Stability, causality, and hyperbolicity in Carter's "regular" theory of relativistic heat-conducting fluids

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Stability and causality are studied for linear perturbations about equilibrium in Carter's "regular" theory of relativistic heat-conducting fluids. The "regular" theory, when linearized around an equilibrium state having vanishing expansion and shear, is shown to be equivalent to the inviscid limit of the linearized Israel-Stewart theory of relativistic dissipative fluids for a particular choice of the second-order coefficients β_1 and γ_2. A set of stability conditions is determined for linear perturbations of a general inviscid Israel-Stewart fluid using a monotonically decreasing energy functional. It is shown that, as in the viscous case, stability implies that the characteristic velocities are subluminal and that perturbations obey hyperbolic equations. The converse theorem is also true. We then apply this analysis to a nonrelativistic Boltzmann gas and to a strongly degenerate free Fermi gas in the "regular" theory. Carter's "regular" theory is shown to be incapable of correctly describing the nonrelativistic Boltzmann gas and the degenerate Fermi gas (at all temperatures).

I. INTRODUCTION

The classic theories of relativistic dissipative fluids developed by Eckart[1-3] and Landau and Lifshitz[4] are now known to be unsatisfactory. In these theories all equilibria are unstable,[5] and linear perturbations propagate acausally.[6] A more complicated theory proposed by Israel and Stewart[7-10] does not have these pathological properties. Their theory admits stable equilibria, and linear perturbations about equilibrium propagate causally and obey hyperbolic differential equations.[11,12]

One difficulty with the application of the Israel-Stewart theory is that the equations of motion contain a number of "second-order coefficients" which cannot be derived from the equation of state of the fluid. These additional thermodynamic functions, which are relaxation times for the various forms of dissipation and coupling constants between the different forms of dissipation, must be specified in order to determine the evolution of the fluid. The second-order coefficients have been explicitly calculated for the Boltzmann gas and for the strongly degenerate free Fermi gas using relativistic kinetic theory.[10,13] They can in principle be determined from experiment for more complicated fluids.

An alternative theory of relativistic heat-conducting fluids, developed by Carter,[14,15] has been proposed[16] as a well behaved and simpler alternative to the Israel-Stewart theory without viscosity. Carter originally developed[14,15] a general formalism for treating relativistic heat-conducting fluids, which included as special cases the pathological theories of Eckart and of Landau and Lifshitz. Carter suggested[14-16] that if a certain function in the formalism, the "anomaly," were set equal to zero, then an acceptable alternative to the Eckart and Landau-Lifshitz theories would result. The zero-anomaly theory was termed the "regular" theory by Carter (we have placed quotation marks around "regular" because we have found the theory to be rather irregular when applied to simple gases). An attractive feature of Carter's "regular" theory is that no unknown second-order coefficients appear in the equations of motion; once the equation of state and the thermal conductivity are specified, the evolution of the fluid is completely determined. The purpose of this paper is to investigate whether or not Carter's "regular" theory is an acceptable alternative to the inviscid Israel-Stewart theory by examining the dynamics of small perturbations about equilibrium in the "regular" theory. We will often hereafter refer to Carter's "regular" theory simply as the "regular theory," despite our conclusions in this paper.

In Sec. II we review Carter's "regular" theory of relativistic heat-conducting fluids. We display the theory in the Eckart frame, where the fluid four-velocity is chosen to be parallel to the particle flux four-vector, to facilitate comparison to the usual presentations[11,12] of the Israel-Stewart theory. We show that both theories have eight degrees of freedom and possess the same equilibrium states. The evolution of the Israel-Stewart fluid is specified once the equation of state of the fluid, two second-order coefficients, and the thermal conductivity, each of which are functions of two independent thermodynamic variables, are known. In contrast, in the "regular" theory there are no unknown second-order coefficients.

In Sec. III we linearize the equations of motion of Carter's "regular" theory about a fiducial background equilibrium state. We show that, for fluids having background equlibrium states with zero expansion and shear, the equations governing linear perturbations in the regular theory are the same as those of the linearized inviscid Israel-Stewart theory when a particular choice of the two second-order coefficients is made. In particular, the second-order coefficient β_1 in the "regular" theory has the specific value $1/Tsn$. The "regular" theory thus has no unknown second-order coefficients only if the entropy is considered to be a known (or at least "knowable") function. This is rather unusual; in most thermodynamic theories only derivatives of the entropy (which are independent of its overall normalization) appear.

The regular theory, when linearized, is thus a special case of the general linearized inviscid Israel-Stewart theory for equilibria with vanishing expansion and shear. For equilibrium states having nonvanishing expansion or shear, the linearized equations of motion of the regular theory are considerably more complicated than those of the inviscid Israel-Stewart theory.

The stability of equilibrium states which have zero expansion and shear in the inviscid Israel-Stewart theory is studied in Sec. IV. An energy (Liapunov) functional is constructed which is quadratic in the perturbations and monotonically decreasing into the future. The energy functional can be expressed as the sum of quadratic terms. If each of these terms is non-negative, then the fluid is stable. The characteristic velocities at which perturbations propagate are also calculated, as are the conditions which must be satisfied for the perturbation equations to form a symmetric hyperbolic system. The stability and hyperbolicity conditions and the characteristic velocities of Carter's "regular" theory follow as a special case of these more general results. An equilibrium state of either theory is shown to be stable if all of the characteristic velocities are less than the speed of light and the perturbation equations form a symmetric hyperbolic system; i.e., just as in the viscous case,[11] the conditions required for stability are identical to those required to guarantee causality and hyperbolicity.

