Singularity of Green's function and effective action in massive Yang-Mills theories

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The singularities of the one-loop Green's function and the effective action in massive Yang-Mills theory are investigated in terms of the original (Proca) variables. Although the result is nonrenormalizable, a high degree of predictive power can be maintained.

I. INTRODUCTION

The standard $SU(2)\times U(1)$ model is known to provide a good description of the electroweak interactions. However, the Higgs potential appears to be rather artificial. Furthermore, just as the four-Fermi theory was nonrenormalizable, there is no guarantee that the effective electroweak Lagrangian at current energies is a renormalizable one.

Once the restriction of renormalizability is removed, a natural candidate for electroweak theory is massive Yang-Mills theory. Indeed, if the Higgs boson is not discovered, it may be the only viable theory existing. It is well known that the theory is renormalizable at one loop on shell.[1-4] To establish this result, however, most authors employ a variant of the Steuckelberg transformation,[5] so that the Green's function becomes unphysical, involving a negative metric. In this paper, we reinvestigate the massive Yang-Mills theory in terms of the physical (Proca) fields.

We first investigate the Green's function in a background field. The result has a quartic singularity, as in the free case. We then turn to the one-loop effective action. We find that the quartic term is ineffective, so that the divergences are only quadratic and logarithmic. Together with the fact that the one-loop effective action is gauge invariant, we find that the theory can maintain considerable predictive power, although it is nonrenormalizable off shell.

Our conventions are as follows. The spacetime metric $g_{\mu\nu}$ is $+---$, and the free massless propagator is defined as

$$D_F(x)=-\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i0}e^{-ik\cdot x}=-\frac{i}{4\pi^2}\frac{1}{x^2-i0}\ . \tag{1.1}$$

Group indices are freely placed. They are often suppressed when using matrix notation, e.g.,

$$A_\mu=A_\mu^aT^a,\quad F_{\mu\nu}=\partial_\mu A_\mu-\partial_\nu A_\mu+[A_\mu,A_\nu]\ , \tag{1.2}$$

$$(T^b)^{ac}=f^{abc},\quad [T^a,T^b]=f^{abc}T^c\ . \tag{1.3}$$

The covariant derivative is denoted as

$$\nabla_\mu(O^aT^a)=(\nabla_\mu O)^aT^a,\quad (\nabla_\mu O)^a=\partial_\mu O^a+f^{abv}A_\mu^bO^c \tag{1.4}$$

rather than as $[\nabla_\mu,O]$, so that the distributive law reads

$$\nabla_\mu(O_1O_2)=(\nabla_\mu O_1)O_2+O_1(\nabla_\mu O_2) \tag{1.5}$$

for any bilinear product.

II. SINGULARITY OF THE GREEN'S FUNCTION

The action for the massive Yang-Mills theory is

$$I_{\rm YM}=\int d^4x\left[-\tfrac{1}{4}(\partial_\mu W_\nu^a-\partial_\nu W_\mu^a+f^{abc}W_\mu^bW_\nu^c)^2+\frac{m^2}{2}W_\mu^aW_a^\mu\right]. \tag{2.1}$$

Splitting W_μ^a into a classical background A_μ^a and a quantum fluctuation Q_μ^a, the one-loop Lagrangian is given by

$$I_{\rm YM}^{(1)}=\tfrac{1}{2}\int d^4x\ Q_a^\mu P_{\mu\nu}^{ab}(\nabla)Q_b^\nu\ , \tag{2.2}$$

$$P_{\mu\nu}(\nabla)=g_{\mu\nu}(\nabla^2+m^2)-\nabla_\mu\nabla_\nu+2F_{\mu\nu}\ , \tag{2.3}$$

where ∇ is the covariant derivative with respect to the background field A_μ^a, and $F_{\mu\nu}$ is a matrix in the adjoint representation.

The propagator is

$$P_{\mu\nu}^{ab}(\nabla)\Delta_{bc}^{\nu\rho}(x,y)=\delta_\mu^\rho\delta^{ac}\delta(x-y)\ . \tag{2.4}$$

Although the propagator has two group indices, we note that only the first one is acted on by the covariant derivatives.

