^2}\sqrt{\alpha^2+1/\gamma^2}} . \end{aligned} \tag{3.18}$$

Substituting these values into (2.71), we find

$$d\tau=\frac{2\pi\, d\alpha}{K(\gamma^2)\sqrt{\alpha^2+\gamma^2}\sqrt{1+\alpha^2\gamma^2}} . \tag{3.19}$$

This is the same as the result in Ref. 3.

Next, let us calculate the ghost factor in (3.5). Substituting the values of γ_i, notice the term

$$\prod_{i=1}^{4}(z-\gamma_i)=(z^2-\alpha^2)\left[z^2-\frac{1}{\alpha^2}\right] . \tag{3.20}$$

Thus, we find

$$A_G=4\left[\frac{1}{\alpha^2}-\alpha^2\right]^2\int_C\frac{dz}{2\pi i}\frac{dz}{d\rho}\frac{1}{(\alpha^2-z^2)\left[\frac{1}{\alpha^2}-z^2\right]} , \tag{3.21}$$

which again agrees with Ref. 3.

Surprisingly, more difficult is the proof that (2.59) and (2.79) reduce the usual end-point (or light-cone) result. This is because, although the limit is smooth and well defined, there are potential infinities which are canceled by zeros, giving us ambiguous $0\times\infty$ terms.

We take the limit that the sum of the α_i is very small, so that the a_i are very close to zero. In other words, the limit is taken as

$$\alpha_4=-\sum_{i=1}^{3}\alpha_i+\epsilon \tag{3.22}$$

as ϵ goes to zero and as δ_1 goes to zero. In this limit, we have, for (2.32),

$$\begin{aligned} A&=b_1-b_2+\frac{1}{2}\frac{(a_1+a_2)^2}{b_1-b_2}+\cdots\,,\\ B&=b_1-b_2+\frac{1}{2}\frac{(a_1-a_2)^2}{b_1-b_2}+\cdots\,. \end{aligned} \tag{3.23}$$

Then we have

$$g_1=\frac{a_1}{b_2-b_1}+\cdots \tag{3.24}$$

and

$$\omega_i=\frac{a_1(b_2-\gamma_i)}{(b_2-b_1)(b_1-\gamma_i)}+\cdots\,. \tag{3.25}$$

This, in turn, gives us the desired expression for $\sin\beta_i$:

$$\sin\beta_i=1-\frac{1}{2}\frac{a_1^2(b_2-\gamma_i)^2}{(b_2-b_1)^2(b_1-\gamma_i)^2}+\cdots\,. \tag{3.26}$$

Before we can insert these expressions into (2.59) and (2.79), we need to know the behavior of δ_1 and τ near the point $k'=1$. Fortunately, the only serious problem is posed by $K(1)=\infty$, which will be canceled when we carefully apply the fact that the sum of α_i equals zero.

The expansion of the various elliptic functions near the dangerous point is given by the following.

From Eq. (904.00),[10] we have

$$\Lambda_0(\phi,k')=\frac{2}{\pi}\left[a_0t_0-\sum_{m=1}^{\infty}a_{2m}(k)t_{2m}(\phi)\right]\,, \tag{3.27}$$

where all we need to know is that $a_0=E$ and $t_0=\phi$. The higher a_{2m} are all higher powers of k.

From Eq. (902.01),[10] we have

$$F(\phi,k')=\sum_{m=0}^{\infty}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}k^{2m}\rho_{2m}(\phi)\,, \tag{3.28}$$

where

$$\rho_0=\ln\frac{1+\sin\phi}{\cos\phi}\,. \tag{3.29}$$

From (900.05),[10] we have

$$K(k')=\sum_{m=0}^{\infty}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\left[\ln\frac{4}{k}-b_m\right]k^{2m}\,, \tag{3.30}$$

where $b_0=0$.

From Eq. (900.10),[10] we have

$$E(k')=1+\frac{1}{2}\left[\ln\frac{4}{k}-\frac{1}{2}\right]k^2+\cdots\,. \tag{3.31}$$

From Eq. (903.01),[10] we have

$$E(\phi,k')=\sum_{m=0}^{\infty}\begin{bmatrix}\frac{1}{2}\\ m\end{bmatrix}k^{2m}d_{2m}(\phi)\,, \tag{3.32}$$

where $d_0(\phi)=\sin\phi$.

For the calculation of δ_1, notice that

$$\delta_1\approx\sum_{i=1}^{4}\alpha_i\beta_i\approx 0 \tag{3.33}$$

because β_i is infinitesimally close to 1 as we take the end-point limit, and the sum of α_i equals zero. Thus, the Riemann sheet becomes planar.

More delicate is the calculation of τ. Here, we know that the divergent term is given by $K(1)=\infty$. This term, and the next-leading term, are given by

$$K(k')E(\beta_i,k')\approx\begin{bmatrix}-\frac{1}{2}\\ 0\end{bmatrix}(\ln 4-\ln k)\sin\beta_i\,, \tag{3.34}$$

where k is very small and $\sin\beta_i$ is very close to 1. Thus, this divergent part is canceled by the summation of α_i being zero. The finite part is given by the expansion

$$E(k')F(\beta_i,k')\approx\ln\frac{1+\sin\beta_i}{\cos\beta_i}\,. \tag{3.35}$$

Using (3.26), we can also calculate the next leading term, which is of order $a_1^2\ln a_1a_2$ which vanishes in this limit.

Let us insert this finite part, disregarding all terms which can be canceled by the fact that the α_i sum to zero. We then find

$$\begin{aligned} \tau&=\sum_{i=1}^{4}\alpha_i\ln\frac{1+\sin\beta_i}{\cos\beta_i}\\ &=\sum_{i=1}^{4}\alpha_i[\ln(b_2-\gamma_i)-\ln(b_1-\gamma_i)]\,. \end{aligned} \tag{3.36}$$

Now compare this result with the usual end-point formalism, which is defined by

$$\rho(z)=\sum_{i=1}^{4}\alpha_i\ln(z-\gamma_i)\,. \tag{3.37}$$

The solution of

$$\frac{d\rho}{dz}=\sum_{i=1}^{4}\frac{\alpha_i}{z-\gamma_i}=0 \tag{3.38}$$

gives us the two turning points z_+ and z_-. But because the two turning points, in our notation, are b_1 and b_2, we now have

$$\tau=\rho(z_+)-\rho(z_-)\,, \tag{3.39}$$

which is precisely the usual end-point result for τ.

Now, let us calculate the ghost contribution for the end-point configuration. In this limit, we find that $g=(b_1-b_2)^{-1}$, that $K(0)=\frac{1}{2}\pi$, and that N is proportional to α_4. Putting all factors into (3.5), we find

$$A_g=\frac{x(1-x)}{(z_+-z_-)(\alpha_1+\alpha_2+\alpha_3)}(\gamma_1-\gamma_3)^{-2}(\gamma_2-\gamma_4)^{-2}\ , \tag{3.40}$$

which is proportional to the result found in Ref. 4. When multiplied by all other factors, we once again trivially obtain the Veneziano model.

