Inhomogeneous quantum cosmology: An infinite-parameter model

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A generalized version of Gowdy T^3 spacetime is studied and a proposal for an approximation scheme made that allows for some study of its quantum behavior in the Hartle-Hawking prescription. The system is reduced in freedom to an infinite-parameter midisuperspace model and the reduced system is analyzed. The measure choice turns out to be difficult. The system is solved up to one loop and the behavior of the partition function with respect to scaling of the metric in the direction transverse to the propagation of gravitational waves is examined. It is found that the scaling of the WKB prefactor depends only on topological factors and that this suggests that the model evolves from a pure state to a mixed one. The relation to the arrow of time is discussed. A crude reduction to dynamical degrees of freedom is made, and the possibility of creating a manifestly convergent Euclidean formalism is discussed. Some open questions as to the nature of a sum over topologies are examined via both the Page and Hawking proposals. Convergence of the partition function is examined. The possibility of a transition in behavior at a critical scale is discussed.

I. INTRODUCTION

In this paper I attempt to set up a toy model that it is hoped will help elucidate the behavior of complicated cosmological systems. It is felt that a model is needed that has a large number of parameters and complicated topological structure, as these features are now being used for various solutions for cosmological systems, particularly in the new arena of wormhole physics. It is hoped that the following can be used as a testbed of these concepts. The primary work of this paper will be to set up the model and examine concepts related to the generalizations of Hawking and Page[1-3] of the idea of a Hartle-Hawking quantum description for our Universe.[4-7]

The Hartle-Hawking prescription for the wave function of the Universe has been studied by many authors. Most of the work has been done on homogeneous systems or purely for perturbations of these spacetimes.[8-10] The majority of the calculations also involve solving the Wheeler-Dewitt equation[9,11] with an assumed boundary condition. I attempt in this paper to work directly with the path-integral formulation of the prescription via an approximation scheme based on the semiclassical expansion, up to the one-loop correction.[12-14] This has been done for the empty Friedmann-Robertson-Walker (FRW) cosmology by Schleich.[12] Most work in the field of quantum cosmology has been done via the *minisuperspace* and *microsuperspace* approximations. The minisuperspace approximation is to reduce the number of degrees of freedom of the three-geometry slices of the spacetime studied down to a finite number, while we reduce the degrees of freedom in the *whole spacetime* to a finite number for the microsuperspace approximation (various researchers use different terminology here; this is based on that used by Hartle). I use an approximation scheme that Kuchař likes to call *midisuperspace.* In the midisuperspace approximation, we reduce the number of degrees of freedom available, but down to a still infinite number, instead of a finite number, thus hoping to produce a system that more accurately models our Universe.

Study of inhomogeneous cosmologies can be motivated by several reasons. In the classical theory, topology change is not possible in a causal Lorentzian background, as has been discussed by Geroch[15] and Gowdy,[16] and cannot even take place in a homogeneous Euclidean background (for a discussion of this problem for strictly nonzero-Ricci-curvature compact manifolds, see Bochner[17]). Hence we are motivated to study inhomogeneous Euclidean systems, especially if we hope to understand modern wormhole physics concepts in any rigorous fashion.[3,18,19] It is also known that a divergence of the path integral in quantum cosmology is directly attributable to inhomogeneity (coming from high-frequency components of the conformal factor[13]), although also possible in purely homogeneous systems (i.e., positive-curvature FRW models). Finally, on an aesthetic basis, gravity is a theory of *shape,* and so we must venture into complicated systems like these if we ever want to fully understand the possibilities of such a quantum object, with its extremely unusual physical states (relative to standard quantum field theory and quantum mechanics).

I study a model *motivated* by the Gowdy metric.[20-22] The latter has previously been studied as a quantum cosmological model,[23-27] but not in the Hartle-Hawking prescription, and not directly from the path-integral formulation. Section II gives a quick description of the canonical formalism for path integrals, both Lorentzian and Euclidean, and discusses the problem of constraints in a manifestly Euclidean system. It is pointed out that the present Euclidean formalism may need to be altered if one wishes to produce consistent calculations for quantum cosmology.

Section III demonstrates a way of formulating a general (Euclideanized) version of this model for a large class of spacetime topologies, and obtains a compact form for the full set of field equations for the Euclideanized Gowdy metric. I indicate problems with finding a smooth spacetime for an arbitrary three-geometry boundary which satisfy these equations.

Section IV looks at the path-integral formulation for the theory and in it I propose a reduction in the freedom of the model to a toy system that produces a consistent basis for calculation of partition functions and eases the matching problem. The new approach is motivated by the superspace path-integral-formalism view that describes quantum cosmology as a theory of *boundaries* to spacetime. The idea, then, is to try to fit some large class of spacetimes to these three-geometries and to perform some appropriate integration. This approach does not usually consider the dynamics of the four-space to be important in the same sense that a worker in Hamiltonian cosmology would and the completed spacetime is viewed only as a mathematical tool in the understanding of superspace, the space of all three-geometries and matter fields thereon. I do not discuss whether one should use a path-integral approach for quantum cosmology, but I will point out that we should take certain dynamical information into account if we wish to perform the integration properly. The reader may refer to the recent paper by Kuchař and Ryan[28] if they wish to see how problems may arise in the use of microsuperspace and minisuperspace approximation schemes. The following work assumes the superspace formalism in the sense of DeWitt, Hartle, Hawking, and Page and tries to construct a model system with sufficient complexity to help us understand the possible future ramifications of studying the quantum nature of inhomogeneous three-geometries. In it, I attempt to start the development of the techniques needed for such a study, albeit with obvious problems with the physical nature. I also consider the path-integral-generated partition function as the basic quantum object, rather than a Hamiltonian (Wheeler-DeWitt operator) approach.

Section V formulates a way of exploring a variation of the concept of the semiclassical expansion up to one-loop order. I discuss what measure should be used in the integral over spacetimes in such an approach (while discussing some potential caveats). This leads to some dramatic results for the small-scale behavior of the model. A crude attempt at reduction to physical variables is made and it is demonstrated that it may be possible to formulate a manifestly convergent path integral for gravity. However, further details are left to later papers.

Section VI demonstrates the use of ζ-function techniques to elucidate the scaling behavior of the partition function. The section then goes on to discuss the expansion of the partition function in terms of a sum over different topologies and it is demonstrated that it is indeed possible to distinguish two proposals for the density matrix.

Section VII examines problems with the general nature of the scale expansion, how it relates to the indefiniteness of the Hilbert action, and the ramifications of the results in the paper. Connections to the Lorentzian transition in the manifestly Euclidean partition function are also discussed briefly.

II. CANONICAL PATH INTEGRALS

Quantum cosmology can be formulated in terms of a path integral over sets of spacetimes. The original form for this, used by the majority of researchers,[11,29–31] is

$$Z_L(\mathcal{C})=\int_{\mathcal{C}}[d^{(4)}\mathcal{G}]e^{iS_T[^{(4)}\mathcal{G}]} , \tag{1}$$

where $\mathcal{C}$ is some collection of Lorentzian spacetimes $^{(4)}\mathcal{G}$'s. The action S_T is the classical Einstein-Hilbert action for general relativity. It can be put into a canonical form by the Arnowitt-Deser-Misner (ADM) 3+1 decomposition of the metric for the spacetime:[32]

$$ds^2=-(N^2-N_iN^i)dt^2+2N_i dt\, dx^i+h_{ij}dx^i dx^j . \tag{2}$$

The x^i (i ranging from 1 to 3) are the coordinates on a three-hypersurface with induced metric h_{ij} and the N and N_i are the lapse and shift of this hypersurface with respect to the spacetime. We then have

$$S_T=\frac{m_P^2}{16\pi}\int N(G^{ijkl}K_{ij}K_{kl}+\sqrt{h}\,{}^{(3)}R)d^3x\, dt . \tag{3}$$

The K_{ij} is the extrinsic curvature tensor of the three-hypersurface, h is the determinant of h_{ij}, $^{(3)}R$ is the curvature scalar of h_{ij}, m_P is the Planck mass, and G^{ijkl} is the DeWitt supermetric, given by

$$G^{ijkl}=\tfrac{1}{2}\sqrt{h}\,(h^{ik}h^{jl}+h^{il}h^{jk}-2h^{ij}h^{kl}) . \tag{4}$$

It is then possible to consider N, N_i, and h_{ij} as canonical variables and rewrite the action as

$$S_T=\int\left[\pi^{ij}\frac{\partial h_{ij}}{\partial t}-NH-N_iH^i\right]d^3x\, dt , \tag{5}$$

where the π^{ij} are the canonically conjugate momenta to the h_{ij}. H and H^i are the super-Hamiltonian and supermomenta, respectively, containing only the π^{ij} and the h_{ij}. The lapse and shift functions have no conjugate momenta and thus are not true dynamical variables, but Lagrange multipliers. In the classical theory, variation of the latter leads to the initial-value problem constraints, given by[11,33]

$$^{(3)}R-K_{ab}K^{ab}+K^a_aK^b_b=0 \tag{6}$$

for the Hamiltonian constraint and

$$D_b(K^{ab}-K^c_c h_{ab})=0 \tag{7}$$

for the momentum constraint. D_b denotes covariant differentiation in the hypersurface. These constraints reduce the dynamical variables in the classical theory from h_{ij} and π^{ij} to a smaller set of true physical variables. The Lorentzian quantum theory then preserves this property, if (1) is now understood to be an integral over canonical coordinates and momenta, by the identity

$$\int_{-\infty}^{+\infty} dN\, e^{-iNH} = 2\pi\delta(H) \tag{8}$$

and similarly for N_i. Hence the partition-function path integral (1) is reduced to an integration over just the physical variables. Thus, the Lorentzian theory is in the form of an already parametrized quantum theory, as discussed by many authors.[29–31,34]

We now turn to the Euclidean (i.e., positive-definite signature) formulation of the theory, as described by Hawking and colleagues. We use a (quantum) world partition function, given in Euclidean form by

$$Z(\mathcal{C}) = \int_{\mathcal{C}} [d^{(4)}\mathcal{G}]\, e^{-S_T[^{(4)}\mathcal{G}]} \tag{9}$$

where $\mathcal{C}$ is some collection of Euclidean spacetimes ${}^{(4)}\mathcal{G}$'s. In the Hartle-Hawking prescription, these are smooth and either compact or compact with *given* three-geometry boundaries. I handle the complicated topologies via the implementations of the prescription of Page[1] and Hawking,[3] which allow for multiply connected spacetimes. This will be used as the basis for the rest of the paper.