As it stands, the highest derivatives in (2.3) are nondiagonal in the Lorentz indices. This is inconvenient, so we proceed as follows. Multiplying (2.4) by ∇^μ on the left, we find

$$(-m^2V_\nu+m^2\nabla_\nu)\Delta^{\nu\rho}(x,y)=\nabla^\rho\delta(x-y)\mathbb{1}\ , \tag{2.5}$$

where **1** is the unit matrix in group space, and

$$V_\nu=-\frac{1}{m^2}\nabla^\mu F_{\mu\nu}\ . \tag{2.6}$$

Further solving for $\nabla_\nu\Delta^{\nu\rho}$ in (2.5) and substituting back into (2.3) and (2.4),

$$\hat P_{\mu\nu}(\nabla)\Delta^{\nu\rho}(x,y)=\left[\delta_\mu^\rho+\frac{1}{m^2}\nabla_\mu\nabla^\rho\right]\delta(x-y)\mathbb{1}\ , \tag{2.7}$$

$$\hat P_{\mu\nu}(\nabla)=g_{\mu\nu}(\nabla^2+m^2)+2F_{\mu\nu}-V_\nu\nabla_\mu-(\nabla_\mu V_\nu)\ . \tag{2.8}$$

Hence introducing a new Green's function $\hat{\Delta}^{ab}_{\mu\nu}(x,y)$ by

$$\hat{P}_{\mu\nu}(\nabla)\hat{\Delta}^{\nu\rho}(x,y)=\delta^{\rho}_{\mu}\delta(x-y)\mathbb{1} \tag{2.9}$$

we may solve (2.7) as

$$\Delta^{\nu\rho}(x,y)=\hat{\Delta}^{\nu\rho}(x,y)+\frac{1}{m^2}\hat{\Delta}^{\nu\alpha}(x,y)\overleftarrow{\nabla}_{\alpha}\overleftarrow{\nabla}^{\rho} , \tag{2.10}$$

where the arrow means that the derivative acts on the left:

$$\overleftarrow{\nabla}_{\alpha}=-\frac{\overleftarrow{\partial}}{\partial y^{\alpha}}+A_{\alpha}(y) . \tag{2.11}$$

Since the highest derivatives in $\hat{P}_{\mu\nu}$ are now diagonal, we may write down the Hadamard expansion

$$\begin{aligned}\hat{\Delta}^{ab}_{\mu\nu}(x,y)=&G^{(0)ab}_{\mu\nu}(x,y)D_F(z)\\&-\frac{i}{16\pi^2}\ln(-z^2+i0)\\&\times\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{z^2}{4}\right]^n G^{(n+1)1b}_{\mu\nu}(x,y)\\&+\text{smooth} ,\end{aligned} \tag{2.12}$$

where $z=x-y$, and the G's are assumed to be smooth at $x=y$. The smooth terms are not identified by the expansion, but it turns out that they are not necessary for our considerations. Substituting (2.12) into (2.9), we obtain the recursion relations

$$z^{\alpha}\nabla_{\alpha}G^{(0)\rho}_{\mu}(x,y)-\frac{z_{\mu}}{2}V_{\nu}(x)G^{(0)\nu\rho}(x,y)=0 , \tag{2.13}$$

$$\begin{aligned}&(z^{\alpha}\nabla_{\alpha}+n+1)G^{(n+1)\rho}_{\mu}(x,y)\\&\quad-\frac{z_{\mu}}{2}V_{\nu}(x)G^{(n+1)\nu\rho}(x,y)+\hat{P}_{\mu\nu}(\nabla)G^{(n)\nu\rho}(x,y)=0 .\end{aligned} \tag{2.14}$$

Equation (2.13) may be absorbed into (2.14) if we define $G^{(-1)}_{\mu\nu}$ to be zero. From Eq. (2.9) we may also derive the initial condition

$$G^{(0)\rho}_{\mu}(y,y)=\delta^{\rho}_{\mu} . \tag{2.15}$$

The solution of Eq. (2.13) can be formally expressed as a path-ordered product. In the case in which the external field satisfies the condition

$$\nabla_{\mu}F^{\mu\nu}=0 \tag{2.16}$$

(2.13) reduces to

$$z^{\alpha}\nabla_{\alpha}G^{(0)}_{\mu\nu}=0 \tag{2.17}$$

which is a homogeneous transport equation in the background connection A_{α}. The solution is given by the path-ordered exponential

$$G^{(0)}_{\mu\nu}(x,y)=P\exp\left[\int_0^{-1}A_{\alpha}[z(s)]\frac{dz^{\alpha}}{ds}ds\right]g_{\mu\nu} \tag{2.18}$$

taken along the straight path $z^{\alpha}(s)=(x-y)^{\alpha}s,\ \ 0<s<1$.