The stability conditions in Carter's "regular" theory are evaluated for a nonrelativistic Boltzmann gas and for a strongly degenerate (but arbitrarily relativistic) free Fermi gas in Sec. V. Since the heat-flow relaxation time β_1 is defined in the "regular" theory to be equal to $1/Tsn$ for all fluids, it is necessary to know the actual numerical value of the entropy in order to evaluate the stability conditions. Using the usual quantum-mechanical normalization of the entropy, we find the regular theory to be hyperbolic for both of these simple gases. We demonstrate that the nonrelativistic Boltzmann gas is predicted to be unstable and acausal by the regular theory; for example, if the particle mass is taken to be the mass of an argon atom, one of the characteristic velocities is roughly equal to 3.6 times the speed of light when the gas is at standard temperature and pressure. We show that the regular theory also predicts the strongly degenerate free Fermi gas to be unstable and acausal at all temperatures. In this case one of the characteristic velocities is shown to approach the speed of light from above as the ratio of the nonrelativistic chemical potential to the rest mass of the fluid particles goes to zero. This is in contrast with the Israel-Stewart theory, which predicts that the Boltzmann gas and the degenerate free Fermi gas are stable with perturbations that propagate causally via hyperbolic equations.[10,11,13] For these reasons we do not believe Carter's "regular" theory to be a generally acceptable alternative to the inviscid Israel-Stewart theory. It is possible that some nonzero choice for the "anomaly" in Carter's formalism (leading to a different theory than the "regular" theory proposed by Carter) will yield an acceptable relativistic theory; we have not explored this possibility here.

We use units in which the speed of light $c=1$.

II. CARTER'S "REGULAR" THEORY

Carter's "regular" theory of relativistic heat-conducting fluids is reviewed in this section, and compared with the inviscid limit of the Israel-Stewart theory. More detailed derivations and discussions of the regular theory are given in Refs. 14–16. We shall follow the presentation of the theory given in Sec. 2.5 of Ref. 15, but will use the notation of Refs. 11 and 12, to facilitate comparison with the Israel-Stewart theory.

In both theories the equations of motion of the fluid include the conservation laws

$$\nabla_a T^{ab}=0 \, , \tag{1}$$

$$\nabla_a n^a=0 \, , \tag{2}$$

where T^{ab} is the stress-energy tensor and n^a is the particle number current. The derivative ∇_a is the covariant derivative compatible with the metric tensor g^{ab}.

Following Carter,[15] we will work in the so-called "Eckart frame," in which the four-velocity u^a is chosen to be parallel to the particle number current:

$$u^a=n^a/n \, , \tag{3}$$

where $n=(-n^a n_a)^{1/2}$ is the number density of particles. With this choice of four-velocity, the stress-energy tensor in the regular theory may be written as

$$T_C^{ab}=\rho u^a u^b+p q^{ab}+q^a u^b+q^b u^a+\frac{q^a q^b}{Tsn} \, , \tag{4}$$

where ρ is the energy density, p is the pressure, T is the temperature, s is the entropy per particle, and q^a is the heat-flow vector field, which is constrained to be orthogonal to u^a: $u^a q_a=0$. Finally, q^{ab} is the projection tensor orthogonal to u^a:

$$q^{ab}=g^{ab}+u^a u^b \, . \tag{5}$$

Deviations from equilibrium are described by the heat-flow vector field q^a. The heat flow is defined in the regular theory by

$$\begin{aligned} -\frac{1}{\kappa}q^a=Tq^{ab}\left[\frac{1}{T}\nabla_b T+u^c\nabla_c u_b\right. & \\ +u^b\nabla_b\left[\frac{q^a}{sn}\right]+\frac{q_b}{sn}\nabla^a u^b & \\ \left.+\frac{2}{sn}q^{a[c}q_b{}^{d]}q^b\left[\nabla_c u_d+\frac{1}{T}\nabla_c\left[\frac{q_d}{sn}\right]\right]\right] \, , & \end{aligned} \tag{6}$$

where κ is the thermal conductivity, and the square brackets around the indices denote antisymmetrization.

The first law of thermodynamics is the regular theory takes the form

$$d\rho=\left[\frac{\rho+p}{n}-\frac{q^2}{Tsn^2}\right]dn+\left[Tn-\frac{q^2}{Ts^2n}\right]ds+\frac{q}{Tsn}dq\ , \tag{7}$$

where $q=(q^aq_a)^{1/2}$. At first glance there appear to be three independent thermodynamic variables (e.g., n, s, and q), rather than the usual two. The dq term in Eq. (7) cannot simply be dropped, as it is necessary if the second law of thermodynamics is to hold in the "regular" theory. One might then wonder how a three-parameter equation of state $\rho(n,s,q)$ is to be specified; as we shall show below, it is possible to use the first and second laws to build up the required three-parameter nonequilibrium equation of state from a known two-parameter (n,s) equilibrium equation of state.

The conservation equations, conductivity equation, and the first law [Eqs. (1), (2), (6), and (7)] imply that the entropy current s^a,

$$s^a=snu^a+\frac{1}{T}q^a\ , \tag{8}$$

satisfies the second law of thermodynamics:

$$\nabla_a s^a=\frac{q^aq_a}{\kappa T^2}\geq 0\ . \tag{9}$$

Equilibrium states are those for which the divergence of the entropy current vanishes. Equation (9) then implies that the fluid is in equilibrium when the heat flow vanishes. The terms involving q in the first law then vanish, and the equation of state takes on its usual form, in which it manifestly only depends on two variables, e.g., $\rho(n,s)$.

An interesting constraint on the three-parameter equation of state arises because both $(\partial\rho/\partial s)_{n,q}$ and $(\partial\rho/\partial q)_{n,s}$ depend only on n, q, s, and T. Since the values of n, q, and s specify a thermodynamic state, the temperature T may be defined using the partial derivative $(\partial\rho/\partial s)_{n,q}$, which gives a quadratic equation to be solved for T:

$$T\left[\frac{\partial\rho}{\partial s}\right]_{n,q}=T^2n-\frac{q^2}{s^2n}\ , \tag{10}$$

which has the roots

$$T=\frac{1}{2n}\left[\frac{\partial\rho}{\partial s}\right]_{n,q}\pm\frac{1}{2n}\left[\left[\frac{\partial\rho}{\partial s}\right]_{n,q}^2+\left[\frac{2q}{s}\right]^2\right]^{1/2}. \tag{11}$$