The new partition function has advantages and disadvantages. The primary advantage is that (1) oscillates, and hence is ill defined, while (9) is not. The major disadvantage is that the latter is manifestly divergent.[13,34] A second problem, not usually discussed, is that we have rotated the part of the integral that enforces the constraint equations, thus causing a problem with the interpretation of the use of (8) to reduce the number of variables integrated over. Therefore, we cannot be sure that we have produced a consistent path integral. It will be suggested later that these two problems are related and that a fully consistent and convergent Euclidean formalism may exist. The latter has already been discussed by Hartle and Schleich and they go on to demonstrate, for the Bianchi type-I cosmology,[34] linearized gravity,[35] and asymptotically flat spacetimes,[36] that the conformal rotation of Gibbons, Hawking, and Perry[13] can be viewed simply as a way of correcting this problem.

III. THE GENERALIZED GOWDY MODEL

Gowdy has studied a set of spacetimes with a G_2 group of motions.[20–22] These have the useful property that a lot is known about their general behavior.[37] The original model (originally derived as a modification of the Einstein-Rosen[38] wave metric) at first seems to have a triple-infinite number of degrees of freedom, but it is known that it only has a single infinity of actually physically independent modes. The great improvement over standard homogeneous models, though, is that a lot of topological information can be included and hence important new properties connected to this fact can be studied. The underlying classical dynamics are not actually important for the study of the quantum dynamics of *extremely small* three-geometries as the action tends to zero in the appropriate limit, as can be seen by the following argument: Under homogeneous conformal scaling, for a *fixed* conformal four-geometry, scalar curvature blows up quadratically as the scale tends to zero and the extrinsic curvature of the boundary blows up linearly, while the spacetime volume tends to zero quartically and the boundary volume tends to zero cubically; thus the action tends to zero quadratically. We would thus expect the contribution to the path integral from the stationary point of the action to tend to zero as the conformal scale decreases. Of course, we are assuming that we can reverse the operation of taking the scale limit with that of taking the integration limit in the path integral (and, hopefully, the strong requirements of the Hartle-Hawking prescription will give us sufficient analytic behavior in the path integral to allow us to do this). This can be seen in Ref. 12, in which three-geometries are filled in with *flat* spacetimes, and fluctuations are all important. We will be concentrating on fitting *compact* spacetimes to given boundary data, and we hope that we can thus ignore divergences in the action that may arise from *classical* solutions. For instance, the Gowdy model we are studying here has a "velocity dominated" singularity at small scales, in the Lorentzian regime, leading to it behaving like a Kasner model at each point in spacetime.[25,37] We would expect this solution to dominate, but the Hartle-Hawking restriction to *smooth* and *compact* spacetimes would hopefully eliminate it from the path integral. It must be admitted, though, that this is largely a problem of *measure;* we cannot be sure what will truly dominate these path integrals until we have a complete understanding of the theory, which is lacking at the moment. For the rest of the paper, we will assume that the WKB contribution tends to zero for small conformal scales. The result is that a lot of behavior can be deduced, even though the completed spacetime is not actually an extremal solution of the field equations.

I will now describe a way of writing the Euclideanized version of these models that allows one to study a wide range of spacetime topologies. The metric is written as

$$ds^2 = \tfrac{1}{2}\omega^z \otimes \omega^{\bar{z}} + \tfrac{1}{2}\omega^{\bar{z}} \otimes \omega^z + \omega^x \otimes \omega^x + \omega^y \otimes \omega^y\ . \tag{10}$$

The complex one-forms ω^z and $\omega^{\bar{z}}$ are conjugate to each other and generate what I call the *dynamical two-space M* with metric

$$ds^2 = \tfrac{1}{2}\omega^z \otimes \omega^{\bar{z}} + \tfrac{1}{2}\omega^{\bar{z}} \otimes \omega^z\ . \tag{11}$$

This (orientable) two-space will have some compact boundary ∂M on which boundary conditions are imposed. The boundary can be viewed simply as a union of closed loops in a two-dimensional space containing M and ∂M. It should also be noted that the *position* of this boundary is itself a boundary condition, an important fact that will be used later. I assume that all the forms depend only on the position in M and that ω^x and ω^y correspond to two transverse, orthogonal, compact, dimensions. This gives us a four-dimensional spacetime of topology $M \times S^1 \times S^1$. The spacetime can be viewed as being that of a closed universe, with "waves" of one polarization propagating in just one direction, and homogeneous transverse to the propagating modes. If M has a boundary of topology $\bigcup S^1$ then the four-space has a three-boundary of topology $\bigcup T^3$. Any component of this boundary can then have an effective topology of T^3,

$S^1 \times S^2$, or S^3 by choosing the transverse scale lengths in such a manner as to pinch off the spacetime in various ways, as is discussed by Gowdy.[20] Hence it is possible to describe a class of spacetimes corresponding to bifurcations and other topological effects by appropriate selection of the structure of M and the transverse scale lengths on the boundaries and in the space. This allows one to study correlation functions between three-geometries with a range of topologies. The complex structure of M is useful, as the global properties of such manifolds have been well classified.[39] One can write the transverse forms as

$$\omega^x = \sqrt{Q(m)}e^{D(m)}dx \ , \tag{12}$$

$$\omega^y = \sqrt{Q(m)}e^{-D(m)}dy \ , \tag{13}$$

with $m \in M$, x and y periodic coordinates with period 2π, and Q and D real. M always has a *local* isothermal representation of the form (corresponding to $\omega^z = e^{-B(m)}dz$)

$$ds^2 = e^{-2B(m)}dz\, d\bar{z} \ . \tag{14}$$

Thus setting $z = t + i\theta$ obviously gives the Euclideanized Gowdy metric locally. One can define derivatives on scalars in the general one-form basis by

$$d\phi = \nabla_z \phi \omega^z + \nabla_{\bar{z}} \phi \omega^{\bar{z}} \tag{15}$$

and represent the connection induced on M by a complex function α defined via

$$d\omega^z = \alpha \omega^z \wedge \omega^{\bar{z}} \ . \tag{16}$$

It is then easily found that $d^2 = 0$ gives the relation

$$\nabla_z \nabla_{\bar{z}} - \bar{\alpha} \nabla_{\bar{z}} = \nabla_{\bar{z}} \nabla_z - \alpha \nabla_z \ . \tag{17}$$

Standard techniques can be used to calculate the Ricci tensor for the space and hence to arrive at the Einstein equations. The resulting equations are

$$\nabla_{\bar{z}} \nabla_z Q - \alpha \nabla_z Q = 0 \ , \tag{18}$$

$$\nabla_{\bar{z}}(Q\nabla_z D) + \nabla_z(Q\nabla_{\bar{z}} D) - \alpha Q \nabla_z D - \bar{\alpha} Q \nabla_{\bar{z}} D = 0 \ , \tag{19}$$

$$\frac{(\nabla_z Q)^2}{2Q^2} - \frac{\nabla_z \nabla_z Q}{Q} - 2(\nabla_z D)^2 - \frac{\bar{\alpha} \nabla_z Q}{Q} = 0 \ , \tag{20}$$

$$\nabla_z \alpha + \nabla_{\bar{z}} \bar{\alpha} - 2\alpha\bar{\alpha} + \frac{(\nabla_z Q)(\nabla_{\bar{z}} Q)}{2Q^2} - 2(\nabla_z D)(\nabla_{\bar{z}} D) = 0 \ . \tag{21}$$

From (17) it is obvious that (18), (19), and (21) are real, while (20) is complex. One therefore has five equations of motion for the system. Classically, it turns out to be easy to demonstrate that (21) is redundant, for points at which $\nabla_z Q \neq 0$, thus giving only four independent relations.

I now introduce a tidy formalism for this model. We define $\tilde{\nabla}_a$, $a = (z, \bar{z})$, to be the two-covariant generalization of ∇_a, acting on one-forms and vectors [in other words, for one-forms of the form $n_z \omega^z + n_{\bar{z}} \omega^{\bar{z}}$, we use (16) and get $\tilde{\nabla}_{\bar{z}} n_z = \nabla_{\bar{z}} n_z - \alpha n_z$]. Once we make this definition, then (18), (19), and (21) can be written in the compact form

$$\tilde{\nabla}^a \nabla_a Q = 0 \ , \tag{22}$$

$$\tilde{\nabla}^a (Q \nabla_a D) = 0 \ , \tag{23}$$

$${}^{(2)}R + \frac{(\nabla^a Q)(\nabla_a Q)}{2Q^2} - 2(\nabla^a D)(\nabla_a D) = 0 \ , \tag{24}$$

where ${}^{(2)}R$ is the scalar curvature of M. We will not simplify (20) here, as it will be discarded soon and (24) used instead, and the above set of equations will become our primary description of the model.