Substituting (2.12) back into (2.10) gives the expansion for the original propagator:

$$\begin{aligned}\Delta^{\nu\rho}(x,y)=&\frac{1}{m^2}B^{\nu\rho}(x,y)+\frac{1}{m^2}C^{(0)\nu\rho}(x,y)D_F(z)\\&-\frac{i}{16\pi^2m^2}\ln(-z^2+i0)\\&\times\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{z^2}{4}\right]^n C^{(n+1)\nu\rho}(x,y)\\&+\text{smooth}\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}B^{\nu\rho}(x,y)=&\frac{z_{\alpha}}{2}G^{(1)\nu\alpha}(x,y)\partial^{\rho}D_F(z)\\&+G^{(0)\nu\alpha}(x,y)\overleftarrow{\nabla}_{\alpha}\partial^{\rho}D_F(z)\\&+G^{(0)\nu\alpha}(x,y)\overleftarrow{\nabla}^{\rho}\partial_{\alpha}D_F(z)\\&+G^{(0)\nu\alpha}(x,y)\partial_{\alpha}\partial^{\rho}D_F(z)\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}C^{(n)\nu\rho}(x,y)=&m^2G^{(n)\nu\rho}(x,y)+G^{(n)\nu\alpha}(x,y)\overleftarrow{\nabla}_{\alpha}\overleftarrow{\nabla}^{\rho}+\frac{z^{\rho}}{2}G^{(n+1)\nu\alpha}(x,y)\overleftarrow{\nabla}_{\alpha}\\&+\frac{z_{\alpha}}{2}G^{(n+1)\nu\alpha}(x,y)\overleftarrow{\nabla}^{\rho}+\tfrac{1}{2}G^{(n-1)\nu\rho}(x,y)+\frac{z_{\alpha}z^{\rho}}{4}G^{(n+2)\nu\alpha}(x,y) .\end{aligned} \tag{2.21}$$

The symmetry of the propagator requires

$$\begin{aligned}B^{\nu\rho}_{ab}(x,y)&=(B^{\rho\nu}_{ba})^T(y,x),\\C^{\nu\rho}_{ab}(x,y)&=(C^{\rho\nu}_{ba})^T(y,x) ,\end{aligned} \tag{2.22}$$

where T denotes the transpose.

The coincidence limits $[G^{(n)}_{\mu\nu}](y)=G^{(n)}_{\mu\nu}(y)=[\nabla_{\alpha}G^{(n)}_{\mu\nu}](x,y)_{x=y}$, etc., may be calculated from the recursion relation (2.13) and (2.14). The details are reported in the Appendix. In order to interchange right and left derivatives we also define a new covariant derivative, denoted as D, acting on matrices as

$$D_{\alpha}T=\partial_{\alpha}T+A_{\alpha}T-TA_{\alpha} . \tag{2.23}$$

For Lie-algebra-valued elements such as the field strength and its multiple covariant derivatives this new definition reduces to the older one. In the coincidence limit,

$$[T(x,y)\overleftarrow{\nabla}_{\alpha}]_{x=y}=-D_{\alpha}[T(x,x)]+(\nabla_{\alpha}T)(x) . \tag{2.24}$$

Formulas (2.15) and (2.24), together with

$$(\nabla_{\beta}G^{(0)\rho}_{\mu})=\tfrac{1}{2}g_{\beta\mu}V^{\rho} , \tag{2.25}$$

give the leading singularity of the propagator:

$$\Delta^{\nu\rho}(x,y)\sim\frac{1}{m^2}\partial^\nu\partial^\rho D_F(z)$$
$$+\frac{1}{m^2}(\tfrac{1}{2}z^\nu V^\alpha+z^\mu A_\mu g^{\nu\alpha})\partial_\alpha\partial^\rho D_F$$
$$+\frac{1}{2m^2}V^\nu\partial\rho D_F(z)+\frac{1}{2m^2}g^{\nu\rho}V^\alpha\partial_\alpha D_F(z)\ , \tag{2.26}$$

which may be checked to be symmetric, despite appearances.