The positive square root is chosen as the temperature because the minus sign yields a temperature which is zero in equilibrium and negative otherwise. The temperature can also be calculated, *independently*, from the first law using the partial derivative $(\partial\rho/\partial q)_{n,s}$:

$$T=\frac{q}{sn}\left[\frac{\partial\rho}{\partial q}\right]_{n,s}^{-1}. \tag{12}$$

Since the temperatures defined in Eqs. (11) and (12) must be the same, a constraint on the thermodynamic derivatives of ρ, and hence on the form of the nonequilibrium equation of state $\rho(n,s,q)$, is implied by the regular theory form of the first law [Eq. (7)]. Setting the two expressions for the temperature equal, we find that the nonequilibrium equation of state must satisfy

$$\frac{2q}{s}\left[\frac{\partial\rho}{\partial q}\right]_{n,s}^{-1}=\left[\frac{\partial\rho}{\partial s}\right]_{n,q}+\left[\left[\frac{\partial\rho}{\partial s}\right]_{n,q}^2+\left[\frac{2q}{s}\right]^2\right]^{1/2}. \tag{13}$$

We now see how to proceed to construct the needed three-variable nonequilibrium equation of state for use in the regular theory. One begins by choosing an equilibrium equation of state $\rho(n,s)$. This function is the $q\to0$ limit of the required three-variable equation of state. The constraint derived from the first law, Eq. (13), may then be integrated with respect to q from zero up to a final value q to yield the required three-parameter equation of state. The need for this rather messy procedure is a result of casting the regular theory into the Eckart frame; fortunately it will prove unnecessary when dealing with linearized perturbations about equilibrium.

Finally, the pressure is defined through the first law as

$$p=-\rho+\frac{q^2}{Tsn}+n\left[\frac{\partial\rho}{\partial n}\right]_{s,q}. \tag{14}$$

The evolution of a fluid in Carter's "regular" theory is specified once the thermal conductivity (which is in general a function of the independent thermodynamic variables) and the equation of state $\rho(n,s,q)$ [subject to Eq. (11)] are known.

The inviscid limit of the Israel-Stewart theory differs from the regular theory in the definitions of the stress-energy tensor, conductivity equation, and the first law. In the inviscid Israel-Stewart theory these are given by

$$T^{ab}=\rho u^au^b+pq^{ab}+q^au^b+q^bu^a\ , \tag{15}$$

$$q^a=-\kappa Tq^{ab}\left[\frac{1}{T}\nabla_bT+u^c\nabla_cu_b+\beta_1u^c\nabla_cq_b+\frac{1}{2}Tq_b\nabla_c\left[\frac{\beta_1u^c}{T}\right]+\gamma_2\nabla_{[b}u_{c]}q^c\right], \tag{16}$$

and

$$d\rho=\frac{\rho+p}{n}dn+Tn\,ds\ . \tag{17}$$

The second-order coefficients β_1 and γ_2 are thermodynamic functions which must be known, along with the equation of state and thermal conductivity, for the evolution equations to be integrable. The temperature and pressure are again determined by the equation of state and the first law:

$$T=\frac{1}{n}\left[\frac{\partial\rho}{\partial s}\right]_n, \tag{18}$$

$$p=-\rho+n\left[\frac{\partial\rho}{\partial n}\right]_s. \tag{19}$$

The second law in the form of Eq. (14) is also satisfied by an Israel-Stewart fluid.

While the equations of the regular theory may here appear to be less simple than in the inviscid Israel-Stewart theory, we should recall that we have chosen here to cast the regular theory in a set of variables which would facilitate comparison with Israel and Stewart. It is possible to write the regular theory in a form which appears much simpler by using a different set of variables.[16] However, that simpler form may not be easily compared to the Israel-Stewart theory. One conceptual advantage to the variables used by Carter in Ref. 16 is that the equation of state there manifestly depends only on two variables, and thus may be taken to be the familiar equilibrium equation of state, without having to resort to integrating the complicated constraint of Eq. (11).

It is easy to see that the Israel-Stewart theory and the regular theory have the same equilibrium states. In both theories, when $\nabla_a s^a=0$, then $q^a=0$, the stress-energy tensor takes the perfect-fluid form

$$T^{ab}=\rho u^a u^b+p q^{ab} \tag{20}$$

and the first law of thermodynamics has the form of Eq. (17). It can be shown [from Eqs. (1) and (2) and either Eqs. (4), (6), and (7) or Eqs. (15)–(17)] that the equilibrium states in both theories satisfy

$$u^a\nabla_a n+n\nabla_a u^a=0 \;, \tag{21}$$

$$u^a\nabla_a\rho+(\rho+p)\nabla_a u^a=0 \;, \tag{22}$$

$$q^{ab}[\nabla_b p+(\rho+p)u^c\nabla_c u_b]=0 \;, \tag{23}$$

$$q^{ab}[\nabla_b T+Tu^c\nabla_c u_b]=0 \;. \tag{24}$$

Both Carter's "regular" theory and the inviscid limit of the Israel-Stewart theory posses eight degrees of freedom. These may be taken to be the three independent components of the four-velocity, u^a, the three independent components of the heat flow, q^a, and two thermodynamic degrees of freedom. In contrast, the older first-order theories of Eckart[1] and Landau and Lifshitz[4] possess only the same five degrees of freedom as a perfect fluid; in those theories the heat-flow vector is determined by an algebraic constraint equation, rather than by a differential equation such as Eq. (6) or Eq. (16).