As we mentioned in the last section, the generalization of these results to the closed-string case is straightforward, except that everything becomes complex and there are missing regions of integration, which can be filled by adding new interactions to the action. For the closed-string case, we first emphasize again that a_i and b_i now are complex, so that we have eight unknowns contained within the z_i.

In addition to τ, we also have the new variable θ, which is the angle of rotation of the intermediate string. These variables are fixed by

$$\begin{aligned}\tau&=\mathrm{Re}[\rho(z_2)-\rho(z_1)]\ ,\\ \theta&=\mathrm{Im}[\rho(z_2)-\rho(z_1)]\ .\end{aligned} \tag{3.41}$$

Let us define a new complex variable $\hat{\tau}=\tau+i\theta$. Then it is now a simple matter to redo the steps between (2.50) and (2.59), remembering that $\hat{\tau}$ is now a complex variable. The only new terms that we must evaluate are the f_i in (2.35) and the inverse tanh in (2.53), because they contribute an imaginary part to $\hat{\tau}$. Fortunately, a simple calculation shows that the imaginary part of the f_i contribution and the arctanh contribution is a constant, so we can neglect these terms. Thus, we have

$$\hat{\tau}=\tau+i\theta=-K(k')\sum_{i=1}^{4}\alpha_i Z(\beta_i,k')\ , \tag{3.42}$$

where the equality holds up to a pure, imaginary constant, and where the right-hand side is now a complex, not real, function.

Now let us calculate $d\tau$. The steps between (2.60) and (2.73) are identical when we make all the a_i and b_i complex, except now the variable x itself is a complex one, which we call $\hat{x}$. We thus have

$$\frac{d\hat{\tau}}{d\hat{x}}=-\frac{N\pi}{2K(k)g\hat{x}(1-\hat{x})(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)}\ . \tag{3.43}$$

Therefore,

$$\det\left[\frac{\partial\hat{\tau}}{\partial\hat{x}}\right]=\left[\frac{\pi N}{2K(k)g\hat{x}(1-\hat{x})(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)}\right]^2\ . \tag{3.44}$$

Because the ghost contribution contained a factor of $|b_0|^2$, we find a similar expression for A_G, which is now the absolute value squared of the previous expression. Putting everything together, we find

$$\det\left[\frac{\partial\hat{\tau}}{\partial\hat{x}}\right]\hat{A}_G=|(\gamma_1-\gamma_3)^2(\gamma_2-\gamma_4)^2|^2\ . \tag{3.45}$$

Combining this fact with the X-dependent factors, which are also complex, we finally find

$$\begin{aligned}A_4&=\int d^2\hat{\tau}\,\hat{A}_G\left\langle\exp\left[i\sum_j P_jX(j)\right]\right\rangle\\ &=\int d^2\hat{x}\,|\hat{x}^{\,2P_1\cdot P_2}(1-\hat{x})^{2P_2\cdot P_3}|^2\ .\end{aligned} \tag{3.46}$$

Although we have now correctly derived the integrand of the Shapiro-Virasoro model, the region of integration for $\hat{x}$ in the interpolating configuration is highly nontrivial; i.e., there are missing regions of integration, depending on the lengths of the external strings. These missing regions of integration (for the open-string field theory) were first discovered in Ref. 1, and correspond to higher-string interactions. (See Refs. 7–9 for a group-theoretical explanation of these missing regions.)

In Fig. 9 we see the integration region of the Shapiro-Virasoro model in the end-point configuration, where the entire complex Koba-Nielsen plane has been stereographically mapped onto the sphere. Notice that the s-, t-, and u-channel graphs correspond to various sections of the sphere, with no missing region. The three regions, in fact, meet at the "north pole." Thus, in the end-point configuration, the three-string vertices by themselves can fill up the entire complex plane.

However, as we slowly go to the interpolating configuration, a missing region begins to open up centered around the north pole (Fig. 10), which gradually gets larger and larger, until it occupies most of the sphere in the midpoint configuration (Fig. 11).

Notice that there are three points where the missing region touches the equator. These points correspond to the "Rubik's cube" points where four aligned closed strings of equal length precisely collide head on.

Unfortunately, an exact analytic expression for the missing region of the complex plane does not exist. Therefore, we must rely on elaborate computer calculations to check that the missing region exists and that it is precisely filled up with the tetrahedron graph. This has been done for the midpoint configuration and several interpolating configurations.

We are still in the process of conducting more ela-

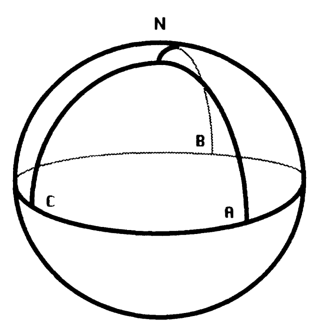

FIG. 9. In the end-point gauge, the Riemann surface for four-closed-string scattering is a sphere, with no missing region.

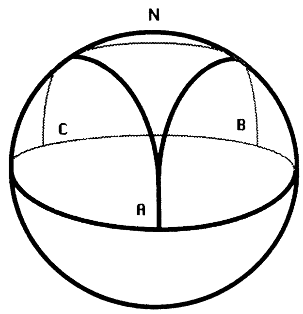

FIG. 10. In the interpolating gauge, a missing region surrounding the north pole begins to open up.

borate and sophisticated computer programs to show in detail how the missing region around the north pole opens up to eventually occupy most of the sphere.

We now turn to the method of Beltrami differentials, which will give us a formal proof of the correctness of our measure.

IV. METHOD OF BELTRAMI DIFFERENTIALS

However, one unsatisfactory feature of this approach is that it depends on a series of "miracles" taking place. As a result, it is difficult to generalize these techniques to N-point and multiloop amplitudes.

In this section, we try to reformulate the problem in terms of Beltrami differentials[4] to solve two problems. First, we wish to reformulate the problem of the measure so that these "miracles" are less obscure and so that we can generalize our results to higher-order amplitudes.