As an example, we may take a monopole background

$$A_a^0=0\ ,\quad A_a^i=-\epsilon_{aik}\frac{x^k}{r^2}f(r) \tag{2.27}$$

(indices running from 1 to 3). We then find

$$V_a^0=0,\quad V_a^j=\epsilon_{ajk}\frac{x^k}{m^2r^4}(2f-3f^2+f^3-r^2f'')\ . \tag{2.28}$$

In particular, $V^\mu=0$ for $f=1$ corresponds to the point monopole (Wu-Yang) solution. The leading singularity of the propagator in that case is given by

$$\Delta_{\nu\rho}^{ab}(x,y)=\frac{1}{m^2}\delta^{ab}\partial_\nu\partial_\rho D_F(z)$$
$$+\frac{-x^ay^b+x^by^a}{m^2r^2}\partial_\nu\partial_\rho D_F(z)\ . \tag{2.29}$$

III. THE EFFECTIVE ACTION

The coincident limits $[G_\mu^{(n)\rho}]$ also appear in the divergent part of the one-loop effective action:

$$\Gamma^{(1)}[A]=\frac{i}{2}\mathrm{Tr}\ln P\ . \tag{3.1}$$

The quadratic Lagrangian (2.2) is invariant under a gauge transformation of the background field A_μ, provided Q_μ transforms as a matter field in the adjoint representation. It follows that the one-loop effective action should be gauge invariant, in contrast with the tree-level action (2.1). In particular, this must also be true of the divergent part.

It is worth remarking that even the divergent part of the effective action is difficult to obtain by conventional methods. Summing diagrams are out of the question, since the divergences become worse and worse. The proper-time (heat-kernel) method also runs into difficulties, since the highest derivatives in (2.3) are nondiagonal in the Lorentz indices.

We therefore make use of our previous results, by taking a variation of (3.1):

$$\delta\Gamma^{(1)}[A]=\frac{i}{2}\mathrm{Tr}(\delta P)P^{-1}=\frac{i}{2}\langle\delta P_{\mu\nu}\Delta^{\nu\mu}(x,y)\delta(z)\rangle\ . \tag{3.2}$$

The angular brackets denote both a trace on group indices and integration over all the independent spacetime coordinates appearing inside the brackets, e.g., in this case x and y. Inserting the explicit expression (2.3) for $P_{\nu\mu}$ into the previous equation, we find, after a little algebra,

$$\begin{aligned}-2i\delta\Gamma^{(1)}[A]&=\langle(\delta\nabla^2)\Delta_\mu^\mu(x,y)\delta(z)\rangle-\langle(\delta\nabla_\mu\nabla_\nu)\Delta^{\nu\mu}(x,y)\delta(z)\rangle+2\langle\delta F_{\mu\nu}\Delta^{\nu\mu}(x,y)\delta(z)\rangle\\&=2\langle\delta A_\alpha\nabla^\alpha\Delta_\mu^\mu(x,y)\delta(z)\rangle-2\langle\delta A_\mu\nabla_\nu\Delta^{\nu\mu}(x,y)\delta(z)\rangle+2\langle\delta F_{\mu\nu}\Delta^{\nu\mu}(x,y)\delta(z)\rangle\ .\end{aligned} \tag{3.3}$$

Derivatives on the propagator can be computed from (2.19):

$$\begin{aligned}m^2\nabla_\alpha\Delta_{\mu\nu}(x,y)&=\nabla_\alpha B_{\mu\nu}(x,y)+\nabla_\alpha C_{\mu\nu}^{(0)}(x,y)D_F(z)+C_{\mu\nu}^{(0)}(x,y)\partial_\alpha D_F\\&\quad+\frac{z_\alpha}{2}D_F C_{\mu\nu}^{(1)}(x,y)\frac{-i}{16\pi^2}\ln(-z^2+i0)\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{z^2}{4}\right]^n\left[\frac{z_\alpha}{2}C_{\mu\nu}^{(n+2)}(x,y)+\nabla_\alpha C_{\mu\nu}^{(n+1)}(x,y)\right]\ .\end{aligned} \tag{3.4}$$

To maintain gauge invariance, we regulate only $D_F(z)$ and its derivatives and $\ln(-z^2+i0)$. Introducing the regulated expressions

$$\begin{aligned}&D_F(z)\to D_F^R(z)\ ,\\&\ln(-z^2+i0)\to\ln^R(-z^2+i0)\ ,\end{aligned} \tag{3.5}$$

and taking the coincidence limit of Eq. (3.4),

$$m^2(\nabla_\alpha\Delta_{\mu\nu})^R=(\nabla_\alpha B_{\mu\nu})^R+(\nabla_\alpha C_{\mu\nu}^{(0)})(D_F^R)+(\nabla_\alpha C_{\mu\nu}^{(1)})(\ln^R)\ . \tag{3.6}$$