III. LINEARIZED "REGULAR" THEORY

In this section we derive the equations for linear perturbations around equilibrium in the regular theory and compare them to the linearized inviscid limit of the Israel-Stewart theory. The complexity of the constraint on the regular theory equation of state [Eq. (13)] may be avoided by expanding the equation of state, $\rho(n,s,q)$, in a Taylor series about $q=0$:

$$\rho(n,s,q)=\rho(n,s,0)+\left[\frac{\partial\rho}{\partial q}\right]_{n,s}q+\frac{1}{2}\left[\frac{\partial^2\rho}{\partial q^2}\right]_{n,s}q^2+\cdots \;, \tag{25}$$

where the thermodynamic derivatives are to be evaluated at $q=0$. There is no term linear in q in this expansion (despite first appearances), since the partial derivative $(\partial\rho/\partial q)_{n,s}$ is linearly proportional to q itself, by Eq. (12). Since the first terms in the expansion involving q are then quadratic in q, they may be neglected in linearizing the equations around $q=0$. All nonequilibrium thermodynamic functions can be then evaluated at $q=0$, and they thus manifestly depend on just two independent thermodynamic variables. We denote by δQ the difference between the nonequilibrium value of a field Q at a given spacetime point and the value of the field at the same spacetime point in the fiducial equilibrium state; i.e., we will work with Eulerian perturbations. The fields $\delta\rho$, δn, δu^a, δq^a, and δg^{ab} describe the perturbations of the fluid about its equilibrium state. Variables which do not include the prefix δ (e.g., n, ρ, u^a, g^{ab}, etc.) refer to the value of the field in the equilibrium state. The fields in the equilibrium state satisfy Eqs. (21)–(24).

We assume that all perturbation quantities δQ are small so that the evolution of these quantities is described well by the linearized equations of motion. For simplicity we assume that $\delta g^{ab}=0$. This limits the applicability of the equations of motion displayed below to special-relativistic fluids, or to short-wavelength perturbations of the fluid; it does not, however, imply that the background geometry is flat. The equations of motion of the regular theory [Eqs. (1), (2), and (6)], linearized about an arbitrary equilibrium state, become

$$\nabla_a\delta T^{ab}=0 \;, \tag{26}$$

$$\nabla_a(n\delta u^a)+\nabla_a(u^a\delta n)=0 \;, \tag{27}$$

$$\begin{aligned}\delta q^a=-\kappa T q^{ab}\Bigg[&\nabla_b\left[\frac{\delta T}{T}\right]+u^c\nabla_c\delta u_b+\delta u^c\nabla_c u_b\\&+\tilde\beta_1 u^c\nabla_c\delta q_b+\frac{1}{2}T\delta q_b\nabla_c\left[\frac{\tilde\beta_1 u^c}{T}\right]\\&+\tilde\gamma_2\delta q^c\nabla_{[b}u_{c]}+\tilde\beta_1\left[\frac{\theta}{3}+\frac{1}{T}u^c\nabla_c T\right]\delta q_b\\&+\tilde\beta_1\sigma_{bc}\delta q^c\Bigg] \;,\end{aligned} \tag{28}$$

where $\tilde\beta_1\equiv 1/Tsn$ and $\tilde\gamma_2\equiv 3/Tsn$ (this notation is chosen so as to reveal the similarity to the Israel-Stewart theory). The expansion of the fluid in the background equilibrium state is $\theta=\nabla_a u^a$, and the equilibrium shear tensor is $\sigma_{ab}=\frac{1}{2}(q^c{}_b\nabla_c u_a+q^c{}_a\nabla_c u_b)-\theta q_{ab}/3$. The perturbed stress-energy tensor is

$$\begin{aligned}\delta T_{ab}=&(\rho+p)(\delta u^a u^b+u^a\delta u^b)\\&+\delta\rho u^a u^b+\delta p q^{ab}+\delta q^a u^b+\delta q^b u^a \;.\end{aligned} \tag{29}$$

The expansion and shear do not appear in the stress-energy tensor because the bulk and shear viscosities are assumed to be zero. The perturbation variables satisfy the constraints $u^a\delta q_a=u^a\delta u_a=0$.

Except for the heat-flow equation [Eq. (28)], these perturbation evolution equations are identical to the equations that result when the Israel-Stewart theory is linearized. In the linearized Israel-Stewart theory the heat-flow vector is

$$\delta q^a = -\kappa T q^{ab}\left[\nabla_b\left[\frac{\delta T}{T}\right] + u^c\nabla_c\delta u_b + \delta u^c\nabla_c u_b + \beta_1 u^c\nabla_c\delta q_b + \frac{1}{2}T\delta q_b\nabla_c\left[\frac{\beta_1 u^c}{T}\right] + \gamma_2\delta q^c\nabla_{[b}u_{c]}\right] . \tag{30}$$

The second-order coefficients β_1 and γ_2 appearing in this equation are unknown thermodynamic functions which must be specified (along with the equation of state and the thermal conductivity) in order for the evolution of the perturbation fields to be determined. They may be determined from kinetic theory or other microscopic considerations, or may be determined in principle from experiment.

If the background equilibrium state has vanishing expansion ($\theta=0$), then this implies that $u^a\nabla_a n = u^a\nabla_a\rho = 0$ by Eqs. (21) and (22), and, using the first law, that $u^a\nabla_a T = 0$. If the background equilibrium state also has vanishing shear ($\sigma_{ab}=0$), then the evolution equations for the perturbation fields in the regular theory are identical to those in the Israel-Stewart theory, if we take as the values of the second-order coefficients $\beta_1=\tilde{\beta}_1, \gamma_2=\tilde{\gamma}_2$. In this case (linearized about a equilibrium state with vanishing expansion and shear), Carter's "regular" theory is a special case of the Israel-Stewart theory, and the predictions of the theories will be identical. However, the second-order coefficients for the simple gases[10,13] are known from kinetic theory to have very different values from the "regular" theory values $\tilde{\beta}_1, \tilde{\gamma}_2$. Using these values for the second-order coefficients will yield predictions for the evolution of the perturbation fields, at least in the case of the simple gases, which disagree with the results of kinetic theory.

If the background equilibrium state does not have vanishing shear or expansion, the predictions of the theories will differ. The Israel-Stewart theory appears to be easier to work with in this case (provided one has some knowledge of the values of β_1 and γ_2) because Eq. (30), which defines the heat flow, is considerably simpler than the equivalent Eq. (28) of Carter's "regular" theory.