Given Q and D on the boundary, ∂M, of M, the solution of (22) and (23) will be unique. To see this, one defines the volume two-form on M by $\epsilon = (i/2)\omega^z \wedge \omega^{\bar{z}}$; then, for $Q = 0$ on ∂M, we have

$$0 = \int_M \epsilon Q \tilde{\nabla}^a \nabla_a Q = -\int_M \epsilon (\nabla^a Q)(\nabla_a Q) \ ,$$

which implies $Q = 0$ throughout M. The solution to (22) is therefore unique from linearity. Furthermore, if Q is now given throughout M, and $Q > 0$ (which must be so, if the metric is real and the boundary has only T^3 components), then $D = 0$ on ∂M will give us

$$0 = \int_M \epsilon D \tilde{\nabla}^a (Q \nabla_a D) = -\int_M \epsilon Q (\nabla^a D)(\nabla_a D) \ ,$$

again giving uniqueness. Things are a little more complicated for the other boundary topologies, as they have a part of the boundary of M for which $Q = 0$ and D is not given, but the same argument still holds, as the boundary term in the above equation still vanishes. It is also found that (22) and (23) are invariant under conformal transformations in M, thus their solution is independent of the underlying local metric of the two-space, only depending on its global properties. One can therefore consider (20) and (24) as being conditions on the structure of M and the embedding of ∂M therein. Unfortunately (20) gives us the latter structure via a first-order differential equation, thus specifying the two-metric from a value at a single point, and so we are not free to choose the metric on ∂M. In principle, it may be possible to move the boundary to allow the selection of any desired three-geometry, but this will obviously be a nontrivial operation at best. This makes it difficult, if not impossible, to implement the Hartle-Hawking condition via a simple classical WKB approximation. One would hope that we could find a way of fixing our three-geometry and then using a different selection scheme (via a microsuperspace technique) to get some complete four-space. This will indeed turn out to be possible, as the next section discusses. It is also important to note that we will not be able, in general, to find a compact solution without boundary to either the full or reduced set of Einstein equations for the system, without just reducing to the homogeneous case.

IV. THE PATH-INTEGRAL FORMULATION

It is a simple, if tedious, task to show that the total action for the system is given by

$$S_T=-\frac{\pi m_P^2}{4}\left[\int_M \epsilon\left[{}^{(2)}RQ+\frac{(\nabla^a Q)(\nabla_a Q)}{2Q}-2Q(\nabla^a D)(\nabla_a D)\right]+2\int_{\partial M} QK\right], \tag{25}$$

where K is the extrinsic curvature of ∂M, induced by its embedding in the dynamical two-space (∂M is a one-space). The integral over ∂M is made in the induced measure. It is then interesting to study which perturbations generate which field equations. We find that $D \to D+\delta D$ implies (23), $\omega^z \to (1+\delta\xi)\omega^z$ with $\delta\xi$ real gives (22), and then $Q \to Q+\delta Q$ implies (24). This leaves (20), which is generated by $\omega^z \to \omega^z+\delta\chi\omega^{\bar{z}}$ with $\delta\chi$ *complex*. This gives us a resolution of the problems with the latter equation, as we shall be able to use it later to remove (20), which is the primary obstacle to finding Hartle-Hawking-type spacetimes to fit to our boundaries. It is also important to notice that once Q and D obey Eqs. (22) and (23) the action is a *conformal invariant*, depending only on the global properties (Teichmüller and topological) of M and the field values on the boundary and having no dependence on the local metric structure of M.

Let us now try to investigate the general nature of the superspace approach and try to understand how midisuperspace, minisuperspace, and microsuperspace techniques influence our calculation. We use the partition function in the form given by (9). In studying Z we usually make some form of reduction to a set of parameters, usually very few, and then effectively perform the integration with respect to these. In the semiclassical approximation we expand the action around solutions that extremize it, as these should be the dominant contributions to it, if no anomalies exist. In fact, for microsuperspace and minisuperspace systems, the extremization will not always generate solutions that satisfy the full Einstein equations, as one cannot be sure that one will recover the equations generated by symmetries of the system and other unvaried parameters. We can see that this happens if we choose the class of spacetimes we integrate over in the path integral in such a way that it does not contain a solution of the Einstein equations for each choice of boundary conditions. In such a case, extrema may still exist, and will dominate the contribution for the *restricted* path integral, but these extrema will not, generally, be solutions of the full set of Einstein equations. So, if we are just interested in the value of that, restricted, path integral, we only solve the *reduced* set of field equations. We can take the point of view that we are interested in the qualitative aspects of Z, primarily in the quantum regime, and that this can be studied in terms of anything that seeks to approximate general relativity (albeit to a certain required accuracy). We therefore only need to obey the field equations that are generated by our actual chosen free parameters.

I propose the following reduction for the generalized Gowdy metric: As a two-dimensional space such as M is primarily determined by its conformal factor, leaving only a finite number of parameters like Teichmüller deformations,[40] I will only look at M-geometries that are in a given conformal class, i.e.,

$$\omega^z=e^{-B}\omega_0^z , \tag{26}$$

where the form ω_0^z has no physical degrees of freedom connected with it. This still gives a class of spaces rather larger than that of the standard Gowdy metric, and of whatever topology one chooses. One can view this as the equivalent of splitting our sum over histories into a sum over (B,Q,D) configurations plus a sum over topologies and parameters associated with the embedding of ∂M in M. If we choose our decomposition such that B is fixed on the boundary, then we will need to change the position of the boundary in addition to the Teichmüller parameters, i.e.,

$$Z_{\text{total}}=\sum_{\text{topologies}}\int[d\mathcal{E}]\int[d\tau]Z(\mathcal{P}) , \tag{27}$$

where $\mathcal{P}$ represents the class of (B,Q,D) fields with a given ω_0^z, τ represents the Teichmüller integration variables, and $\mathcal{E}$ represents the extra embedding freedoms. The ω_0^z is seen as representing the global properties of M that one has fixed upon.

In general one must be sure that we are not summing over configurations that are diffeomorphic to one another. In most cases this will force one to introduce Faddeev-Popov ghost terms into the action. This can be avoided if one chooses the fields to reflect the physical degrees of freedom or, in the spirit of minisuperspace techniques, the *reduced* degrees of freedom. It is possible to reduce the boundary freedoms (unphysically) so that the spacetime metric freedoms are reduced, and diffeomorphisms are not possible. We can see this if we let i denote the trivial inclusion map of a component of ∂M into M and i^* denote the corresponding pullback on one-forms from the spacetime to the boundary manifold. We can then define $\gamma_0^a=i^*\omega_0^a$, while using the original notation for ω^x and ω^y. This allows us to write the three-boundary metric as

$$ds^2=\frac{e^{-2B}}{2}\gamma_0^z\otimes\gamma_0^{\bar{z}}+\frac{e^{-2B}}{2}\gamma_0^{\bar{z}}\otimes\gamma_0^z+\omega^x\otimes\omega^x+\omega^y\otimes\omega^y . \tag{28}$$

But the first part of this is a metric on a one-dimensional space and hence we know that the above can simply be written as

$$ds^2=L^2d\beta^2+Q(\beta)e^{2D(\beta)}dx^2+Q(\beta)e^{-2D(\beta)}dy^2 , \tag{29}$$

where L is a constant and β is a cyclic coordinate for T^3 boundaries, and takes values over some interval of the real line for the other topologies. Hence most of the freedom of B is due to selecting different diffeomorphic representations of the boundary components, reducing down to just one L for every component. Therefore, for a boundary fixed by field value *and* by M position, and fixed ω_0^z, we can remove the freedom of making boundary diffeomorphisms by choosing B to have only one free parameter on each of the components of ∂M, corresponding to the scale lengths in each one-space and fixing the boundary embedding. Misner also reduces B down to one degree of freedom in his paper on the Gowdy metric,[26] but he uses a direct Hamiltonian cosmology based approach that uses the extra inhomogeneous degrees of freedom to help solve the constraint equations while we are reducing the freedom based on geometric arguments. We can see that this has reduced the freedom possible in the four-space from the fact that a four-diffeomorphism of the form

$$g_{ab} \rightarrow g_{ab} + \tilde{\nabla}_{(a}\xi_{b)} \tag{30}$$

will not satisfy (26) unless it is a local conformal mapping, obeying (31) and (32), and that such a map will move the boundary [if an analytic function $y(z)$ is equal to z on a closed loop, it is equal to z throughout the region bounded by that loop]. It is this point at which we arrive at an approximation scheme that requires a loss of physical behavior. Rather than ensuring gauge (diffeomorphism) invariance of the partition function (with diffeomorphisms acting on the three-boundary), we have *projected* down to a rather *ad hoc* representation of those boundaries, effectively reducing the class of spacetimes we fit to the boundaries.

From the above argument, we can see that, in general, we can consider $Z(\mathcal{P})$ to be the contribution we get if we only perform *part* of the integration in (9). Such a calculation does not include the integration over the $\delta\chi$ parameter described above. It is hoped that $Z(\mathcal{P})$ would represent most of the physics for our system. The idea is to try and produce a model that we can solve and then maybe this will lead us onto correct formulations in the future.