Second, we wish to solve a long-standing problem with regard to the measure of the four-string interaction. The original Jacobian that takes us from τ in the light-cone configuration to the Koba-Nielsen variable x was calculated in its entirety by Cremmer and Gervais and proven to all orders by Mandelstam[11,12] using the theory of conformal determinants. It reads, for the open-string process $1+2\rightarrow 5\rightarrow 6\rightarrow 3+4$,

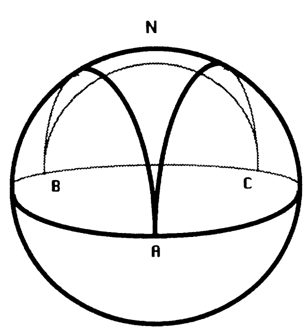

FIG. 11. In the midpoint gauge, the missing region now occupies most of the northern (and southern) hemispheres. This missing region must be filled with a tetrahedron graph.

$$\frac{d\tau}{|\alpha_5|}\det(I-\tfrac{1}{4}\bar{N}\,{}^{\tau_1}_{66}\bar{N}\,{}^{\tau_2}_{55})^{-12}\exp\left[\sum_{i=1}^{6}\frac{\tau_i-\theta_i}{\alpha_i}\right] = \prod_{i<j}|z_i-z_j|^{\alpha_i/\alpha_j+\alpha_j/\alpha_i}\frac{\prod_i dz_i}{dV_{abc}} = \frac{\prod_i dz_i e^{-N_{ii,00}}}{dV_{abc}}\,, \tag{4.1}$$

where

$$dV_{abc}=\frac{dz_a dz_b dz_c}{|z_a-z_b||z_b-z_c||z_c-z_a|}\,,$$

where N is the Neumann function of the various three-string vertices:

$$N_{ij,nm}=\frac{1}{nm}\frac{1}{(2\pi i)^2}\oint_{z_i}\oint_{z_j}\frac{dz\,dz'}{(z-z')^2}e^{-n\xi_i(z')m\xi_j(z)}$$

(see Refs. 13 and 14 for notation and details).

A covariant generalization of this is obviously possible for the end-point configuration, using the fact that the ghost contribution to the measure reduces the determinant of the Laplacian from the 13th power down to the 12th power. All the other factors remain the same.

The puzzling fact, however, is that the Jacobian of the transformation from τ down to x is totally different in form from the Jacobian found earlier, yet the final result is the same.

This problem is not just confined to the end-point or light-cone configuration. In the midpoint configuration, for example, we could have contracted the oscillators for the four-string interaction directly, and we would also have found factors of the determinant of the Laplacian to the 13th power. Therefore, it is puzzling that there appears to be two entirely different deviations of the Jacobian, using different equations and different assumptions, yet giving us the same result.

We will resolve this problem using Beltrami differentials. First, quasiconformal transformations differ from conformal transformations in an essential way. While configuration transformations infinitesimally map circles to circles, quasiconformal transformations map circles to ellipses. In fact, the ratio of the minor to major axes of this ellipses gives us a way of measuring different quasiconformal transformations.

Quasiconformal transformations occur naturally in string-field theory because the stretch variables τ_i twist variables θ_i, and expansions α_i all generate quasiconformal transformations. For example, the rule used in Nambu-Goto or Polyakov quantization is to sum over all inequivalent conformal surfaces. Since τ,θ,α_i all generate quasiconformal transformations, they take us from one conformal surface to a conformally inequivalent one. Thus, these moduli must be integrated over.

Conformal transformation, of course, are generated by

$$z\rightarrow z+v(z)\,,$$

where $v(z)$ is a function of z, not $\bar{z}$. Quasiconformal transformations can be generated if we allow $v(z)$ to be a function of both z and its complex conjugate. Thus, the "size" of a quasiconformal transformation can be given by a mixed tensor

$$\mu_{\bar{z}}^{z}=\partial_{\bar{z}}v^{z} . \tag{4.2}$$

If the transformation is conformal, then $\mu_{\bar{z}}^{z}$ is equal to zero, since v^z is a function of z only.

Quasiconformal transformations are easy to visualize. For example, a transformation which takes a square and stretches it into a rectangle cannot be a conformal transformation. It must be quasiconformal, since the transformation of a stretch in the x direction is given by

$$z \rightarrow z + \mathrm{Re}z \tag{4.3}$$

and since Rez is not a function of z.

These results generalize for an arbitrary Riemann surface. For a genus g surface (there are slight changes for a surface of $g=0$), there are g quadratic differentials which transform as ϕ_{zz} which allow us to define the surface itself. Let us now use $3g-3$ complex moduli dm_i to characterize the conformal class of this surface. The question is: is it possible to define on this surface a measure over dm_i which is conformally invariant?

Let us start with the Polyakov form for the measure of integration for the multiloop amplitude (there is a slight difference when we include punctures):

$$d\mu=\prod_{i=1}^{6g-6} dm_i \frac{\det\langle \mu_i|\phi_k\rangle}{\det^{1/2}\langle \phi_m|\phi_n\rangle}\left[\frac{\det\nabla^2}{\int d^2\sigma\sqrt{g}}\right]^{-D/2} \times \det{}^{1/2}P_1^{\dagger}P_1 , \tag{4.4}$$

where $\langle \mu_i|$ are the Beltrami differentials corresponding to a set of quasiconformal transformations, where m_i are the $3g-3$ complex moduli parameters and $|\phi_k\rangle$ are the quadratic differentials. The advantage of writing the multiloop measure in this form is that we can analyze its conformal structure all at once.

Let us calculate how each term in the measure transforms under a conformal transformation:

$$\left[\frac{\det P_1^{\dagger}P_1}{\det\langle \phi_j|\phi_k\rangle_g}\right]^{1/2}=\left[\frac{\det\hat{P}_1^{\dagger}\hat{P}_1}{\det\langle \hat{\phi}_j|\hat{\phi}_k\rangle_{\hat{g}}}\right]^{1/2} e^{-26S_L} , \tag{4.5}$$

$$\left[\frac{\det'\nabla^2}{\int d^2\sigma\sqrt{g}}\right]^{-D/2}=\left[\frac{\det'\nabla^2}{\int d^2\sigma\sqrt{\hat{g}}}\right]^{-D/2} e^{DS_L/2} , \tag{4.6}$$

where we have made the conformal transformation from the metric g_{ab} to $\hat{g}_{ab}$. Notice that in 26 dimensions the anomalous term involving the Liouville action S_L vanishes. Furthermore, the Jacobian of the transformation of dm_i cancels against the determinant of the Beltrami differentials.

The object of writing the measure in this form is that we can now make a conformal transformations to any configuration we please. The measure, however, is not quite in the form that we wish. Let us use the identity

$$\int Db\, Dc\, D\bar{b}\, D\bar{c} \prod_{i=1}^{3g-3} b(z_i)\bar{b}(z_i)\exp\left[i\int L_{\mathrm{gh}}d\sigma\, d\tau\right] =(\det{}^{1/2}P_1^{\dagger}P_1)\frac{|\det\phi_k(z_j)|^2}{\det\langle \phi_m|\phi_n\rangle} , \tag{4.7}$$

where L_{gh} is the ghost action. From this identity, we can derive

$$\left[\int db\, Dc \prod_{i=1}^{3g-3}\langle \mu_i|b\rangle\exp\left[i\int L_{\mathrm{gh}}d\sigma\, d\tau\right]\right]^2 =\det{}^{1/2}P_1^{\dagger}P_1\frac{\det\langle \mu_j|\phi_k\rangle}{\det^{1/2}\langle \phi_m|\phi_n\rangle} \tag{4.8}$$

and

$$d\mu=\prod_i dm_i\left[\int DX\, Db\, Dc \prod_{i=1}^{3g-3}\langle \mu_i|b\rangle \times\exp\left[i\int (L_X d\sigma\, d\tau+L_{\mathrm{gh}}d\sigma\, d\tau)\right]\right]^2 . \tag{4.9}$$