After some manipulations we obtain the following form for the variation of the effective action:

$$\begin{aligned}-2im^2\delta\Gamma^{(1)}[A]&=2\langle\delta A_\alpha(x)(\nabla_\alpha C_\mu^{(0)\mu})(x)\rangle(D_F^R)-2\langle\delta A_\mu(x)(\nabla_\nu C^{(0)\nu\mu})(x)\rangle(D_F^R)\\&\quad+2\langle\delta A_\mu(x)(\nabla^\alpha C_\mu^{(1)\mu})(x)\rangle(\ln^R)-2\langle\delta A_\mu(x)(\nabla_\nu C^{(1)\nu\mu})(x)\rangle(\ln^R)\\&\quad+\frac{2}{m^2}\langle\delta F_{\mu\nu}(C^{(0)\nu\mu})(x)\rangle(D_F^R)+2\langle\delta F_{\mu\nu}(x)(C^{(1)\nu\mu})(x)\rangle(\ln^R)\ .\end{aligned} \tag{3.7}$$

We note that there is no contribution from $B_{\mu\nu}$, so the divergence is quadratic rather than quartic.

As we can see from these equations, at the one-loop level, the divergences come in well-defined combinations which give to the theory a high degree of predictive power. It is our aim to discuss the contribution coming from the pole (quadratic) singularity by explicitly computing the coefficients which appear in the previous equation. For the logarithmic singularity one requires the evaluation of the derivatives of the strength equation (2.13) up to fifth order. It is a straightforward but exceedingly lengthy computation and it will not be given in this paper.

From Eq. (2.21) specialized to $n=0$ we get

$$(C_{\mu\nu}^{(0)})(x)=m^2(G_{\mu\nu}^{(0)})(x)+[G_{\mu\alpha}^{(0)}(x,y)\overleftarrow{\nabla}^{\alpha}\overleftarrow{\nabla}_{\nu}]_{x=y}+\tfrac{1}{2}(G_{\mu\nu}^{(1)})(x)\ . \tag{3.8}$$

Using (2.24) and the results quoted in the Appendix we find the expression of $C^{(0)}$,

$$(C_{\mu\nu}^{(0)})=\frac{m^2}{2}g_{\mu\nu}+\tfrac{1}{4}(\nabla_\mu V_\nu-\nabla_\nu V_\mu)-\tfrac{1}{2}F_{\mu\nu}+\tfrac{1}{8}V^2 g_{\mu\nu}+\tfrac{1}{8}(V_\mu V_\nu+V_\nu V_\mu)\ , \tag{3.9}$$

which is symmetric in the simultaneous interchange of the Lorentz indices and, respectively, of the group indices, as expected.

As a last step we evaluate the first derivative of $C^{(0)}$ in the coincidence limit. We find

$$\begin{aligned}(\nabla_\mu C_{\nu\rho}^{(0)})={}&\frac{m^2}{2}g_{\mu\nu}V_\rho-D_\rho(\nabla_\alpha\nabla_\mu G_\nu^{(0)\alpha})-D_\alpha(\nabla_\rho\nabla_\mu G_\nu^{(0)\alpha})\\&+(\nabla_\alpha\nabla_\rho\nabla_\mu G_\nu^{(0)\alpha})+\tfrac{1}{2}g_{\rho\mu}[G_{\nu\alpha}^{(1)}(x,y)\overleftarrow{\nabla}^{\alpha}]_{x=y}+\tfrac{1}{2}[G_{\nu\mu}^{(1)}(x,y)\overleftarrow{\nabla}_{\rho}]_{x=y}+\tfrac{1}{2}(\nabla_\mu G_{\nu\rho}^{(1)})\ .\end{aligned} \tag{3.10}$$