It is only needed to satisfy (22), (23), and (24) to find the dominant contribution to $Z(\mathcal{P})$, henceforth simply denoted by Z. The remaining equation (20) will no longer be generated. These three equations are very nice, as they have a unique solution for (B,Q,D) given on the boundary. The following dependencies exist: Q depends only on its value on ∂M, D depends only on D on ∂M and Q in M, while B is arrived at by solving a Poisson equation with a source generated by D and Q. I call the reduced set of equations the *proximal equations* and the corresponding solution the *proximal solution* in what follows. The proximal system performs the same function as the classical system does in standard semiclassical techniques. In principle, the idea is that one hopes to add more integration variables to the system and correct the measure at each stage, thus eventually tending to a correct description of the system, while having some description of physical reality in the meantime, albeit only to some approximate degree.

V. THE SEMICLASSICAL EXPANSION

We wish to find an appropriate form of semiclassical expansion around our proximal solutions. To do this, we need to decide what the basic objects of our path integral are. The standard Euclidean idea is to integrate over all nondiffeomorphic spacetimes, paying no heed to dynamics. This is rather a crude technique, as we can see from the discussion in Sec. II about the nature of the transition from Eq. (1) to Eq. (9). The standard Euclidean program seems to damage the part of the integration that maintains the constraints, suggesting that maybe we should solve the constraints first. One of the most important results that comes from studying the dynamics of the system before the choice of variables comes from a rather subtle, but very important, property of this model. As pointed out by Berger,[23–25] Misner[26], and Kuchař,[29] the Lorentzian Gowdy models, the original Einstein-Rosen metric, and thus this model have a rather important *dynamical* symmetry; the local metric (14) retains its form under local analytic mappings $z \rightarrow y(z)$ (mutatis mutandis for the Lorentzian case). The dynamical invariance comes from writing $y = u(\theta,t) + iv(\theta,t)$, and writing down the Cauchy-Riemann equations

$$\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial t} \tag{31}$$

and

$$\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial \theta} . \tag{32}$$

These equations imply that u and v then satisfy (18), by simply substituting the local form of the metric. Hence, for *classical* solutions, we can set u or v equal to Q locally and then choose the other variable to satisfy (31) and (32) by integration. Thus we can use the dynamics of the *classical* system to remove all physical freedom, up to choices in local coordinate patches, from Q. For the Lorentzian case, as remarked by Gowdy,[20] we cannot always choose Q to be a time coordinate, as the transformation corresponding to the above cannot alter the global properties of the Q solution. Hence Q may have timelike or spacelike level surfaces. In the Lorentzian T^3 case, though, we can use a theorem by Gowdy called the *corner* theorem[20] [a *corner,* in Gowdy's terminology, being the (null) junction hypersurface between two regions of spacetime, where the propagation of Q changes from spacelike to timelike]. The theorem restricts the type of corner that can occur in a G_2 spacetime and, in the Lorentzian T^3 case, it tells us that we can always choose $Q = t$, thus allowing us to consider it as an *intrinsic* time coordinate.[23–26] When Q has timelike level surfaces, we can use (31) and (32) to generate an *extrinsic* time, which is formed out of a combination of the momenta and canonical coordinates, when viewed in a 3+1 formalism.[29] However, in general, the Euclidean case is complicated, in that the behavior the Lorenztian model has with junction conditions at corners is no longer present and the choice of Q as a time or space coordinate becomes a choice on coordinate patches, rather than one on regions with different causal properties, as one might expect of a system in which there is no defined time. The elliptic nature of the equations also drastically alters their properties over their Lorentzian counterparts. One example of this is the result that solutions to (18) have saddle points rather than maxima or minima. The latter removes the ability, in general, to choose three-space hypersurfaces along or orthogonal to Q equipotentials (Gowdy T^3 and Einstein-Rosen behavior) and still choose the boundary to be compact. The latter follows from that fact that a regular harmonic function does not, in general, have a nonsingular set of compact equipotentials. The result of this is that we will not be able to maintain the choice $Q=t$ or $Q=\theta$ and have compact t hypersurfaces, and, furthermore, we will generally not even have *one* compact constant t hypersurface (except in the trivial case of Q being constant throughout the whole of M). This obviously complicates the understanding of our system as having a single dynamic freedom in the Euclidean sense and the interpretation of Q as being a coordinate wave. This leads to two possible approaches for formulating a path-integral quantization for this model. The first is to ignore dynamical effects and just integrate over physically different spacetimes. The second approach is to *reduce* the freedoms in our model to match, at least partially, the dynamical freedoms. This latter choice may seem a rather drastic and difficult step, but it can be motivated by studying the excellent paper on the quantization of Einstein-Rosen space by Kuchař.[29] In that paper, it was shown that the system could be reduced down to a theory of a massless scalar field in a fixed flat Minkowski background, and that the full action for the theory was obtained by integrating over all possible coordinate systems with appropriate constraints added. This is a rather amazing result, and raises the question as to how the Euclidean path integral should be performed; what *should* we take as the physical variables? Both cases will be considered in this paper but, unfortunately, in both cases, the calculation we will make will not be as rigorous or complete as one might hope. Both quantization procedures introduce complexities in the path integral, and I am primarily concerned with the scaling properties of the system and how it effects our ideas about quantum mechanics. The author hopes to describe the details of a full, consistent, Euclidean quantization program for this system in a later paper.

I start by looking at the naive integration over all spacetimes, without concern for dynamical symmetries. One can expand around a solution to the proximal equations with respect to a chosen set of perturbations. In general one wants to define which perturbations can be considered an orthogonal set. I choose

$$J\to J-j+\sigma\ , \tag{33}$$

$$D\to D+w\ , \tag{34}$$

$$\omega^z\to e^{-\sigma}\omega^z\ , \tag{35}$$

where $J=\ln Q$. This can be motivated by studying the metric that can be defined on superspace. This is given by integrating the DeWitt metric over a given hypersurface, the integrated object being

$$ds^2_{\rm local}=\frac{\sqrt{h}}{2N}(h^{ik}h^{jl}+h^{il}h^{jk}-2h^{ij}h^{kl})\gamma_{ij}\gamma_{kl}\ . \tag{36}$$

I make the natural choice that γ_{ij} is just the metric perturbation. For ω_0^z representing flat space one gets the following for the local patch representation:

$$ds^2_{\rm local}=-2Q\left[\frac{2\delta B\delta Q}{Q}-\frac{\delta Q^2}{Q^2}+4\delta D^2\right]\ , \tag{37}$$

which implies defining $V=B-J$ with the local form of Gowdy becoming

$$ds^2=e^{-2B}(dt^2+d\theta^2)+e^{B-V}(e^{2D}dx^2+e^{-2D}dy^2)\ . \tag{38}$$

It is obvious that this diagonalizes (37) and that (33), (34), and (35) correspond to the perturbations in B, V, and D. It is found that the calculations are then simplified if one goes to a set of "null" coordinates given by $a=\sigma+j$, $b=\sigma-j$. The Jacobian for this transformation just introduces a normalization constant, which can be discarded and will be in all future calculations (this is a partition function calculation, so we are not interested in global normalization). Define $S_T=\pi m_P^2 I$ and write

$$I=I^{(0)}+I^{(1)}+I^{(2)}+\cdots\ , \tag{39}$$

where $I^{(n)}$ contains only perturbations of order n in the fields. $I^{(1)}=0$, as (B,Q,D) is a proximal solution; therefore one can write the approximation

$$Z\approx e^{-\pi m_P^2 I^{(0)}}Z^{(2)}\ , \tag{40}$$

where we have

$$Z^{(2)}=\int[da][db][dw]e^{-\pi m_P^2 I^{(2)}} \tag{41}$$

with some measure on the perturbations (which is discussed later). We now consider scaling properties of this approximation. The usual scaling that one is interested in is *conformal* scaling of the four-metric. However, as the action and field equations are invariant under scaling of the metric of the dynamical two-space (B in the above local form), we will concentrate on Q rescaling in the rest of the paper and mean $Q\to kQ$ (k const) when we use the term *scaling*. At small scales we find that $I^{(0)}\to 0$, in the sense of maintaining all the other parameters constant and just varying the total normalization of Q. Therefore $Z^{(2)}$ represents the small-scale partition function. This is, of course, the standard one-loop approximation.[41] Evaluating $I^{(2)}$ gives

$$I^{(2)}=-\tfrac{1}{8}\int_M Q\epsilon(-3\nabla^a b\nabla_a b-2\nabla^a w\nabla_a w$$
$$-4b\nabla^a D\nabla_a w+4b\tilde{\nabla}^a\nabla_a a)\ . \tag{42}$$

Following standard methods,[12–14] I will write $I^{(2)}$ in terms of a Hermitian operator A acting on the perturbation

$$I^{(2)}=\langle(a,b,w)|A(a,b,w)\rangle\ , \tag{43}$$

where the inner product will be chosen so that it generates a measure on the perturbations by defining a norm

$$\|g\|^2=\langle g|g\rangle\ . \tag{44}$$

The corresponding Lebesgue measure is formally defined by writing a given perturbation as

$$dg=\sum_i(da_i)g_i\ , \tag{45}$$

where we have

$$\langle g_i|g_j\rangle=\delta_{ij}\ , \tag{46}$$

and then defining $\int[dg]$ by

$$\int[dg]=\prod_i\int_{-\infty}^{+\infty}da_i\ . \tag{47}$$

The orthogonality is chosen to give the structure of (41). Appendix A contains the technical details on how to obtain this operator from the perturbations in $I^{(2)}$. But, from direct observation of the structure of (42), we can see that A has no terms connecting a to a or a to w. Thus A must be of the form

$$A=\begin{bmatrix}0 & 0 & B_1\\ 0 & B_2 & B_3\\ B_1^\dagger & B_3^\dagger & B_4\end{bmatrix} \tag{48}$$

with B_2 and B_4 Hermitian and the matrix ordered (a,w,b). This implies that we can formally write

$$\det A=\det(B_1B_1^\dagger)\det B_2 \tag{49}$$

and that B_3 and B_4 do not contribute to the one-loop approximation. Hence $Z^{(2)}$ can be calculated from the effective perturbation action

$$I_e^{(2)}=\tfrac{1}{8}\int_M Q\epsilon(2\nabla^a w\nabla_a w-4b\tilde{\nabla}^a\nabla_a a)\ . \tag{50}$$