Now we use the fact that the ith Beltrami differential can be written as

$$\mu_{i,\bar{z}}^{z}=\nabla_{\bar{z}}v_i^z , \tag{4.10}$$

where v_i^z generates a quasiconformal transformation. Now we use the fact that we can integrate by parts:

$$\begin{aligned}\langle \mu_i|b\rangle &=\int d^2z\sqrt{g}\,g^{z\bar{z}}\mu_{i,\bar{z}}^{z}b_{zz}\\ &=\int d^2z\sqrt{g}\,\nabla^z v_i^z b_{zz}=\oint_{C_i}dz\, b_{zz} ,\end{aligned} \tag{4.11}$$

where the contour C_i corresponds to a slice on the Riemann surface along where we stretch or twist the surface. The final formula we wish is therefore

$$d\mu=\prod_{i=1}^{6g-6} dm_i\int DX\int Db\, Dc\, D\bar{b}\, D\bar{c} \times\exp\left[i\int (L_X+L_{\mathrm{gh}})d\sigma\, d\tau\right] \times\prod_{j=1}^{3g-3}\oint_{C_j}dz\, b_{zz}\oint_{C_j}d\bar{z}\, b_{\bar{z}\bar{z}} . \tag{4.12}$$

This is the formula we desire. In the midpoint configuration, notice that the slice C_i corresponds to vertical slices along the various horizontal propagators in a multiloop diagram. The integral over these slices, in turn, simply reproduce the various factors of b_0 in the gauge-fixed propagator.

When we make the transition to the end-point configuration, however, the Beltrami differentials are defined along a much different set of slices. The remarkable fact about the previous formula is that we can show that $d\mu$ is invariant under the very complicated conformal transformation which takes us from the midpoint configuration to the end-point configuration. Although the variables in the various configurations quite dissimilar, this formula guarantees that the measure of integration is the same. For example, the measure in the midpoint configuration is given by the various propagation distances τ_i, while the end-point configuration is given by distances τ_i and changes in string length α_i. Nevertheless, the measure is conformally invariant.

The point is that the measure of integration, including the ghost contribution, is insensitive to the height of the external strings. Thus, it should be insensitive to whether we are working in the midpoint, end-point, or interpolating configurations. In fact, we see that the measure is only a function of the various slices of the Riemann surface corresponding to the quasiconformal transformations, not the parametrization lengths of the external strings. In fact, we see that the *only* restriction on the Riemann surface is that it be constructed out of rectangles (cylinders) for the propagators and vertical slices for the vertex functions.

Thus, the final result for the Veneziano and Shapiro-Virasoro model should be identical independent of whether we are in the end-point, midpoint, or interpolating configurations.

Although this method is powerful and generalizes to all orders in perturbation theory, we should also note an important defect. Namely, this method says nothing about the region of integration and missing regions. Because all arguments were infinitesimal, we can accidentally overlook large portions of the region of integration corresponding to four-string-type interactions. Thus, we still must, in the final analysis, resort to conformal maps and computer calculations to ultimately determine the region of integration.

Last, we note that Beltrami differentials also give us the solution to the problem mentioned earlier, the fact that historically a different solution was given to the measure problem, involving the determinant of the Laplacian over the world-sheet surface, while the current derivation of the Jacobian makes no mention of these determinants.

The solution of the problem is now more transparent. First, we note that the determinant of the Laplacian over the upper-half place or the entire complex plane is a constant. Unlike the world sheet of the four interacting strings, the complex plane has no structure, and hence the determinant over the complex plane cannot be anything else but a constant factor.

Second, we note that the determinant of the Laplacian is covariant, not invariant, under a conformal transformation. For the world sheet of the interacting strings, the determinant is highly nontrivial, depending on the turning point and string lengths, while the determinant of the Laplacian over the complex plane is a constant. Thus, to solve the problem of a conformal measure we need to form a conformally invariant object.

The measure actually consists of the product of two conformally invariant object. First, clearly only the product of (4.5) and (4.6) yields a conformally invariant object. Second, the $d\tau$ integration term can be made conformally invariant by multiplying by $\det\langle \mu_i|\phi_k\rangle$. Thus, (4.4) is the conformally invariant measure that we desire.

Third, we see how to explain the N_{00}^{ii} term in (4.1). In the calculation of the Veneziano amplitude, we must necessarily truncate the divergent part coming from the X-functional integral for the $i=j$ string. This can be seen, for example, in (3.7), where we have explicitly deleted the divergent part corresponding to $i=j$. When we conformally map the upper-half plane back to the string world sheet, however, this $i=j$ truncation is created by the N_{00}^{ii} term. Thus, the N_{00}^{ii} term is automatically included in (3.7) when we made the conformal transformation from the string world sheet to the complex plane.

Let us now make the transformation of (4.4) from the midpoint, end-point, and interpolating configurations to the complex plane. Notice that we immediately get the identity (4.1).

V. METHOD OF REFLECTIONS

Last, we give an operatorial derivation of our results which generalizes to all orders in perturbation theory.

We expect that

$$\langle V_{\alpha_1\alpha_2\alpha_5}|D_5|V_{\alpha_5\alpha_3\alpha_4}\rangle \tag{5.1}$$

defined in the interpolating configuration should yield the Veneziano amplitude in the midpoint configuration or, for closed strings, the Shapiro-Virasoro amplitude in the midpoint configuration. It is easy to check that this is so.

Let us define a U operator which changes the parametrization length of a string:

$$U=\exp\left[\sum_{n=1}^{\infty}\epsilon^n(L_n-L_{-n})\right]. \tag{5.2}$$

By inserting the expression UU^{-1} at each vertex, we have the operatorial freedom of changing the configuration of each vertex (as long as we keep careful track of the leftover U factors). Thus, (5.1) can be written as

$$\left\langle V_{\alpha_1\alpha_2\alpha_3}\right|\exp\left[-\sum_{n=1}^{\infty}\sum_{r=1,2,5}\epsilon_r^n(L_n^r-L_{-n}^r)\right] \times D_5\exp\left[\sum_{n=1}^{\infty}\sum_{s=5,3,4}\epsilon_s^n(L_n^s-L_{-n}^s)\right]\left|V_{\alpha_5\alpha_3\alpha_4}\right\rangle, \tag{5.3}$$

where we have now changed the values of the string lengths. Now let us remove the factors of ϵ^n by using the method of reflections. We use the reflection identity

$$(L_n^r/\alpha_r-n\tilde{N}_{nm}^{rs}L_{-m}^s/\alpha_s-c_n^r)|V\rangle=0 \tag{5.4}$$

and the transmission identity

$$L_n x^{L_0-1}=x^n x^{L_0-1}L_n, \tag{5.5}$$

where

$$c_n^r=\tfrac{1}{2}D\sum_{p=1}^{n-1}p(n-p)N_{p,n-p}^{rr}+\sum_{n=1}^{n-1}(p^2-n^2)\tilde{N}_{p,n-p}^{rr} -n^2\tilde{N}_{n0}^{rr}.$$

All factors of L_n-L_{-n} in turn can be converted into factors of L_n or L_{-n} via these identities. For example, we have infinitesimally

$$\left[1+\sum_{n=1}^{\infty}(L_n^r-L_{-n}^r)/\alpha_r\right]|V\rangle = \left[1+\sum_{n=0}^{\infty}(\delta\alpha_n^r\tilde{N}_{nm}^{rs}L_{-m}^s/\alpha_s)\right]|V\rangle\ .$$

(See Ref. 7 for a more complete definition of the notation.) If we treat the L_n infinitesimally, then we see that these factors bounce back and forth between the two vertices, like a light ray which reflects an infinite number of times between two parallel mirrors. Each time a reflection takes place, the intensity of the original beam gets diminished, because it picks up factors of x, which is less than one. Finally it disappears altogether after an infinite number of iterations.