IV. STABILITY, CAUSALITY, AND HYPERBOLICITY

The connection between stability, causality, and hyperbolicity in the linearized inviscid Israel-Stewart theory and in the linearized regular theory will be studied in this section. We will only consider equilibrium states having vanishing expansion and shear so that the energy functional techniques of treating stability developed in Ref. 11 may be used, and so that the regular theory is obtained as a special case of the Israel-Stewart theory by setting $\beta_1=\tilde{\beta}_1$ (the value of γ_2 is not constrained by stability considerations). Our treatment closely follows that of Ref. 11, in which stability of the viscous Israel-Stewart fluid was analyzed. Owing to the way in which thermal conductivity and viscosity are coupled in the more general theory, however, the separate derivation given below is necessary to treat the inviscid case properly.

The conditions which an equilibrium state must satisfy for stability will be derived first. An energy (or Liapunov) functional is a nonincreasing function of time which depends quadratically on the perturbation fields. If this functional is non-negative for all possible values of the perturbation fields the equilibrium state is stable, since in this case the energy functional is bounded below by zero. If, instead, the energy functional is negative for some values of the perturbation fields, then (since the energy functional is nonincreasing into the future) the energy functional is unbounded below into the future in this case, implying that some of the perturbation fields could evolve towards infinity. This implies (at the level of linear perturbation theory) that the equilibrium state is unstable. These arguments are made in a more rigorous form in Refs. 11 and 12.

An energy functional for linear perturbations in the inviscid Israel-Stewart theory can be constructed from an energy current vector E^a defined by

$$TE^a = \delta T^a{}_b\delta u^b - \frac{1}{2}(\rho+p)u^a\delta u^b\delta u_b + \frac{\delta T}{T}\delta q^a + \frac{1}{2}\beta_1 u^a\delta q^b\delta q_b + \frac{1}{2}(\rho+p)^{-1}\left[\left[\frac{\partial\rho}{\partial p}\right]_s(\delta p)^2 + \left[\frac{\partial\rho}{\partial s}\right]_p\left[\frac{\partial p}{\partial s}\right]_\Theta(\delta s)^2\right]u^a , \tag{31}$$

where $\Theta=(\rho+p)/Tn - s$. The total energy $E(\Sigma)$ associated with a spacelike surface Σ is

$$E(\Sigma) = \int_\Sigma E^a d\Sigma_a . \tag{32}$$

The energy functional $E(\Sigma)$ will be nonincreasing into the future provided

$$E(\Sigma') - E(\Sigma) = \int \nabla_a E^a dV \le 0 \tag{33}$$

for every spacelike surface Σ' to the future of Σ. Thus the energy functional will be a nonincreasing function of time as long as the divergence of the energy current is nonpositive. The divergence of the energy current in Eq. (31) may be calculated with the aid of the linearized equations [Eq. (26), (27), (29), and (30)]. For the case of vanishing expansion and shear the result is

$$\nabla_a E^a = -\frac{\delta q^a\delta q_a}{\kappa T^2} \le 0 . \tag{34}$$

As required, the energy functional is therefore nonincreasing into the future.

The energy functional may be expressed in terms of an energy density:

$$E(\Sigma) = \int_\Sigma e\frac{u^a}{T}d^3x_a , \tag{35}$$

where the energy density e is defined by

$$e=TE^a t_a/u^b t_b \ . \tag{36}$$

The vector t^a is the future-directed unit normal to the spacelike surface Σ. Thus the energy functional will be non-negative for all possible values of the perturbation fields if and only if the energy density e is a non-negative functional of the perturbation fields at all points in the fluid.

It is possible to factor the energy density into the form

$$e=\tfrac{1}{2}\sum_A \Omega_A(\delta Z_A)^2 \ . \tag{37}$$

The Ω_A are functions of the thermodynamic variables of the equilibrium state given by

$$\Omega_1=(\rho+p)^{-1}\left[\frac{\partial p}{\partial \rho}\right]_s , \tag{38}$$

$$\Omega_2=(\rho+p)^{-1}\left[\frac{\partial p}{\partial s}\right]_p\left[\frac{\partial p}{\partial s}\right]_\Theta , \tag{39}$$

$$\Omega_3=(\rho+p)\left[1-\lambda^2\left[\frac{\partial p}{\partial \rho}\right]_s\right] -\frac{\lambda^2}{\Omega_5}\left[\frac{1}{\lambda^2}-\frac{n}{T}\left[\frac{\partial T}{\partial n}\right]_s\right]^2 , \tag{40}$$

$$\Omega_4=\rho+p-\frac{1}{\beta_1} , \tag{41}$$

$$\Omega_5=\frac{\beta_1}{\lambda^2}-\frac{1}{T^2 n}\left[\frac{\partial T}{\partial s}\right]_n , \tag{42}$$

$$\Omega_6=\beta_1 \ . \tag{43}$$

The δZ^a are linearly independent combinations of the perturbation fields:

$$\delta Z_1=\delta p+(\rho+p)\left[\frac{\partial p}{\partial \rho}\right]_s \times\left[\lambda_a\delta u^a+\frac{1}{T}\left[\frac{\partial T}{\partial p}\right]_s\lambda_a\delta q^a\right] , \tag{44}$$

$$\delta Z_2=\delta s+\frac{\rho+p}{T}\left[\frac{\partial T}{\partial \rho}\right]_p\left[\frac{\partial s}{\partial p}\right]_\Theta\lambda_a\delta q^a , \tag{45}$$

$$\delta Z_3=\frac{1}{\lambda}\lambda_a\delta u^a , \tag{46}$$

$$\delta Z_4{}^a=\gamma^a{}_b\delta u^b , \tag{47}$$

$$\delta Z_5=\lambda_a\delta q^a+\frac{1}{\Omega_5}\left[\frac{1}{\lambda^2}-\frac{n}{T}\left[\frac{\partial T}{\partial n}\right]_s\right]\lambda^a\delta u_a , \tag{48}$$

$$\delta Z_6{}^a=\gamma^a{}_b\delta q^b+\frac{1}{\beta_1}\gamma^a{}_b\delta u^b \ . \tag{49}$$

In these equations λ^a is the velocity of observers moving relative to the rest frame of the background equilibrium state along t^a,

$$\lambda^a=q^{ab}t_b/u_c t^c , \tag{50}$$

λ is the norm of λ^a (which can be shown to be constrained to $0\le\lambda^2\le 1$), and the projection tensor γ^{ab} is given by

$$\gamma^{ab}=q^{ab}-\frac{1}{\lambda^2}\lambda^a\lambda^b \ . \tag{51}$$

The requirement that the energy functional be non-negative is then equivalent to the condition that each of the Ω_A be non-negative:

$$\Omega_A\ge 0 \ . \tag{52}$$

Equation (52) must be satisfied for all values of λ between zero and one (the speed of light) for the equilibrium state to be stable. Setting $\lambda=1$ gives the most restrictive conditions. These inequalities are also the stability conditions for equilibrium states with vanishing expansion and shear in the regular theory, provided that β_1 is replaced by $\tilde{\beta}_1$ in the expressions for the Ω_A.