This resulting structure, corresponding to operators that directly represent the two wave equations (as will be seen later), is also good motivation for the chosen perturbation expansion. This action is conformally invariant and $I^{(0)}$ is scale invariant. One thus expects the measure to be, at least, scale invariant. If it is not scale invariant then anomalies can be expected. This motivates one to choose a measure that is independent of the physical parameters. The work by Faddeev and Popov[42] would suggest a measure for the system, which is of exponential form, to have a structure

$$[da][db][dw]=\left[\prod_x da(x)\right]\left[\prod_x db(x)\right]\left[\prod_x dw(x)\right]\ , \tag{51}$$

where $\prod_x$ indicates that the fields are considered to be orthogonal variables at each point, and the functional integral is then understood via some discretization of M. We must thus consider the manifold as a collection of points, each acting like a field index, rather than as a collection of patches. It is very difficult to formulate such a prescription in terms of an inner-product-generated Lebesgue measure without running into dependence on the fine details of the choice of metric structure that the product generates. This can give unphysical dependencies in the system. The problem can be solved in principle by either integrating over all background structures (with the inclusion of ghost terms to avoid overcounting) or by using complicated correction methods (cf. Refs. 43 and 44). It is felt by the author that at this point the problem can only be made tractable by using a measure that is strictly two-covariant and involves either a fixed choice of background (which generates an arbitrariness in the one-loop approximation, after regularization) or uses the metric structure induced on M by the solution (which could give problems with anomalies). The two possibilities can be expressed by stating that the measure is defined by the following inner product, acting on single components:

$$\langle l|m\rangle=\int_M\rho lm\ , \tag{52}$$

where the volume two-form is either the one on the base (w_0^z) space

$$\rho=\frac{i}{2}w_0^z\wedge\omega_0^{\bar z} \tag{53}$$

or the volume form of the dynamical two-space (the w^z space), with $\rho=\epsilon$. Another measure would be the "natural" measure on the whole four-geometry, given by $\rho=Q\epsilon$. The latter measure has explicit dependence on Q and will thus generate anomalies. It should be pointed out at this point that these measures will not make sense once we allow the boundary to move; in that case, we have to use ghost techniques to remove problems with overcounting, and Faddeev-Popov determinants will enter the calculation. One must now define operators covariant in the background space, and Hermitian in the above norm. For the first case we define new derivatives via

$$d\phi=\partial_z\phi\omega_0^z+\partial_{\bar z}\phi\omega_0^{\bar z} \tag{54}$$

and for the second case set $\partial_a=\nabla_a$. The following forms for the remaining entries in (48) are then obtained:

$$B_1=-\tfrac14 Q\tilde{\partial}^a\partial_a\ , \tag{55}$$

$$B_2=-\tfrac14\tilde{\partial}^a(Q\partial_a)\ . \tag{56}$$

In the following we define Laplacians by $\Box=\tilde{\partial}^a\partial_a$, with obvious generalizations to other spaces. These operators have the highly redeeming quality of not having any zero modes for perturbations that vanish on the boundary, an important result as these modes can cause great problems with anomalies.[40,41] It should be remarked that, unlike in Refs. 12 and 41, our measure is now encoded into our operators and that we do not have to make corrections for measure rescaling. Using the results in Appendix A, and noticing that our perturbation fields are dimensionless, then gives the one-loop correction as

$$Z^{(2)}=\det{}^{-1/2}\left[-\frac{\pi m_P^2}{4}(\tilde{\partial}^a Q\partial_a)\right]\times\det{}^{-1/2}\left[\frac{\pi^2 m_P^4}{16}(Q\tilde{\partial}^a\partial_a)(\tilde{\partial}^a\partial_a Q)\right]\ . \tag{57}$$

One can then go even further by noticing from (50) that (as $\Box Q=0$) we can reexpress the determinant for the w-w interaction as

$$\det B_2=|\det B_1|\ . \tag{58}$$

We now define an *effective* eigenvector space (see Appendix B), conformal to (M,ρ), with inner product defined by the volume form $\nu=Q^{-1}\rho$. In this space we find that B_1 just reduces down to a Laplacian:

$$B_1 = -\frac{\Box_{EE}}{4} . \tag{59}$$

Hence we can calculate the determinant in this space and get

$$Z^{(2)} = \left| \det{}^{-3/2} \left[-\frac{\pi m_P^2}{4} (\Box_{EE}) \right] \right| . \tag{60}$$

It is interesting to note that for the first case, in which ∂_a has no implicit dependence on B and D, $Z^{(2)}$ is only a function of Q. Thus, if we take the view of Kuchař that we should view Q as some form of coordinate, $Z^{(2)}$ is only a function of how the "dynamical" part of the boundary data is embedded in the spacetime (which is specified in part by the *nondynamical* Q) and is not a function of the values of the dynamical variables themselves. If we do not take this view, or consider the case for which the integration measure is that for (M,ϵ) (and $Z^{(2)}$ thus has an implicit dependence on B and D), then the above property vanishes.

I now consider a second way in which to handle the path integral. It was noted in Sec. II that the Euclidean path integral may not generate the proper constraints on the dynamical variables, and hence we may be integrating incorrectly over the redundant variables. Also, in our approximation, it is obvious that we have not varied the embedding freedoms fully in our spacetime; therefore, we have not even integrated over the full range of lapse and shift. Thus, even if (9) does give the correct form of the integral, our previous calculation probably has an integration over redundant variables. It seems sensible to try and "deparametrize" the theory before we attempt the integration, and then only integrate over the true dynamical variables. Unfortunately, the complexity of our model makes this difficult, as was pointed out in Sec. III, but we can try a crude calculation and use the result by Kuchař that cylindrical Einstein-Rosen waves provide us with a model which can indeed be deparametrized. The resulting theory is then reduced down to the study of the cylindrically symmetry massless scalar field. The Euclidean formulation of such a field is manifestly convergent, which is a hopeful sign that the divergence of the Euclidean path integral for gravity only comes from a mishandling of the redundant variables in the theory. I now consider the possibility of incorporating the dynamical symmetry for Q into the system. This is rather a complicated affair in practice, due to the Euclidean nature of the system (as described earlier) and so I will attempt an extremely crude technique at this point, to help us to understand how things change in this approach. I will simply state that we no longer consider Q to be a dynamical variable in our calculations, roughly associating it with an underlying intrinsic coordinate system. Thus we basically set $b=0$ in all the above calculations. This may seem to only reduce the system by one degree of freedom, but it actually reduces it by *two;* as stated before, the action is conformally invariant once Q is harmonic in M, and this can be seen from (50), in which a only appears with b. It is also important to notice that $a=b=0$ reduces our freedom on three-surfaces to one *traceless* degree, as one might expect from linearized gravity theory,[35] considering we only have a single polarization of gravity waves. Setting $b=0$ reduces the effective one-loop perturbation to the *exact* perturbation. Furthermore (50) is *positive* and hence the path integral for w is *manifestly convergent.* We then have

$$Z^{(2)}_{b=0} = \left| \det{}^{-1/2} \left[-\frac{\pi m_P^2}{4} (\Box_{EE}) \right] \right| . \tag{61}$$

In other words, we have only changed the power of the determinant that we get, at least in this crude calculation. We still have to integrate over *embeddings,* however. These are encoded into Q in this formalism. Hence it seems possible to associate the conformal divergence in the theory with the integration over the *redundant* degrees of freedom of the theory. This has already been pointed out by Schleich[34] for the Bianchi type-I cosmology. She went further to rotate the redundant degree of freedom and produce a manifestly convergent path integral and was able to handle the one (Hamiltonian) constraint in her model (as well as making a sensible, skeletonized, choice of measure). It is hoped that the author will be able to extend the present research to reach a higher degree of technical accuracy, such as is found in Schleich's fascinating paper[34] on her Bianchi type-I model (a subclass of this model). However, we may run into problems, as the ADM decomposition which allows us to identify the physical variables cannot be used throughout a spacetime with a complicated topology. In general, we do not have an ADM time, intrinsic or extrinsic, that extends all the way throughout the closed manifold.

VI. SCALING PROPERTIES OF THE PARTITION FUNCTION

For the rest of this paper I will combine the above results and discuss the properties of an assumed generalization of (60) to

$$Z^{(2)} = \left| \det{}^{-f/2} \left[-\frac{\pi m_P^2}{4} (\Box_{EE}) \right] \right| . \tag{62}$$

We then assume only that f is greater than zero. Obviously (62) will be difficult to evaluate in practice, but I will demonstrate that the potential exists for one to be able to extract significant information from it.