However, all that remains are the various factors of L_0 and various measure terms that are created after each reflection process. Fortunately, we can sum all these terms exactly. In summary, we have

$$I=\left\langle V_{\alpha_1\alpha_2\alpha_5}\left|\int_{R_l}x^{L_0-2}dx\right|V_{\alpha_3\alpha_4\alpha_6}\right\rangle_{\text{midpoint}} =\left\langle V_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_5}\left|\int_{R_l}(x')^{L_0}f(\alpha_i,x)\frac{dx}{x^2}\right|V_{\bar{\alpha}_3\bar{\alpha}_4\bar{\alpha}_6}\right\rangle_{\text{end point}}, \tag{5.6}$$

where

$$x'=x[1+\xi(x)],\quad f(\alpha_i,x)=1+\eta(x)\ , \tag{5.7}$$

where

$$\xi(x)=\sum_{r=1,2,5}\sum_{i=2}^{\infty}\sum_{n_i=1}^{\infty}\delta\alpha^{rn_1}\tilde{N}_{n_1n_2}^{r5}\alpha_5x^{n_2}n_2\tilde{N}_{n_2n_3}^{66}\cdots x^{n_i}n_i\tilde{N}_{n_i0}^{r_ir_i}\frac{1}{\alpha_{r_i}} +\sum_{s=3,4,6}\sum_{j=2}^{\infty}\sum_{n_j=1}^{\infty}\delta\alpha^{sn_1}\tilde{N}_{n_1n_2}^{s6}\alpha_6x^{n_2}n_2\tilde{N}_{n_2n_3}^{55}\cdots x^{n_j}n_j\tilde{N}_{n_jo}^{r_ir_i}\frac{1}{\alpha_{r_i}} \tag{5.8}$$

and

$$\begin{aligned}\eta(x)=&\sum_p\sum_{r=1,2,5}\sum_{i=2}^{\infty}\sum_{n_i=1}^{\infty}\delta\alpha^{rn_1}\tilde{N}_{n_1n_2}^{r5}\alpha_5x^{n_2}\cdots\tilde{N}_{n_io}^{r_ip}\frac{1}{\alpha_p}+\sum_q\sum_{s=2,3,6}\sum_{j=2}^{\infty}\delta\alpha^{sn_1}\tilde{N}_{n_1n_2}^{s6}\alpha_6x^{n_2}\cdots\tilde{N}_{n_j0}^{s_jq}\frac{1}{\alpha_q}\\ &+\sum_{r=1,2,5}\sum_{i=2}^{\infty}\sum_{l=2}^{i}\sum_{n_i=1}^{\infty}\delta\alpha^{rn_1}\tilde{N}_{n_1n_2}^{r5}\alpha_5x^{n_2}\cdots x^{n_l}n_l^2\tilde{N}_{n_ln_{l+1}}^{r_lr_{l+1}}\cdots x^{n_i}n_i\tilde{N}_{n_io}^{r_ir_i}\frac{1}{\alpha_{r_i}}\\ &+\sum_{s=3,4,6}\sum_{j=2}^{\infty}\sum_{l=2}^{j}\sum_{n_j=1}^{\infty}\delta\alpha^{sn_1}\tilde{N}_{n_1n_2}^{s6}\alpha_6x^{n_2}\cdots x^{n_l}n_l^2\tilde{N}_{n_1n_{l+1}}^{r_lr_{l+1}}\cdots x^{n_j}n_j\tilde{N}_{n_jo}^{r_jr_j}\frac{1}{\alpha_{r_j}}\ ,\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}&i=\text{even}\to r_i=6,\quad s_i=5,\quad p=3,4,\quad q=1,2\ ,\\ &j=\text{odd}\to r_i=6,\quad s_i=5,\quad p=1,2,\quad q=3,4\ .\end{aligned} \tag{5.10}$$

(Notice that the last two terms appearing in the expression for η differ from the previous two terms because they have factors of n^2; i.e., they come from the b_0 contribution to the power series.)

Notice that the power expansion in the Neumann functions of three-string vertices precisely reproduces the Neumann functions defined over the entire four-string interaction. To see this, let us actually contract over the intermediate indices of the scattering amplitude to construct the four-string interaction in operator language. We need to use the following identity, which uses the coherent-state basis:

$$\begin{aligned}\langle V_{125}|x^R|V_{634}\rangle&=\langle 0|\exp[\tfrac{1}{2}(a|M_2|a)+(a|L_2)]\exp[\tfrac{1}{2}(a^\dagger|M_1|a^\dagger)+(a^\dagger|L_1)]|0\rangle\\ &=\det{}^{-D/2}(1-M_2M_1)\exp\left[\left[L_1\left|\frac{1}{1-M_2M_1}\right|L_2\right]+\frac{1}{2}\left[L_2M_1\left|\frac{1}{1-M_2-M_1}\right|L_2\right]\right.\\ &\qquad\left.+\frac{1}{2}\left[L_1\left|\frac{1}{1-M_2M_1}\right|M_2L_1\right]\right]\ .\end{aligned} \tag{5.11}$$

Notice that the power expansion resulting from $(1-M_2M_1)^{-1}$ is precisely the infinite power expansion found in the method of reflections.

Next, to make sense out of this result, we now use the formula

$$\delta\tau=\frac{1}{2\pi}\sum_i\int N(\sigma,\tau;\sigma_i',\tau_i')\delta\, d\sigma_i \ . \tag{5.12}$$

This is a rather remarkable formula, which captures the implicit nature of conformal transformations. It states that if we make a conformal transformation such that we rescale the parametrization lengths of the external strings at infinity, then the distance τ between the turning points of a four-string interaction can be computed exactly in the above formula. The above formula gives us an exact way in which to see how the τ parameter of the interpolating theory changes as we smoothly change α_i in the external legs.

Putting everything together, we now find that $\xi(z)$, which measures the change in the τ parameter as we change the external legs, can be given equivalently by the power expansion of the Neumann function, or the above expression for $\delta\tau$. Inspecting the series term for term, we find an exact correspondence, both the change in the τ variable in the propagator and also the measure of integration.