The stability conditions require the square of the adiabatic sound speed $(\partial p/\partial\rho)_s$ to be bounded between zero and the speed of light by the Ω_1 and Ω_3 conditions. The Ω_2 condition is the relativistic Schwarzschild criterion for stability against convection.[17] The Ω_4 and Ω_6 conditions require β_1 to be bounded below by $(\rho+p)^{-1}$. In the regular theory the Ω_4 and Ω_6 conditions and the definition of $\tilde{\beta}_1$ may be used to show that the relativistic chemical potential

$$\mu=\frac{\rho+p}{n}-Ts \tag{53}$$

is required to be non-negative. The Ω_5 condition in the regular theory evaluated at $\lambda=1$ is a constraint on the specific heat at constant volume $T(\partial s/\partial T)_n$:

$$c_V\ge s \ . \tag{54}$$

The causal properties of the two theories will now be studied. The system of equations for the linear perturbations in the inviscid Israel-Stewart theory [Eqs. (26), (27), (29), and (30)] has the form

$$A^A{}_B{}^a\nabla_a\delta Y^B+B^A{}_B\delta Y^B=0 , \tag{55}$$

where the components of δY^B are the eight perturbation fields δn, $\delta\rho$, δu^a, δq^a, and the index A labels one of the eight equations for the perturbation fields. The components of the matrices $A^A{}_B{}^a$ and $B^A{}_B$ are functions of the variables of the equilibrium state. The level surfaces of a scalar function ϕ satisfying

$$\det(A^A{}_B{}^a\nabla_a\phi)=0 \tag{56}$$

are the characteristic surfaces for these equations; the fields δY^B cannot be freely specified on these surfaces [see Ref. 18, p. 170]. The characteristic velocities are the slopes of these surfaces; information propagates along the characteristic surfaces with these velocities.

The characteristic velocities are most easily found by solving Eq. (56) with the following choice of coordinates. At some point in the fluid choose a Cartesian coordinate system which is comoving with the fluid and which has the x^1 direction along the direction of spatial variation of

ϕ. Then at this point

$$g^{ab}\partial_a\partial_b = -(\partial_0)^2+(\partial_1)^2+(\partial_2)^2+(\partial_3)^2 , \tag{57}$$

$$u^a\partial_a = \partial_0 , \tag{58}$$

and

$$\phi = \phi(x^0, x^1) . \tag{59}$$

In this coordinate system the equation for the characteristic velocities [Eq. (56)] becomes

$$\det(v A^A{}_B{}^0 - A^A{}_B{}^1) = 0 , \tag{60}$$

where v is the characteristic velocity defined by

$$v = -\frac{\partial_0\phi}{\partial_1\phi} . \tag{61}$$

When the set of eight perturbation fields is chosen as

$$\delta Y^B = \{T\delta\Theta, \delta T/T, \delta u^1, \delta q^1, \delta u^2, \delta q^2, \delta u^3, \delta q^3\} , \tag{62}$$

the characteristic matrix block diagonalizes:

$$v A^A{}_B{}^0 - A^A{}_B{}^1 = \begin{bmatrix} \mathbf{Q} & 0 & 0 \\ 0 & \mathbf{R} & 0 \\ 0 & 0 & \mathbf{R} \end{bmatrix} . \tag{63}$$

The matrices $\mathbf{Q}$ and $\mathbf{R}$ are defined as

$$\mathbf{Q} = \begin{bmatrix} \frac{v}{T}\left[\frac{\partial n}{\partial\Theta}\right]_T & \frac{v}{T}\left[\frac{\partial\rho}{\partial\Theta}\right]_T & -n & 0 \\ \frac{v}{T}\left[\frac{\partial\rho}{\partial\Theta}\right]_T & vT\left[\frac{\partial\rho}{\partial T}\right]_\Theta & -(\rho+p) & -1 \\ -n & -(\rho+p) & v(\rho+p) & v \\ 0 & -1 & v & v\beta_1 \end{bmatrix} , \tag{64}$$

$$\mathbf{R} = \begin{bmatrix} v(\rho+p) & v \\ v & v\beta_1 \end{bmatrix} . \tag{65}$$

Thus the equation for the characteristic velocities [Eq. (60)] takes the form

$$\det\mathbf{Q}(\det\mathbf{R})^2 = 0 , \tag{66}$$

where the determinants are

$$\det\mathbf{Q} = (Av_L{}^4 + Bv_L{}^2 + C) \times \left[\left[\frac{\partial n}{\partial\Theta}\right]_T \left[\frac{\partial\rho}{\partial T}\right]_\Theta - \left[\frac{\partial n}{\partial T}\right]_\Theta \left[\frac{\partial\rho}{\partial\Theta}\right]_T\right] , \tag{67}$$

$$\det\mathbf{R} = v_T{}^2[(\rho+p)\beta_1 - 1] . \tag{68}$$

The coefficients of the quartic in Eq. (67) are

$$A = (\rho+p)\beta_1 - 1 , \tag{69}$$

$$B = [1-(\rho+p)\beta_1]\left[\frac{\partial p}{\partial\rho}\right]_s + \left[\frac{\partial\Theta}{\partial s}\right]_\rho , \tag{70}$$

$$C = \frac{1}{T^2}\left[\frac{\partial T}{\partial s}\right]_p \left[\frac{\partial p}{\partial n}\right]_s . \tag{71}$$

The set of characteristic velocities is found by setting $\det\mathbf{R}$ and $\det\mathbf{Q}$ to zero. The two characteristic velocities found by setting $\det\mathbf{R}$ to zero both vanish. These are the transverse characteristic velocities since the matrix $\mathbf{R}$ describes perturbations tangent to the characteristic surfaces. The matrix $\mathbf{Q}$ has four nonvanishing characteristic velocities which satisfy the quartic

$$Av_L{}^4 + Bv_L{}^2 + C = 0 . \tag{72}$$

These are the longitudinal characteristic velocities. The characteristic velocities for the regular theory are obtained by replacing β_1 by the specific value $\tilde{\beta}_1$ in Eqs. (69) and (70).