I use ζ-function regularization techniques here to obtain finite scaling behavior for the system. One wishes to be able to evaluate the determinant of an operator G that is manifestly infinite. We can give meaning to this procedure and obtain a finite result, though, by defining the generalized ζ function for G, the technical details of which are explained in Appendix B. One can then obtain information on the scaling properties of $Z^{(2)}(Q)$, as one notices by separating the scale part of Q from its "shape" part: Define $Q=kQ_0$ with Q_0 being chosen via some normalization condition on ∂M, then $B_1(Q)=kB_1(Q_0)$ and $B_2(Q)=kB_2(Q_0)$. Appendix B then details how to get *regularized* scaling for determinants of operators that have this property. From the latter we get the following result for the scale dependence of the one-loop correction:

$$Z^{(2)}(Q)=k^{\gamma}Z^{(2)}(Q_0)\ , \tag{63}$$

where the scaling exponent γ is defined via

$$\gamma(Q_0)=-\tfrac{1}{2}f\zeta(-\Box_{EE}(Q_0),0) \tag{64}$$

or, using the notation of Appendix B, from the heat-kernel expansion,

$$\gamma(Q_0)=-\tfrac{1}{2}fC_2[-\Box_{EE}(Q_0)]\ . \tag{65}$$

But it turns out that C_2 is a topological invariant in this case. In fact, we have[45–47]

$$C_2=\tfrac{1}{6}\chi\ , \tag{66}$$

where χ is the Euler number of the effective (dynamical) manifold (M,v) (in other words, the Euler number of the two-geometry). For T^3 boundaries this is obviously just the Euler number of M. So, using modern terminology,[18,19]

$$\gamma(Q_0)=\frac{2n_{\text{worms}}+n_{\text{babies}}-1}{12}f\ , \tag{67}$$

where n_{worms} is the number of wormholes in the spacetime and n_{babies} is the number of baby universes (here I define a baby universe as a closed boundary to the four-space or as a "handle" connecting the Euclidean past and future of a given junction hypersurface in the case of closed spacetimes, with wormholes being handles connecting the *same* side). In general, however, the baby universes can be the same size, or bigger, than the "parent"; hence, maybe they should be described as *adolescent* universes. One can extend the scaling range even further, by noticing that if (B,Q,D) is a proximal solution with action S_T then (B,kQ,D) is also a proximal solution, but with action kS_T. Hence the one-loop approximation is

$$Z(B,Q,D)\approx(e^{-\pi m_P^2 I^{(0)}(Q_0)})^k k^{-\chi f/12}Z^{(2)}(Q_0)\ . \tag{68}$$

At small scales the scaling exponent, $\gamma=-\frac{1}{12}\chi f$, completely determines the behavior, as a geometry with one value of the exponent will always dominate completely over another one with a smaller exponent, at sufficiently small k. To realize the importance of the above scaling rule, we will digress into a discussion of the relation between spacetime topology and our ability, or inability to represent the quantum behavior of our model via a pure quantum state. The scaling law gives us a connection between topological sums and a scale expansion of the partition functions. I now look at the prescriptions of Hawking and Page for the density matrix of the Universe and use this to construct the necessary formalism for the expansion. One wants to consider a system comprised of distinct disconnected three-geometries. This has been called the *Googolplexus.*[18,19] Both prescriptions define probabilities in terms of *closed* four-geometries, and we should therefore consider the topological sums of our manifolds, in other words,

$$\tilde{M}=M^+\cup M^-\ , \tag{69}$$

where from here onwards we will always be talking about a two-manifold with a Q and D field defined thereon, and M^+ and M^- are two manifolds with boundaries with $\partial M^+=\partial M^-$ (the junction hypersurface). This implies that they have the same value of n_{babies}. This basic object is considered in a fundamentally different way in each proposal. I will now show that Page[1] views it as a single four-geometry contribution to the density matrix, the only term for that specific four-topology, while Hawking[3] views it as one spacetime affixed to a specific set of three-boundaries, one of many terms that perform that function. Hawking thus counts via three-geometries, and not four-geometries. He argues that the two are equivalent, but we will see that the two do seem to give different looking prescriptions, unless certain factors are added to the Hawking prescription.

It is useful to give a quick description of each prescription so that the difference can be seen. The original Hartle-Hawking proposal can be described in two ways, and the two mixed-state prescriptions correspond to generalizing the two descriptions. Hawking's prescription is the natural extension of viewing the original proposal as taking the set of spacetimes $\mathcal{C}$ to be four-geometries bounded by a particular three-geometry, the resulting path integral being considered to be a wave function. His extension is simply to allow the boundary to have disconnected components, the wave function then becoming a density matrix by tracing out the inaccessible states of the other universes. The Page prescription starts with the probabilistic interpretation of the original proposal, in that $\mathcal{C}$ is taken to be the set of four-geometries *without* boundary and containing a specific three-geometry, but with the three-geometry splitting the four-geometry into two *disconnected* components. The author will then consider the Page extension as being simply to remove the last constraint and consider $\mathcal{C}$ to be the set of spacetimes containing a specific single-component three-geometry. The latter prescription seems to be very natural, and the author tends to prefer it. Page himself describes his prescription as being for the density matrix for the system by considering $\mathcal{C}$ to be those (possibly disconnected) four-geometries that have *two* given three-geometry boundaries, thus forming the two "variables" in the density matrix. This would be a preferable statement, as the density-matrix formalism is something that we know how to do, whereas partition functions for arbitrary classes of histories do not yet have a good theoretical basis. Unfortunately, we do not know if a Hilbert-space structure exists for the interpretation of a density matrix, so I will assume the former description of the Page prescription for now.

It is assumed in the following that we are only looking at *connected* four-geometry contributions to the partition function, which we further assume can be described by a split into two regions given by (69). We denote the number of wormholes in each region by n^+_{worms} and n^-_{worms}, respectively, and define $h=n^+_{\text{worms}}+n^-_{\text{worms}}+n_{\text{babies}}$, the number of handles in the four-geometry. The scaling law gives us the following structure for the partition function:

$$Z_{\text{total}}=\sum_{h=0}^{h=\infty}\tilde{Z}(h,k)k^{hf/6} . \tag{70}$$

The k dependence is put in the corrected partition function for h handles, $\tilde{Z}$, to allow for other corrections apart from the small-scale results. We will not write the functional dependence of Z_{total} and $\tilde{Z}$ on the class of spacetimes chosen in that which follows, to simplify the notation. We replace the topological expansion with a *scale* expansion motivated by the form of the proximal expansion and above. The expansion is written as

$$Z_{\text{total}}=k^{-g}\sum_{n=0}^{n=\infty}G_n k^{nf/6} , \tag{71}$$

with G_n having dependence on the field values on the midsection slice ∂M and g representing any change that may occur after the embedding integration. For a particular $\tilde{M}$ we can define a representative set of functions $H_n(\tilde{M})$ by the scale expansion of its partition function

$$\tilde{Z}(\tilde{M})=k^{-g}\sum_{n=0}^{n=\infty}H_n(\tilde{M})k^{nf/6} . \tag{72}$$

The assumption is made that the scaling law as embodied in Eq. (70) is still true, which then implies that $H_n(\tilde{M})=0$ for $n<h$. For the proximal expansion, to one-loop, we will also have $H_n(\tilde{M})=0$ for $n-h$ odd, and the other terms will just come from the expansion of the exponential in Eq. (68). The one-loop proximal expansion will not be used in the following, apart from the two assumptions about the H_n. We then have

$$G_n=\sum_{\tilde{M}}H_n(\tilde{M}) . \tag{73}$$

One can now define an *indicator* function on the $\tilde{M}$ by

$$I_n(\tilde{M})=\begin{cases}1 & \text{if } \exists m\le n \text{ with } H_m(\tilde{M})\neq 0 ,\\ 0 & \text{otherwise} ,\end{cases} \tag{74}$$

which is zero for $n<h$ and one otherwise. So the scale expansion does indeed allow us to identify which topologies contribute to a given order by defining

$$N(n)=\sum_{\tilde{M}}\mathrm{I}_n(\tilde{M}) . \tag{75}$$

This is simply the number of manifolds with $h\le n$ and is also a *physical* object, being the number of diagrams for all contributions up to a given scale order. For $h>0$ we have that $N(h)-N(h-1)$ is just the number of contributing manifolds with h handles. It is this number that the prescriptions of Hawking and Page would consider different. In the Page prescription we want to know how many four-geometries we have contributing (I assume in the following that we count the time reverse of $\tilde{M}$ as being a different term). For $h>0$ and $n_{\text{babies}}>0$, we have at least one handle that connects either side of the three-geometry in question, ∂M. It is therefore possible to slide all the other handles, along the connecting handle, to any side of ∂M that we want to, and thus there is only one such term, the others all being topologically equivalent. For $n_{\text{babies}}=0$ there is no such handle, and the number of handles on either side of the three-geometry cannot be changed, so we thus have $h+1$ such terms. This gives a total of $h+2$ terms for $h>0$ and one term for $h=0$ in this prescription. For the Hawking prescription we count each value of n^+_{worms}, n^-_{worms}, and n_{babies} as being a *different* term, and thus the number of terms is just the number of ways of finding three integers such that $n^+_{\text{worms}}+n^-_{\text{worms}}+n_{\text{babies}}=h$. This gives us $\frac{1}{2}(h+1)(h+2)$ terms for this prescription. Thus it does seem that the Hawking prescription overcounts, relative to the Page prescription. The Hawking prescription could be made equal to the Page prescription by dividing by appropriate factors, above and beyond those needed to correct for overcounting from sliding three-geometries around in the same four-geometry,[1] and obviously both interpretations are useful in different situations.