Using repeatedly the results quoted in the Appendix, we finally get

$$\begin{aligned}(\nabla_\mu C_{\nu\rho}^{(0)})={}&m^2(\tfrac{1}{6}g_{\nu\mu}V_\rho+\tfrac{1}{6}g_{\mu\rho}V_\nu-\tfrac{1}{3}g_{\nu\rho}V_\mu)+(-\tfrac{1}{3}\nabla_\rho F_{\mu\nu}+\tfrac{1}{6}\nabla_\nu F_{\mu\rho}-\tfrac{1}{2}\nabla_\mu F_{\nu\rho})\\&-\tfrac{1}{24}(g_{\mu\rho}\nabla^2V_\nu+g_{\mu\nu}\nabla^2V_\rho+g_{\nu\rho}\nabla^2V_\mu)+\tfrac{1}{12}(-\nabla_\rho\nabla_\mu V_\nu+2\nabla_\mu\nabla_\nu V_\rho+2\nabla_\rho\nabla_\nu V_\mu-3\nabla_\rho\nabla_\nu V_\mu)\\&-\tfrac{1}{8}\nabla_\rho(V_\mu V_\nu+V_\nu V_\mu)-\tfrac{1}{8}(g_{\mu\nu}\nabla_\rho V^2-\tfrac{1}{3}g_{\mu\nu}V_\alpha\nabla^\alpha V_\rho+g_{\rho\nu}\nabla_\alpha V_\mu V^\alpha+g_{\mu\nu}\nabla_\alpha V_\rho V^\alpha+\tfrac{1}{3}g_{\mu\rho}\nabla_\alpha V_\nu V_\alpha)\\&-\tfrac{1}{24}(g_{\nu\rho}V_\alpha\nabla^\alpha V_\mu+V_\nu\nabla_\rho V_\mu+V_\rho\nabla_\nu V_\mu)+\tfrac{1}{24}(2\nabla_\nu V^\alpha V_\alpha+V^\alpha\nabla_\alpha+V_\alpha\nabla_\nu V^\alpha)\\&+\tfrac{1}{24}(V_\nu\nabla_\mu V_\rho+V_\mu\nabla_\nu V_\rho+2\nabla_\mu V_\nu V_\rho+2\nabla_\nu V_\mu V_\rho+2\nabla_\nu V_\rho V_\mu+2\nabla_\rho V_\nu V_\mu)\\&+\tfrac{1}{6}(F_{\mu\rho}V_\nu+\tfrac{1}{2}g_{\nu\rho}F_{\mu\alpha}V^\alpha+\tfrac{1}{2}g_{\mu\nu}F_{\rho\alpha}V^\alpha+\tfrac{1}{2}g_{\rho\mu}F_{\nu\alpha}V^\alpha+F_{\nu\rho}V_{\nu\rho}+F_{\nu\rho}V_\mu+F_{\nu\mu}V_\rho)\\&-\tfrac{1}{12}(V_\nu F_{\mu\rho}+g_{\nu\rho}V^\alpha F_{\mu\alpha}g_{\mu\nu}V^\alpha F_{\rho\alpha}-2g_{\rho\mu}V^\alpha F_{\nu\alpha}+3g_{\nu\rho}V^\alpha F_{\alpha\mu}+V_\mu F_{\nu\rho}+V_\rho F_{\nu\mu}+3g_{\mu\nu}V^\alpha F_{\alpha\rho})\\&+\tfrac{1}{48}(g_{\rho\mu}V^2V_\nu+g_{\rho\mu}V_\nu V^2+g_{\rho\mu}V_\alpha V_\nu V^\alpha+g_{\nu\rho}V^2V_\mu+g_{\mu\nu}V^2V_\rho\\&\quad+V_\nu V_\mu V_\rho+V_\mu V_\nu V_\rho+V_\nu V_\rho V_\rho+V_\rho V_\nu V_\mu)+\chi^{\alpha}_{\alpha\rho\mu,\nu}\ ,\end{aligned} \tag{3.11}$$

where $\chi^{\rho}_{\beta\gamma\delta,\mu}$ is a tensor symmetric in $\beta\gamma\delta$ which is defined in the Appendix [Eq. (A18)].