The conditions for the system of linear perturbation equations [Eq. (55)] to form a hyperbolic system will now be determined. The most common definition of hyperbolicity, which requires the characteristic velocities to satisfy conditions such as reality, cannot be used because the characteristic velocities are not all distinct. Since the matrices $A^A{}_B{}^a$ are symmetric, the system of perturbation equations would form a symmetric hyperbolic system if some linear combination of the $A^A{}_B{}^a$ were positive definite. The system of perturbation equations would then have a well-posed initial-value problem (see Ref. 18, p. 593).

A necessary and sufficient condition for a matrix to be positive definite is that the determinant of each leading principal submatrix be positive. The matrix $A^A{}_B{}^0$ is itself positive definite if and only if the following inequalities are satisfied:

$$\left[\frac{\partial n}{\partial\Theta}\right]_T > 0 , \tag{73}$$

$$\left[\frac{\partial\rho}{\partial T}\right]_n > 0 , \tag{74}$$

$$\beta_1(\rho+p) - 1 > 0 . \tag{75}$$

Thus a sufficient condition for a positive-definite linear combination of the $A^A{}_B{}^a$ to exist is for each of these inequalities to be satisfied. Equations (73) and (74) also imply that $(\partial\rho/\partial T)_\Theta > 0$.

Satisfaction of the inequalities in Eqs. (73)–(75) is also a necessary condition for a positive-definite linear combination of the $A^A{}_B{}^a$ to exist. The matrix $A^A{}_B{}^0$ must be positive definite because only $A^A{}_B{}^0$ has nonzero diagonal elements and none of the other $A^A{}_B{}^a$ have nonzero off-diagonal elements in the locations where the off-diagonal elements of $A^A{}_B{}^0$ are nonzero. Therefore, the perturbation equations [Eq. (55)] form a symmetric hyperbolic system if and only if the inequalities in Eqs. (73)–(75) are satisfied. It is conceivable that these are not the weakest conditions necessary for the theory to be hyperbolic; they are, however, probably the weakest conditions which guarantee hyperbolicity and imply stability.[12] Equations (73)–(75) are also the hyperbolicity conditions for the regular theory provided β_1 is replaced by $\tilde{\beta}_1$ in Eq. (75).

The connection between stability, causality, and hyperbolicity will now be established. In what follows it will be assumed that the stability conditions are strictly satisfied, $\Omega_A(\lambda^2) > 0$, for all values of λ bounded by $0 \le \lambda \le 1$. One relationship is that stability implies causality and hyperbolicity.

For equilibrium states having zero expansion and shear, the linear perturbations of a fluid described by the inviscid Israel-Stewart theory (or the regular theory) propagate causally (subliminally) and obey a symmetric hyperbolic system of differential equations when the stability conditions are satisfied.

This is demonstrated by showing that the longitudinal characteristic velocities are bounded by $0 \le v_L{}^2 \le 1$ and that hyperbolicity conditions are satisfied when the stability conditions are satisfied. Rather than reproducing the details of the proof here we refer the interested reader to Ref. 11 because the proof of the corresponding theorem in the Israel-Stewart theory with viscosity is nearly identical.

The converse theorem also holds.

In the inviscid Israel-Stewart theory (and hence also in Carter's "regular" theory), equilibrium states which have zero expansion and shear are stable against the unbounded growth of linear perturbations if these perturbations propagate causally and obey a symmetric hyperbolic system of equations, in the sense of Eqs. (73)–(75).

Again, we refer the reader to Ref. 11 for details of how this may be demonstrated.

We have shown that stability is equivalent to causality and hyperbolicity for equilibrium states with vanishing expansion and shear in both the linearized inviscid Israel-Stewart and the linearized regular theories.

V. APPLICATION TO SOME SIMPLE GASES

In Secs. III and IV it was shown that the significant difference between the linearized regular theory and the linearized inviscid Israel-Stewart theory for equilibrium states having zero expansion and shear is the value chosen for the second-order coefficient β_1. In this section we shall evaluate the stability conditions and characteristic velocities in the regular theory for a nonrelativistic Boltzmann gas and a strongly degenerate free Fermi gas.

For a nonrelativistic Boltzmann gas the value of β_1 appropriate to the Israel-Stewart theory has been calculated from kinetic theory:[10]

$$\beta_1 = \frac{2\beta}{5p} , \tag{76}$$

where the dimensionless parameter $\beta = m/kT$, and m is the rest mass of the fluid particles. The value of β_1 for a strongly degenerate free Fermi gas has also been determined:[13]

$$\beta_1 = \frac{9(1+\nu)(\beta\nu)^2}{A_0 \pi^2 \nu^2 (\nu^2+2\nu)^{3/2}} + O((\beta\nu)^0) , \tag{77}$$

where

$$A_0 = \frac{m^4 g}{2\pi^2 \hbar^3} , \tag{78}$$

g is the spin weight, and the dimensionless potential ν is defined by $\nu = \mu/m - 1$ [where μ is the relativistic chemical potential defined in Eq. (53)]. Equation (77) is valid as the leading term in an asymptotic expansion at any temperature (value of β), as long as the degenerate limit, $\beta\nu \gg 1$, is satisfied; both nonrelativistic and relativistic degenerate Fermi gases are thus included.