To understand the implications of the prescriptions of Hawking and Page, and the scaling law, we need to be able to discuss the concept of a *pure* state versus a *mixed* one in purely partition-function language. In both prescriptions we define the probability of finding the present time hypersurface in a particular three-geometry configuration ${}^{(3)}\mathcal{G}$ as the partition function evaluated for a class of closed compact spacetimes containing the wanted ${}^{(3)}\mathcal{G}$, with the counting being defined by whichever prescription we use. We can then go on to define the probability of finding two universes in a particular state as that which we get by evaluating the partition function for classes of spacetimes that contain those three-geometries, and so on. Everett shows us how to use a pure probability description to obtain the concepts of correlation and information for our system.[48] I will not go into the details of this, but refer the reader to this paper. We can then describe a pure state for a particular three-geometry as being one for which the probability distribution of one universe is uncorrelated to the probability distribution of another, a mixed state simply being one that is not pure. We can now see how the Page and Hawking prescriptions can give us a mixed state by the following argument: Both prescriptions have four-geometries that contain *both* three-geometries (in other words, the path components of all the compact four-geometries are not all simply connected and thus the junction boundary has more than one path component). The latter terms are obviously correlation terms between the two universes in question. The size of these terms determines how mixed (correlated to another universe) our Universe is.

One important point is that if we *do* consider the prescriptions in the above sense, then the Hawking prescription, by construction, gives us a pure state for the Googolplexus, or more precisely, applying it for a *finite* number of components does, but the Page prescription gives us a seemingly *mixed* state. Thus there would be no single "universal wave function" in the Everett sense,[48] but only some kind of density matrix at best. Of course, both prescriptions would tend to suggest a mixed state for our own path component, and it is here that wormhole-type arguments may matter.[19]

Hence, if we regard a boundary as being a separate universe, the contributions in our model with holes in them correspond to correlations to disconnected regions of our three-geometry. The above scaling result thus implies that spacetimes with no holes (i.e., terms not representing correlation effects) will dominate over those that have holes, at small scales. Thus a single component of the Universe can be described by a pure state while small, but needs to be described by a mixed state once larger (at least until one reaches the point at which WKB dominates and our approximations break down). This would imply a local arrow of time for small geometries, at least in the Euclidean regime,[49–51] with an increase of entropy in the direction of increasing conformal time. It is important to note that this result is independent of what choice we make for the measure, and follows from the implicit Q-scale independence of it. This gives the potentially very useful result that one can expand the partition function for the Universe at small scales in powers of k, each term then corresponding to a specific group of spacetime topologies. One hence has a Feynman-like prescription for the quantum state of our world, with very close relations to the standard Polyakov formalism for bosonic strings.[52]

How good are the predictions of the model on these points? I believe that it is here that we can place rather a lot of faith. The model contains a roughly consistent description of the basic fluctuations one would expect and it is hard to see how a truly consistent model could destroy the scaling properties (at least at a qualitative level) and still have general relativity as a semiclassical expectation theory limit (i.e., be anomaly-free). The addition of the removed $\omega^z \rightarrow \omega^z + \delta\chi\omega^{\bar{z}}$ fluctuations, which would almost all be canceled by ghost corrections, does not seem to have the ability to drastically alter the qualitative aspects in this regard. As long as the operators that one gets from the perturbations are roughly Laplacian in nature (in our case, conformally Laplacian) the scaling law should have the same sign and basic effect.

VII. DISCUSSION

We will try and discuss some of the possible implications of what we have learned in this crude model in this section. The (rather speculative) discussion will primarily concentrate on what we may or may not expect if the properties of this model are representative of the properties of the real Universe.

Our major result seems to be the mixing of the quantum state as the scale of our Universe increases. It is *extremely important* for standard quantum cosmology, based on general relativity without higher-order corrections to the standard Einstein-Hilbert action, to get time as a *correlative* effect. From the Wheeler-DeWitt equation, and its attendant momentum constraints, we know that gravity, as a quantum theory, seems to have a description as a theory of *three-geometries* and does not have time as a physical observable (even as the basis for Schrödinger time, this variable may be "inaccessible," as has been noted by Page[53]). Though many theorists attempt to use WKB solutions as an attempt to place time back into their calculations, we know that this is not an approach that we should be happy with, if we want a truly fundamental way of explaining how and why we *observe* time in the way that we do. Time seems to need an explanation in terms of how we correlate to various "clock" systems in our Universe, and is deeply entwined with the second law of thermodynamics; it is not a parameter that one can identify in a classical solution. It is gratifying to see that this, more complicated, system does indeed possess the properties needed for this type of explanation of the nature of time.

The reader may wonder if a correlation between the degree of mixing in the Universe and scale would imply that the arrow of time will reverse as the Universe collapses.[49–51] It is here that one must point out that the arrow of time I am discussing here is a *global* arrow and hence rather rough. We have purely put a time index on superspace and should be careful, as the properties of the classical solutions we find ourselves in will dictate how we observe this arrow. If we are truly in a chaotic inflation bubble in a larger system,[54] for instance, our region could expand and collapse while the larger system increased monotonically in conformal time. Thus, we would have no reverse in the locally observed arrow of time. A second, more general, argument due to Page[50,55] is that in a solution which truly undergoes collapse, one has *two* WKB solutions for each three-geometry. One of these corresponds to the Universe expanding to that three-geometry, and the other corresponds to the collapse. One solution can have extremely large four-volume, especially if we allow ourselves to look at Lorentzian and/or complexified classical solutions, and the other one can be quite small. For instance, in the last moments of collapse, we have a large, mainly Lorentzian, solution corresponding to our past, and small, probably Euclidean, one for our future. The scale of the large solution *increases* as the three-geometry shrinks, thus confusing our view of there being a decreasing scale as we approach collapse. Thus the matter of the observed arrow of time is complicated. It is hard to discuss these problems here, until one focuses on the case in which one has other boundary topologies, apart from T^3, as it is known that T^3 cannot recollapse ($Q=t$). We are also defeated by the fact that the large solution may not in fact be *smooth*, even if Euclidean, and we are thus thwarted in a WKB analysis.

The next problem is *convergence*. Although (68) and (70) seem to formally give us the partition function, we do still have problems. The problems are centered around the fact that probabilities are formed from *closed* topologies. We have talked about a spacetime $\tilde{M}$ as having a given number of handles; unfortunately we cannot be sure that a smooth solution exists for these many-handle spacetimes (unlike the many-hole spacetimes). The problems associated with the fact that (25) is unbounded below were only solved for the purely one-loop part, but in general one wants to sum over baby universe states, and the zeroth-order (WKB) part will then reenter the problem. One can put an arbitrarily convoluted Q condition at these free boundaries, and hence drive S_T as negative as one desires, at least when one does not reduce to the true physical variables. Thus it is seen that the $\tilde{Z}$ may be formally divergent.

A further realization of the importance of the convergence of the scale expansion comes from understanding that the partition function will be of exponential form as long as the sum is convergent. It is manifestly positive for $k^{f/2}$ real and hence is described by a radius of convergence, k_{crit}. The purely one-loop part will *converge* for all k with relatively mild conditions on the regularized $\tilde{Z}(h,0)$. If the WKB part of the sum diverged for large k, as would happen if we can create handles with some negative or at least arbitrarily near zero action, then we could expect that k_{crit} would indeed be *finite* rather than trivially zero or infinite. Unfortunately (25) suggests that the proximal solutions do indeed have a positive minimum action for handles and that this action is proportional to k (for homogeneous boundary conditions, the action is proportional to the Euler characteristic of M). This would give an expansion that is convergent for all k, and may even cause the state to become pure at large scales (at which handle suppression would be exponentially suppressed). The latter is probably just a result of the use of an extremal technique. It is obvious that one should be able to add handles with an arbitrarily small action, and that the number should *increase* for large k, not decrease. The model fails to take this into account as it stands, as Q always lies between the extrema set at the boundary, thus forcing wormholes to have a large transverse area, which is not something one would expect for a true topological fluctuation. We could argue that a truly physical model will have our Universe as a bubble in a larger system, and hence the Q value in the space could get very small, thus allowing wormholes with small actions. In general, however, one would still expect to be able to produce small wormholes, even in quite smooth spacetimes, so a more powerful technique is needed if we wish to extend these predictions to large k. We are motivated to want the expansion to diverge for large k for the following reason: For $k < k_{\text{crit}}$ we are in a manifestly Euclidean regime. For $k > k_{\text{crit}}$, an infinite number of terms are needed for the expansion, and this seems to indicate the possibility that Z_{total} could now oscillate. This may correspond to spacetime undergoing a phase transition to Lorentzian behavior, described by classical solutions that correspond to imaginary Euclidean time. This would give one a *Euclidean* description of the transition, without the need for a direct complexification (other than as a mathematical tool). This is obviously a complicated issue, and concrete results are not easily available at this time, though one would hope that the question could be resolved in the near future, with its important implications.

One major caveat in the whole approach may spring from our choice of measure to be invariant under changes in Q. This does seem a natural choice, if one wishes to have an *anomaly-free* approach, but it was pointed out earlier that there exists a "natural" measure, $\rho = Q\epsilon$ (i.e., the normal integration measure for a four-dimensional space). This measure will produce *scale invariant* operators for B_1 and B_2, thus drastically altering the predictions in this paper. However, this measure is banned as long as we maintain the anomaly-free approach and then scaling seems automatic.

Our major problem, though, is the harsh nature of the proximal approximation. The rough nature of the removal of a whole class of fluctuations seems to have been forced upon us, to allow us to get any results at this early stage. This is a generic problem in quantum cosmology, caused by the fact that we are dealing with a parametrized and constrained system.[11] The invariance of the wave function under three-diffeomorphisms is generated by the general-relativistic momentum constraints and it is often difficult to solve these and the Wheeler-DeWitt equation at the same time. This manifests itself in the path integral as the fact that we often fit a three-metric to a four-metric and, for convenience, work with perturbations that do not change the boundary three-metric. Hence we end up with a reduced class of diffeomorphisms on the four-space that do not extend to boundary diffeomorphisms. This can leave us with a solution that is not three-covariant. The only solution is to choose an appropriate integration over the extra freedoms, requiring a possibly complicated measure or to fully take into account perturbations which correspond to boundary gauge freedoms from the beginning, again requiring a complicated measure. I did not attack such problems here, and attempted to create a toy system in which one can try and understand how the complicated spacetime topologies allowed by inhomogeneous spacetimes could alter the underlying behavior of the quantum dynamics of spacetime.