Thus, we find that we can write

$$x'^{L_0}f(\alpha_i,x)\frac{dx}{x^2}=x'^{L_0}\frac{dx'}{x^2} \ .$$

This is a nontrivial identity. It shows that, after an infinite number of reflections, we can reassemble all terms and show that the propagator defined over x becomes modified to a propagator defined over x', defined by (5.7), such that the region of integration is modified. This is an operatorial proof that can be generalized to show that we can always choose arbitrary external lengths for our N-string (and even multiloop) amplitudes. This confirms the conclusions that we have found in this paper using third elliptic integrals and using the method of Beltrami differentials.

The price to pay, as we have emphasized is that there may be crucial missing regions of integration. We can show that, for the s- and t-channel graphs in the interpolating configuration, there is no missing region. The sum of the s- and t-channel graphs add to give the usual Veneziano amplitude. However, for the t-u graphs, we can show that this is not true, that there is a missing region with the topology of a four-string interaction. The same applies for the Shapiro-Virasoro amplitude, where we always find missing regions in all of the channels.

To see how these missing regions yield higher-point interactions, let us make the transition from the midpoint configuration to the end-point configuration for the four-string scattering amplitude for the t-u graphs. Because the propagator is not invariant under a universal string group (USG) transformation, let us break up the amplitude into two pieces: I and II. If the s channel is represented by $1+2\to3+4$, then the graphs in question are

$$\mathrm{I}+\mathrm{II}=\langle V_{145}|D_5|V_{235}\rangle_{\text{midpoint}}=\left\langle V_{145}\,\middle|\,\left[\int_T^\infty+\int_0^T\right]e^{-\tau(L_0-1)}d\tau\,\middle|\,V_{235}\right\rangle_{\text{midpoint}} , \tag{5.13}$$

where

$$\begin{aligned}\mathrm{I}&=\left\langle V_{145}\,\middle|\,\int_0^T d\tau\, e^{-\tau(L_0-1)}\,\middle|\,V_{235}\right\rangle_{\text{midpoint}}=\left\langle V_{\alpha_1\alpha_4\alpha_5}\,\middle|\,\int_0^T e^{-\tau(L_0-1)}f(\tau)d\tau\,\middle|\,V_{\alpha_2\alpha_3\alpha_5}\right\rangle_{\text{end point}}\\&=\langle V_{\alpha_1\alpha_4\alpha_5}|D_5|V_{\alpha_2\alpha_3\alpha_5}\rangle_{\text{end point}}\end{aligned} \tag{5.14}$$

so that the term I alone allows us to retrieve the amplitude in the end-point configuration. However, the integration region is modified, so that II contains the missing region which corresponds to the four-string interaction.

To see this, let us now make the transition from the midpoint configuration to the interpolating configuration. Let us define α to be between 0 and α_5. Then

$$\mathrm{II}=\left\langle V_{145}\,\middle|\,\int_0^T d\tau\, e^{-\tau(L_0-1)}\,\middle|\,V_{235}\right\rangle_{\text{midpoint}}=\left\langle V_{\alpha_1\alpha_4\alpha}\,\middle|\,\int_0^T d\tau\, e^{-\bar{\tau}L_0}\bar{f}(\bar{\tau})\,\middle|\,V_{\alpha\alpha_2\alpha_3}\right\rangle_{\text{interpolating}} , \tag{5.15}$$

where we set $e^{-\bar{\tau}}=1$, which in turn fixes the value of α. Thus, the propagator actually vanishes, leaving us with a contact term.

Thus, *the four-string interaction, first found in Ref. 1, is just the square of the interpolating vertex.*

Putting the t-u graphs together, we can now show

$$\langle V_{145}|D_5|V_{235}\rangle_{\text{midpoint}}+u\text{ channel}=\langle V_{\alpha_1\alpha_4\alpha_5}|D_5|V_{\alpha_2\alpha_3\alpha_5}\rangle_{\text{end point}}+u\text{ channel}+\int_{\alpha_1-|\alpha_4|}^{\alpha_1+|\alpha_4|}d\alpha\,\mu\langle V_{\alpha_1\alpha_4\alpha}|V_{\alpha\alpha_2\alpha_3}\rangle \ , \tag{5.16}$$

where we now have an explicit representation for μ:

$$\mu=\bar{f}(\bar{\tau})\frac{d\tau}{d\alpha} .$$

(For further details, see Refs. 7 and 8.)

VI. CONCLUSION

Conventional wisdom holds that the Veneziano amplitude can be derived from string field theory only in two ways: from light-cone string field theory[1] and Witten's string field theory.[2] In this paper, we demonstrate several novel results. First, we show that this belief is actually incorrect, that we can derive the Veneziano amplitude using strings of *arbitrary* length. Using the interpolating vertex, we can, in fact, smoothly interpolate the Veneziano amplitude from the light cone to the Witten scattering amplitude. The calculation is highly nontrivial, and we show the correctness of our result in three totally independent ways.

First, we wrote down the conformal map that takes us from the complex plane to the world sheet of the four interacting strings and showed, using third elliptic integrals, that we have the correct measure of integration when we replace τ with the Koba-Nielsen variable x. The calculation is quite long because the external strings have arbitrary length. We cannot use the cyclic symmetries that simplified enormously Giddings's original calculation for open strings. However, a large number of surprising cancellations occur which allow us to write a rather simple expression for τ in (3.42),

$$\hat{\tau}=\tau+i\theta=-K(k')\sum_{i=1}^{4}\alpha_i Z(\beta_i,k') ,$$

and finally a remarkable relationship between the Jacobian and the ghost contribution in (3.46):

$$\det\left[\frac{\partial\hat{\tau}}{\partial\hat{x}}\right]\hat{A}_G=|(\gamma_1-\gamma_3)^2(\gamma_2-\gamma_4)^2|^2 .$$

Second, we used the theory of Beltrami differentials to give the origin of these "miracles." We can show that the measure transforms correctly under conformal transformations. In particular, the original derivation of Giddings makes no reference to the length of the external strings. The only constraint on the original calculation is that the propagators are strips or cylinders and the vertices are vertical slices. The quasiconformal transformations associated with stretches, twists, and expansions are unaffected by changing the lengths of the external legs. In this fashion, we can show the correctness of our results to all orders in perturbation theory.

Third, we used the "method of reflections" to show, at the level of operators, that we obtain the correct measure of integration.

To further show the correctness of our results, we have calculated the region of integration on computer. Numerical studies of the missing region for open (closed) strings show that it vanishes for the midpoint (end-point) configuration, becomes larger and larger in the interpolating configuration, and assumes the maximum volume in Koba-Nielsen space in the end-point (midpoint) configuration.

Higher amplitudes are now being looked at. The theory (with midpoint vertices) is nonpolynomial,[9] as in Einstein's theory.

In hindsight, we should expect many surprises for the closed-string field theory, as a perturbative theory, because it gives us a new triangulation of moduli space. In general, the fundamental region for the moduli space of Riemann surfaces is unknown to the mathematicians, even for the simple case of $g=2$. Surprisingly, string field theory actually solves this problem and gives us a successful triangulation of moduli space and its fundamental region[10] in the end-point configuration. Thus, by performing Diff(S) transformations on the field theory, we are in essence creating more and more triangulations of moduli space, which in general are not known.