In the regular theory, however, for any fluid, $\tilde{\beta}_1 = 1/Tsn$. For a nonrelativistic Boltzmann gas

$$\tilde{\beta}_1 = p^{-1} \left\{ \ln \left[\frac{g}{n} \left[\frac{mkT}{2\pi\hbar^2} \right]^{3/2} \right] + \frac{5}{2} \right\}^{-1} , \tag{79}$$

where the standard quantum-mechanical arguments have been used to fix the zero of the entropy, and thus $\tilde{\beta}_1/\beta_1 \simeq \beta^{-1}$. For the free Fermi gas in the regular theory the results of Ref. 13 may be used to show that

$$\tilde{\beta}_1 = \frac{3(\beta\nu)^2}{A_0 \pi^2 \nu^2 (\nu+1)(\nu^2+2\nu)^{1/2}} - \frac{7(2\nu^2+4\nu-1)}{10 A_0 (\nu+1)(\nu^2+2\nu)^{5/2}} + O((\beta\nu)^{-2}) . \tag{80}$$

In most physical applications of the degenerate free Fermi gas (for example, metals[19] and white dwarf and neutron stars[20]), $\nu \lesssim 0.1$. In the limit $\nu \ll 1$, $\tilde{\beta}_1/\beta_1 \simeq 2\nu/3$. Since the second-order coefficient β_1 is essentially a relaxation time,[21,22] we see that for a nonrelativistic Boltzmann gas and for the physically interesting degenerate free Fermi gases the value for the relaxation time of the heat flow in the regular theory is much shorter than that predicted by kinetic theory. Thus the predictions of Carter's "regular" theory in these applications will be very different from the Israel-Stewart theory.

The Boltzmann gas and the strongly degenerate free Fermi gas described by the Israel-Stewart theory have been previously shown to be stable, with perturbations that propagate causally via hyperbolic differential equations when kinetic theory values for the second-order coefficients are used.[10,11,13] In contrast, we find that in Carter's "regular" theory, applied to a nonrelativistic Boltzmann gas, one of the stability conditions is violated:

$$\Omega_5(1) \simeq -\frac{2}{3p} \frac{\ln \left[\frac{g}{n} \left[\frac{mkT}{2\pi\hbar^2} \right]^{3/2} \right] + 1}{\ln \left[\frac{g}{n} \left[\frac{mkT}{2\pi\hbar^2} \right]^{3/2} \right] + \frac{5}{2}} < 0 . \tag{81}$$

The inequality in Eq. (81) follows because the argument of the logarithm must be large compared to one in order for a free Fermi or Bose gas to be well approximated as a Boltzmann gas. Thus, whenever conditions are such that the nonrelativistic Boltzmann approximation applies, Carter's "regular" theory predicts the gas to be unstable. Since stability is equivalent to causality and hyperbolicity, we should ask which of these conditions is violated. It is easy to see that Eqs. (73)–(75) are satisfied, so the regular theory applied to the nonrelativistic Boltzmann gas is hyperbolic, which then implies that it must be acausal. Carter's "regular" theory consequently predicts that perturbations can propagate faster than the speed of light in the nonrelativistic Boltzmann gas. For example, when

Eq. (72) is solved for the characteristic velocities of perturbations at atmospheric pressure and room temperature in Carter's "regular" theory, with the particle mass equal to the mass of an argon atom, the two distinct nonzero characteristic velocities are found to be $v_L \simeq 10^{-6}$ (which is close to the adiabatic sound speed) and $v_L \simeq 3.6$ (i.e., 3.6 times the speed of light).

The Ω_5 stability condition is also violated when Carter's "regular" theory is applied to the strongly degenerate free Fermi gas:

$$\Omega_5(1) = -\frac{6\nu^4+24\nu^3+37\nu^2+26\nu+12}{5A_0(\nu+1)^3(\nu+2\nu)^{5/2}} + O((\beta\nu)^{-2}) \ . \tag{82}$$

Since ν is positive for a degenerate free Fermi gas, $\Omega_5(1) < 0$. Thus the "regular" theory predicts all strongly degenerate free Fermi gases to be unstable. Equations (73)–(75) are satisfied so that the regular theory is also hyperbolic in this case. The solution of Eq. (72) again yields an acausal characteristic velocity:

$$v_L = 1 + \frac{2\pi^2\nu^2(8\nu^4+32\nu^3+40\nu^2+16\nu+9)}{15(\beta\nu)^2(\nu^2+2\nu)^2(2\nu^2+4\nu+3)} + O((\beta\nu)^{-4}) > 1 \ . \tag{83}$$

So Carter's "regular" theory also predicts strongly degenerate free Fermi gases to be acausal.

Because even these simple gases are predicted by Carter's "regular" theory to be unstable and acausal, we do not believe that it is, in general, an acceptable alternative to the inviscid Israel-Stewart theory. Carter's "regular" theory was designed[16] to act as a simpler alternative to the Israel-Stewart theory, which would obviate the difficulty of determining the functional form of the second-order coefficients. Unfortunately, as we have shown here, the specific thermodynamic values chosen in Carter's "regular" theory for the second-order coefficients are unacceptable even when applied to a nonrelativistic Boltzmann gas or to a strongly degenerate free Fermi gas. One could attempt to "fix" the problems with describing the Boltzmann and Fermi gases in the "regular" theory by changing the normalization of the entropy function (and hence changing the numerical value of β_1). This is an unattractive possibility, as it changes the entropy from being a microscopically calculable quantity (with overall normalization set by the usual quantum-mechanical arguments at $T=0$) to a function whose overall normalization is chosen so as to give reasonable dynamics to the "regular" theory. Furthermore, any choice of normalization, e.g., the Boltzmann gas, will give acceptable (causal) dynamics only over a finite range of densities: e.g., as the density of a nonrelativistic Boltzmann gas is lowered, any fixed normalization of the entropy will eventually yield acausal behavior.

There may be specific equations of state for which Carter's "regular" theory is stable (and hence causal and hyperbolic); proposed equations of state can be easily tested using the stability conditions derived here [Eqs.(38)–(43)]. It may also be possible to find an acceptable theory of relativistic heat-conducting fluids by working within the Carter formalism, but abandoning the suggested value of zero for the anomaly.

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