VIII. CONCLUSION

We have started to formulate the basic structure needed for a Euclidean cosmological model with enough degrees of freedom to allow us to discuss some of the major issues in quantum gravity, as it relates to cosmology. We have found an extremely simple law for the scaling behavior of the partition function and have been able to suggest a link between it and the arrow of time. The model has several defects. The main problem is that we have not attacked the constraint problem properly, taking the "low road" to a simple solution. A second problem is that we have obviously not come up with a satisfactory measure for the path integral. The third, and maybe fundamental, problem is that we have primarily based the work on the Hartle-Hawking proposal in its raw, naive, form whereas the author tends to feel that we really need to concentrate on developing a formalism for quantum gravity that takes the nature of the constraint manifold into account. Such a program may be built on the work on the Einstein-Rosen model by Kuchař.

Finally, in conclusion, this seems to be the beginning of a useful formalism. It gives one the ability to look at very complicated situations, but with a highly reduced level of technical complexity in the manipulations. The author hopes that it can be used to provide a more rigorous framework for questions ranging from baby universes to the nature of time and the density matrix of the universe.

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APPENDIX A

Hawking[41] explains the use of ζ-function and heat equation techniques in curved space. His description is useful for the case of simple models where the spacetime parameters are essentially *global*, but one must add a few details for a practical calculation in a general, inhomogeneous, curved space. I attempt to do this in the next two sections. The theory for general four-geometries is presented in Refs. 13 and 14, but in a form that has inherent anomaly problems. Some variation of that technique, however, must be used if one wishes to add the correct ghost terms to the system.

In general one is interested in an N-dimensional dynamical space M of volume form ϵ, with a bilinear quadratic expression A_1 acting on some field of the form

$$F=(f_1,\ldots,f_{n_p}) . \tag{A1}$$

The f_i are considered to be n_p orthogonal fields. We are interested in calculating expressions of the form

$$Z=\int [dF]\exp\left[-\pi m_P^2\int_M \epsilon A_1(F,F)\right] . \tag{A2}$$

We assume the measure on the perturbations corresponds to a Lebesgue measure generated by an inner product of the form, for fields F and G with components f_i and g_j,

$$\langle F|G\rangle=\int_M \rho\left[\sum_i f_i g_i\right] , \tag{A3}$$

where the effective volume form ρ does not necessarily equal ϵ. In general the f_i will transform as scalar densities of weight β and ρ will thus transform as an N-form density of weight -2β.

We next generate a self-adjoint representation of A_1 in the (M,ρ) inner product space, henceforth called the effective product space. Thus we define an operator A via

$$\langle\phi|A\psi\rangle=\tfrac{1}{2}[\langle(\phi+\psi)|A(\phi+\psi)\rangle -\langle\phi|A\phi\rangle-\langle\psi|A\psi\rangle] , \tag{A4}$$

where the diagonal terms of A come from

$$\langle F|AF\rangle=\int_M \epsilon A_1(F,F) . \tag{A5}$$

It is these two formulas that uniquely specify A, given the inner product. It is important to note that this operator's property of being self-adjoint is *chosen* via the inner product, something Hawking[41] does not make quite clear. The A operator can therefore be viewed as an $n_p\times n_p$ matrix of operators A^{ij} with $A^{ji}=(A^{ij})^\dagger$. We then expand F in the form

$$F=\sum c_n\psi_n , \tag{A6}$$

where the ψ_n obey the relations

$$A\psi_n=\lambda_n\psi_n , \tag{A7}$$

$$\langle\psi_n|\psi_m\rangle=\delta_{nm} . \tag{A8}$$

One then has

$$\int_M \epsilon A_1(F,F)=\sum_n \lambda_n c_n^2 \tag{A9}$$

and the measure is therefore

$$[dF]=\prod_n \mu dc_n . \tag{A10}$$

The normalization parameter μ is needed to make the measure dimensionless and must have units of one over the dimension of the f_i. Thus we get the usual Gaussian integral

$$Z=\prod \mu\int_{-\infty}^{+\infty} dc_n e^{-\pi m_P^2\lambda_n c_n^2} . \tag{A11}$$

Therefore one gets the important relation

$$Z=\det{}^{-1/2}\left[\frac{m_P^2 A}{\mu^2}\right] . \tag{A12}$$

The dependence of Z on the parameters of the background space will thus depend on the chosen set of f_i and β. Unlike matter, gravity generates its own background, so the integration measure is encoded into the A operator. Matter does not have this property in general, and usually scaling must be taken into account.[41]

We should notice that Eq. (A12) allows us to formally handle the divergent case of negative eigenvalues. In this case, the Z defined above will still be finite, but could pick up a factor of i. We can then view the above equation as automatically implementing the ideas of Gibbons, Hawking, and Perry[13] and "rotating" the divergent integrals in Eq. (A11) to the complex plane.

APPENDIX B

We now go on to the development of generalized ζ-function techniques for inhomogeneous local parameter spacetimes. We will be interested in evaluating $\det G$ for some operator G, acting on scalars, which is self-adjoint in some space with an inner product of the form

$$\langle\psi|\phi\rangle=\int_M \nu\psi\phi . \tag{B1}$$

We use a different volume form here to emphasize the fact that G could be one component of a self-adjoint operator defined in some other inner product space, and therefore not self-adjoint in that space (see Sec. V for an example). We call this space the effective eigenvector space, or EE-space. We note that the eigenvalue problem

$$G\phi_n=\lambda_n\phi_n \tag{B2}$$

defines the eigenvalues of G independent of the inner product space, but that the eigenvector problem is well defined in EE-space via

$$\langle\phi_n|\phi_m\rangle=\delta_{nm} . \tag{B3}$$

We define the generalized ζ function via

$$\zeta(G,s)=\sum_n \lambda_n^{-s} . \tag{B4}$$

This function will generally be defined for $\text{Re}(s)>N/2$ and can be analytically continued to a meromorphic function, regular at $s=0$. Differentiating (B4) with respect to s, and evaluating at $s=0$ formally gives

$$\zeta'(G,0)=-\ln \det G . \tag{B5}$$

Hence this gives one a finite, regularized, value for $\det G$. Also, if one evaluates the ζ function for kG, where k is some constant, one has

$$\zeta(kG,s)=k^{-s}\zeta(G,s) . \tag{B6}$$

Differentiating this gives

$$\zeta'(kG,0)=\zeta'(G,0)-\zeta(G,0)\ln k . \tag{B7}$$

In other words, from knowledge of $\zeta(G,s)$ near $s=0$ one can obtain the regularized determinant, and its scaling behavior

$$\det(kG)=k^{\zeta(G,0)}\det(G) . \tag{B8}$$

One wants to evaluate ζ near zero, which can be accomplished via the heat equation

$$\frac{d}{dt}F(x,y,t)+G_xF(x,y,t)=0 , \tag{B9}$$

where G_x acts only on x and $x,y\in M$. The problem is defined by the initial condition on the heat kernel F:

$$F(x,y,0)=\delta(x,y) . \tag{B10}$$

Where the δ function is defined relative to the EE-space integral equation:

$$\int_{y\in M}\nu(y)A(y)\delta(x,y)=A(x) . \tag{B11}$$

This shows where the dependence on the EE-space comes in. The solution is easily found to be

$$F(x,y,t)=\sum_n e^{-\lambda_n t}\phi_n(x)\phi_n(y) . \tag{B12}$$

We define the trace function $Y(G,t)$ by

$$Y(G,t)=\int_{x\in M}\nu(x)F(x,x,t) . \tag{B13}$$

We thus have

$$Y(G,t)=\sum_n \exp(-\lambda_n t) , \tag{B14}$$

independent of the choice of EE-space. We then find that $\zeta(G,s)$ is related to $Y(G,t)$ via a Mellin transform

$$\zeta(G,s)=\frac{1}{\Gamma(s)}\int_{0+}^{\infty}t^{s-1}Y(G,t)dt . \tag{B15}$$

The $0+$ in the integration denotes that this limit is to be taken last. Now, $\Gamma(s)$ has a pole at $s=0$, and $Y(G,t)$ exponentially decays for large t, hence we only need to expand Y in an asymptotic expansion valid as $t\to 0+$ if we are interested in the ζ function near zero. Gilkey[56] shows that we can expand Y as follows:

$$Y(G,t)\sim\sum_{n=0}^{n=\infty}C_n(G)t^{(n-N)/2} \tag{B16}$$

for $t\to 0+$. I use C_n instead of the usual B_n to avoid confusion with the operators discussed earlier. Thus we can formally write

$$\zeta(G,0)=\lim_{s\to 0}\left[\frac{1}{\Gamma(s)}\sum_{n=0}^{n=\infty}\frac{C_n(G)}{\frac{n}{2}+s-\frac{N}{2}}\right] . \tag{B17}$$

The poles of $\Gamma(s)$ cancel the poles in the sum for s a nonpositive integer. The result is therefore finite, as mentioned earlier, and we have

$$\zeta(G,0)=C_N(G) , \tag{B18}$$

which is just the constant term in the asymptotic expansion of $Y(G,t)$.

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