The motivation for pursuing this line of approach comes from geometric-string field theory,[7–9] which is reparametrization independent and hence can easily accommodate strings of arbitrary length. However, because geometric-string field theory is well beyond the scope of this article, the reader is encouraged to consult the references on how gauge-fixing geometric string field theory leads to scattering amplitudes with arbitrary string lengths. We have deliberately written this paper so that it can stand by itself, independent of geometric-string field theory.

ACKNOWLEDGMENTS

We wish to acknowledge fruitful discussions with Bunji Sakita. We also would like to acknowledge extensive discussions with Joseph Lykken in the early stages of this calculation. M.K. was supported in part by Grants Nos. NSF-PHY-86-15538, CUNY-FRAP-RF-13873, and DE-AC02-76ER0220.

APPENDIX A

In this appendix, we give an independent derivation of (2.31) and (2.33). This is important, because for the closed-string case we will be taking complex values of the various arguments, and it is important to check that all analytic continuations are possible and well defined.

We begin with the definition

$$F(\phi,k)=\int_0^y\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} . \tag{A1}$$

It is important to note that this expression is finite for all real or complex values of y, including infinity.

Let us now make the change of variables

$$x=\frac{At+B}{Ct+D} . \tag{A2}$$

Then the Jacobian of the change of variables is

$$dx=\frac{AD-BC}{(Ct+D)^2}dt . \tag{A3}$$

Then the integral becomes

$$F(\phi,k)=\int_0^y (AD-BC)dt\,[(Ct+D)^2-(At+B)^2]^{-1/2}\times[(Ct+D)^2-k^2(At+B)^2]^{-1/2}$$
$$=(AD-BC)(C^2-A^2)^{-1/2}(C^2-k^2A^2)^{-1/2}\times\int dt\prod_{i=1}^{4}(t-p_i)^{-1/2}\,, \tag{A4}$$

where

$$p_1=-\frac{B+D}{A+C},\quad p_2=-\frac{D-B}{C-A}\,,$$
$$p_3=-\frac{D+kB}{C+kA},\quad p_4=-\frac{D-kB}{C-kA}\,. \tag{A5}$$

We can now make an exact correspondence between A, B, C, and D and the variables given in the paper in (2.32):

$$g=\frac{\sqrt{(C^2-A^2)(C^2-k^2A^2)}}{AD-BC}\,,$$
$$p_1=b_1+ia_1,\quad p_2=b_2-ia_1\,,$$
$$p_3=b_2+ia_2,\quad p_4=b_2-ia_2\,, \tag{A6}$$
$$b_1=\tfrac{1}{2}(p_1+p_2)=\frac{AB-CD}{C^2-A^2}\,,$$
$$b_2=\tfrac{1}{2}(p_3+p_4)=\frac{k^2AB-CD}{C^2-k^2A^2}\,.$$

More difficult is the calculation of the integral

$$\mathrm{I}=\int\frac{dz}{(z-\gamma_i)\sqrt{(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)}}\,. \tag{A7}$$

To perform this integration, let us once again make a transformation of variables:

$$z=\frac{ax+b}{cx+d}\,, \tag{A8}$$

where this time we solve for a, b, c, and d because this transformation maps the points

$$z=p_i=(z_1,\bar{z}_1,z_2,\bar{z}_2)\longleftrightarrow x=\pm 1,\pm\frac{1}{k}\,. \tag{A9}$$

Then we can write

$$\prod_{i=1}^{4}(z-p_i)^{1/2}=k^{-1}(cx+d)^{-2}\prod_{i=1}^{4}(a-p_ic)^{1/2}\sqrt{(1-x^2)(1-k^2x^2)}\,. \tag{A10}$$

Let us now replace z everywhere with x. Then we find

$$\mathrm{I}=\frac{k(ad-bc)}{\prod_{i=1}^{4}(a-p_ic)^{1/2}}\times\int dx\frac{cx+d}{[(a-\gamma_ic)x+b-\gamma_id]\sqrt{(1-x^2)(1-k^2x^2)}}\,. \tag{A11}$$

Let

$$\alpha=\frac{a-\gamma_ic}{b-\gamma_id}\,. \tag{A12}$$

Let us also define

$$d\mu=\frac{dx}{(1-\alpha^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}} \tag{A13}$$

and

$$Q_i=-\frac{(ad-bc)\alpha k}{\prod_{i=1}^{4}(a-p_ic)^{1/2}(a-\gamma_ic)}\,. \tag{A14}$$

Then our results simplify to

$$\mathrm{I}=Q_i\int d\mu\left[cx^2+x\left[d-\frac{c}{\alpha}\right]-\frac{d}{\alpha}\right]\,. \tag{A15}$$

Now we can express everything in terms of elliptic integrals and rational functions. Let us first rewrite

$$\frac{x^2}{1-\alpha^2x^2}=-\frac{1}{\alpha^2}\left[1-\frac{1}{1-\alpha^2x^2}\right]\,. \tag{A16}$$

This formula, we see, allows us to rewrite the x^2 term in terms of the difference between first and third elliptic integrals. Thus, the x^2 term and the constant term appearing in the expression for I can be recombined as the difference of first and third elliptic integrals. The remaining linear term in x can be rewritten as the integral of a rational function by making the substitution $\nu=x^2$. Finally, we have

$$\mathrm{I}=Q_i(-c\alpha^{-2}-d\alpha^{-1})F(\phi,k)+Q_ic\alpha^{-2}\Pi(\phi,\alpha^2,k)+\tfrac{1}{2}(d-c\alpha^{-1})Q_i\int\frac{d\nu}{(1-\alpha^2\nu)\sqrt{(1-\nu)(1-k^2\nu)}}\,. \tag{A17}$$

Notice that the last term gives us the term f_i in (2.35).

APPENDIX B

To prove (2.48), we first note that, with a little work, we can rewrite g_1 as

$$g_1=\frac{4a_1(b_2-b_1)}{(A+B)^2-4a_1^2} \tag{B1}$$

and

$$\frac{\omega_i(\omega_i-g_1)}{a_1+b_1g_1-\gamma_ig_1}=\frac{a_1(1+g_1^2)}{(b_1-a_1g_1-\gamma_i)^2} \tag{B2}$$

and

$$1+\omega_i^2=\frac{[(b_1-\gamma_i)^2+a_1^2](1+g_1^2)}{(b_1-a_1g_1-\gamma_i)^2} \tag{B3}$$

and

$$k^2+\omega_i^2=\frac{(A-B)^2}{(A+B)^2}+\left[\frac{a_1+b_1g_1-\gamma_ig_1}{b_1-a_1g_1-\gamma_i}\right]^2\,. \tag{B4}$$

Expanding this out, we find

$$k^2+\omega_i^2=2(1+g_1^2)(A+B)^{-2}(b_1-a_1g_1-\gamma_i)^2 \times\{[(b_1-\gamma_i)^2+a_1^2][(b_1-b_2)^2+a_1^2+a_2^2] -[(b_1-b_2)^2-a_1^2+a_2^2][(b_1-\gamma_i)^2-a_1^2] -4a_1^2(b_1-\gamma_i)(b_1-b_2)\} . \tag{B5}$$

After a bit of work, we find

$$\sqrt{1+\omega_i^2}\sqrt{k^2+\omega_i^2}=\frac{(\omega_i-g_1)\omega_i}{a_1+b_1g_1-g_1\gamma_i} \times g\sqrt{(\gamma_i-b_1)^2+a_1^2} \times\sqrt{(\gamma_i-b_2)^2+a_2^2} , \tag{B6}$$

which is the desired result.