The integration of the variational Eq. (3.2) can be carried through straightforwardly by taking the following parametrization for the background field $A_\mu(x)$ and for its variation

$$A_\mu^{(t)}(x)=tA_\mu(x),\quad \delta A_\mu^{(t)}(x)=dt\,A_\mu(x),\quad 0<t<1\ . \tag{3.12}$$

At the same time we define rescaled tensors such as

$$L_\alpha^{(t)}=(\nabla_\alpha C_\mu^{(0)\mu})|_{A\to tA}\ , \tag{3.13}$$

$$M_\alpha^{(t)}=(\nabla_\nu C_\alpha^{(0)\nu})|_{A\to tA}\ , \tag{3.14}$$

$$N_{\mu\nu}^{(t)}=(C_{\mu\nu}^{(0)})|_{A\to tA} \tag{3.15}$$

in which the functional dependence on the background field has been rescaled by the same parameter t. The pole contribution to the effective action can be expressed as

$$\Gamma^{(1)}_{\text{pole}}[A]=\frac{i(D_F^R)}{m^2}\int_0^1 dt\int d^4x\ \mathrm{tr}[A^\alpha(L_\alpha^{(t)}-M_\alpha^{(t)})+(\partial_\mu A_\nu-\partial_\nu A_\mu+2t[A_\mu,A_\nu])N^{(t)\nu\mu}]\ . \tag{3.16}$$

With a point monopole, the result is linearly divergent at the origin:

$$\Gamma^{(1)}_{\text{pole}}[A]=\frac{i(D_F^R)}{m^2}\int dx^0d^3x\frac{7}{3r^4}\ . \tag{3.17}$$

Hence, for an extended monopole of mass $M \sim m/e^2$,

$$\Gamma^{(1)}_{\text{pole}}[A] = \frac{i(D_F^R)}{m^2} M \int dx^0 \ , \tag{3.18}$$

which may be identified as a quadratically divergent mass renormalization.

An analogous procedure allows one to compute also the logarithmic contribution if necessary.

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APPENDIX: EVALUATION OF COINCIDENCE LIMITS

From the recursion relation

$$(z^\alpha \nabla_\alpha + n) G_\mu^{(n)\rho}(x,y) - \frac{z_\mu}{2} V_\nu(x) G^{(n)\nu\rho}(x,y) + \hat{P}_{\mu\nu}(\nabla) G^{(n-1)\nu\rho}(x,y) = 0 \ , \tag{A1}$$

$$\hat{P}_{\mu\nu}(\nabla) = g_{\mu\nu}\nabla^2 - V_\nu \nabla_\mu + m^2 g_{\mu\nu} + 2F_{\mu\nu} - \nabla_\mu V_\nu \tag{A2}$$

and the initial conditions

$$(G_\mu^{(0)\rho})(y) = G_\mu^{(0)\rho}(y,y) = \delta_\mu^\rho \mathbb{1}, \quad G^{(-1)}(x,y) = 0 \tag{A3}$$

the coincidence limit for G follows immediately:

$$(G_\mu^{(n)\rho}) = -\frac{1}{n}(\hat{P}_{\mu\nu} G^{(n-1)\nu\rho}) \quad (n \geq 1) \ . \tag{A4}$$

Multiplying (A1) by ∇_β we find

$$[z^\alpha \nabla_\beta \nabla_\alpha + (n+1)\nabla_\beta] G_\mu^{(n)\rho}(x,y) - \frac{z_\mu}{2} \nabla_\beta [V_\nu(x) G^{(n)\nu\rho}(x,y)] - \tfrac{1}{2} g_{\mu\beta} V_\nu(x) G^{(n)\nu\rho}(x,y) + \nabla_\beta \hat{P}_{\mu\nu}(\nabla) G^{(n-1)\nu\rho}(x,y) = 0 \ . \tag{A5}$$

Hence,

$$\begin{aligned}
(\nabla_\beta G_\mu^{(n)\rho}) &= \frac{1}{2(n+1)} g_{\mu\beta} V_\nu (G^{(n)\nu\rho}) - \frac{1}{n+1}(\nabla_\beta \hat{P}_{\mu\nu} G^{(n-1)\nu\rho}) \\
&= \begin{cases} -\dfrac{1}{2(n+1)n} g_{\mu\beta} V^\nu (\hat{P}_{\nu\alpha} G^{(n-1)\alpha\rho}) - \dfrac{1}{n+1}(\nabla_\beta \hat{P}_{\mu\nu} G^{(n-1)\nu\rho}) \quad (n \geq 1) \ , \\ \tfrac{1}{2} g_{\mu\beta} V^\rho \quad (n=0) \ . \end{cases}
\end{aligned} \tag{A6}$$