APPENDIX C

We wish to calculate the tangent of ϕ_2 in (2.51) which can be written as

$$\tan\phi_2=\frac{b_2+ia_2-b_1+a_1g_1}{a_1+g_1b_1-g_1(b_2+ia_2)} . \tag{C1}$$

Let us take the real part of this equation:

$$\mathrm{Re}\tan\phi_2=\{4a_1(b_2-b_1)[a_1^2-a_2^2-(b_2-b_1)^2] +2a_1(b_2-b_1)[A^2+B^2-4a_1^2]\} \times\{[a_1+g_1(b_1-b_2)]^2+g_1^2a_2^2\}^{-1} \times[(A+B)^2-4a_1^2]^{-1}=0 . \tag{C2}$$

Now that we have shown that the tangent function is pure imaginary, we now show

$$\begin{aligned}\tan\phi_2&=i\frac{a_1a_2(1+g_1^2)}{[a_1+g_1(b_1-b_2)]^2+g_1^2a_2^2}\\ &=ia_1a_2(a_1^2+\tfrac{1}{2}[(b_1-b_2)^2-a_1^2+a_2^2]-\{[8a_1^2(b_2-b_1)]^2+2[(b_1-b_2)^2-a_1^2+a_2^2]^2\}(4AB)^{-1})^{-1}\\ &=i(4a_1a_2)(A-B)^{-2}=\frac{i}{k} .\end{aligned} \tag{C3}$$

APPENDIX D

We now prove (2.66):

$$\sum_{i=1}^{4}\frac{\alpha_i\sin\beta_i\cos\beta_i}{r(\beta_i,k')}=0 . \tag{D1}$$

From (2.45), we can write

$$\sin\beta_i\cos\beta_i=\frac{\omega_i}{1+\omega_i^2} . \tag{D2}$$

This means

$$\sum_{i=1}^{4}\alpha_i\frac{\sin\beta_i\cos\beta_i}{r(\beta_i,k')}=\sum_{i=1}^{4}\frac{\alpha_i\omega_i}{\sqrt{(1+\omega_i^2)(k^2+\omega_i^2)}} . \tag{D3}$$

We now use the result of Appendix B, so that we can replace

$$\sqrt{(1+\omega_i^2)(k^2+\omega_i^2)} =g\omega_i\frac{\omega_i-g_l}{a_1+b_1g_1-\gamma_i} \times\sqrt{(\gamma_i-b_1)^2+a_1^2}\sqrt{(\gamma_i-b_2)^2+a_2^2} . \tag{D4}$$

Substituting the value for ω_i, we can thus write (D1) as

$$\begin{aligned}&\sum_{i=1}^{4}\frac{\alpha_i(a_1+b_1g_1-\gamma_ig_1)(b_1-a_1g_1-\gamma_i)}{ga_1(1+g_1^2)\sqrt{(\gamma_i-b_1)^2+a_1^2}\sqrt{(\gamma_i-b_2)^2+a_2^2}}\\ &\quad=\sum_{i=1}^{4}\frac{(a_1+b_1-\gamma_ig_1)(b_1-a_1g_1-\gamma_i)}{Nga_1(1+g_1^2)\prod_{j=1,\,j\neq i}^{4}(\gamma_i-\gamma_j)}\\ &\quad=\frac{1}{ga_1(1+g_1^2)}\sum_{i=1}^{4}\frac{1}{\prod_{j=1,\,j\neq i}^{4}(\gamma_i-\gamma_j)}(c_1+c_2\gamma_i+c_3\gamma_i^2) ,\end{aligned} \tag{D5}$$

where we have eliminated α_i by its boundary condition (2.14) and c_i are constants independent of the index i.

But we also have identities

$$\sum_{i=1}^{4}\frac{1}{\prod_{j=1,\,j\neq i}^{4}(\gamma_i-\gamma_j)}=\sum_{i=1}^{4}\frac{\gamma_i}{\prod_{j=1,\,j\neq i}^{4}(\gamma_i-\gamma_j)} =\sum_{i=1}^{4}\frac{\gamma_i^2}{\prod_{j=1,\,j\neq i}^{4}(\gamma_i-\gamma_j)}=0 . \tag{D6}$$

Thus, because each term in (D5) vanished, we arrive at (D1).

[1]M. Kaku and K. Kikkawa, Phys. Rev. D **10**, 1110 (1974); **10**, 1823 (1974).
[2]E. Witten, Nucl. Phys. **B268**, 253 (1986).
[3]S. Giddings, Nucl. Phys. **B278**, 242 (1986).
[4]S. Giddings and E. Martinec, Nucl. Phys. **B278**, 91 (1986).
[5]B. Zwiebach, Ann. Phys. (N.Y.) **186**, 111 (1988); Phys. Lett. B **213**, 25 (1988).
[6]B. Zwiebach, Nucl. Phys. **B317**, 147 (1989).
[7]M. Kaku, *Introduction to Superstrings* (Springer, New York, 1988), Chap. 8; Int. J. Mod. Phys. A **2**, 1 (1987); Phys. Lett. B **200**, 22 (1988); Report No. CCNY-HEP-14-1986 (unpublished); Report No. CCNY-HEP-3-1987, 1987 (unpublished); Report No. CCNY-HEP-7-1988 (unpublished); CCNY report (unpublished).
[8]M. Kaku, Phys. Rev. D **38**, 3052 (1988); M. Kaku and J. Lykken, *ibid.* **38**, 3067 (1988).
[9]M. Kaku, in *Functional Integration, Geometry and Strings,* proceedings of the 25th Karpacz Winter School, Karpacz, Poland, 1989 (Birkhauser, Berlin, 1989). Also M. Kaku, this issue, Phys. Rev. D **41**, 3734 (1990); Report No. OU-HET-121 (unpublished); See also T. Kugo, H. Kunitomo, and K. Suehiro, Phys. Lett. B **226**, 48 (1989).
[10]P. F. Bryd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer, New York, 1971).
[11]E. Cremmer and J. L. Gervais, Nucl. Phys. **B90**, 410 (1975).
[12]S. Mandelstam, in *Unified String Theories,* edited by M. Green and D. Gross (World Scientific, Singapore, 1986).