Similarly, further differentiation yields

$$\begin{aligned}
&[z^\alpha \nabla_\gamma \nabla_\beta \nabla_\alpha + (n+1)\nabla_\gamma \nabla_\beta + \nabla_\beta \nabla_\gamma] G_\mu^{(n)\rho}(x,y) - \frac{z_\mu}{2} \nabla_\gamma \nabla_\beta [V_\alpha(x) G^{(n)\alpha\rho}(x,y)] \\
&\qquad - \tfrac{1}{2}(g_{\mu\gamma}\nabla_\beta + g_{\mu\beta}\nabla_\gamma)[V_\alpha(x) G^{(n)\alpha\rho}(x,y)] + \nabla_\gamma \nabla_\beta \hat{P}_{\mu\alpha}(\nabla) G^{(n-1)\alpha\rho}(x,y) = 0 \ .
\end{aligned} \tag{A7}$$

Using the commutation relation

$$(\nabla_\mu \nabla_\nu O - \nabla_\nu \nabla_\mu O)^a = (F_{\mu\nu})^{ac} O^c \tag{A8}$$

we find

$$(\nabla_\gamma \nabla_\beta G_\mu^{(0)\rho}) = -\tfrac{1}{2} F_{\beta\gamma} \delta_\mu^\rho + \tfrac{1}{4} g_{\gamma\mu} [\nabla_\beta (V_\nu G^{(0)\nu\rho})] + \tfrac{1}{4} g_{\beta\mu} [\nabla_\gamma (V_\nu G^{(0)\nu\rho})] \ , \tag{A9}$$

$$\begin{aligned}
(\nabla_\gamma \nabla_\beta G_\mu^{(1)\rho}) &= -\tfrac{1}{3} F_{\beta\gamma} (G_\mu^{(1)\rho}) + \tfrac{1}{6}(g_{\gamma\mu} \nabla_\beta V_\alpha + g_{\beta\mu} \nabla_\gamma V_\alpha)(G^{(1)\alpha\rho}) \\
&\quad + \tfrac{1}{6} V_\alpha [(g_{\gamma\mu}\nabla_\beta + g_{\beta\mu}\nabla_\gamma) G^{(1)\alpha\rho}] - \tfrac{1}{3}(\nabla_\gamma \nabla_\beta \hat{P}_{\mu\alpha} G^{(0)\alpha\rho}) \ .
\end{aligned} \tag{A10}$$

Substitution of (A6) then gives

$$(\nabla_\gamma \nabla_\beta G_\mu^{(0)\rho}) = -\tfrac{1}{2} F_{\beta\gamma} \delta_\mu^\rho + \tfrac{1}{4}(g_{\gamma\mu} \nabla_\beta V^\rho + g_{\beta\mu} \nabla_\gamma V^\rho) + \tfrac{1}{8}(g_{\gamma\mu} V_\beta V^\rho + g_{\beta\mu} V_\gamma V^\rho) \ , \tag{A11}$$

$$\begin{aligned}(\nabla_\gamma\nabla_\beta G_\mu^{(1)\rho}) = {} & -\tfrac{1}{3}(\nabla_\gamma\nabla_\beta\hat{P}_{\mu\alpha}G^{(0)\alpha\rho}) - \tfrac{1}{12}[g_{\gamma\mu}V^\nu(\nabla_\beta\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) + (\gamma\leftrightarrow\beta)] \\ & - \tfrac{1}{24}(4g_{\gamma\mu}\nabla_\beta V^\nu + 4g_{\beta\mu}\nabla_\gamma V^\nu - 8F_{\beta\gamma}\delta_\mu^\nu + g_{\gamma\mu}V_\beta V^\nu + g_{\beta\mu}V_\gamma V^\nu)(\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) \end{aligned} \tag{A12}$$

and

$$(\nabla^2 G_\mu^{(0)\rho}) = \tfrac{1}{2}\nabla_\mu V^\rho + \tfrac{1}{4}V_\mu V^\rho , \tag{A13}$$

$$(\nabla^2 G_\mu^{(1)\rho}) = -\tfrac{1}{3}(\nabla^2\hat{P}_{\mu\alpha}G^{(0)\alpha\rho}) - \tfrac{1}{6}V^\nu(\nabla_\mu\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) - \tfrac{1}{12}(4\nabla_\mu V^\nu + V_\mu V^\nu)(\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) . \tag{A14}$$

Collecting all terms we get

$$(G_\mu^{(1)\rho}) = -(\hat{P}_{\mu\nu}G^{(0)\nu\rho}) = \tfrac{1}{2}\nabla_\mu V^\rho + \tfrac{1}{4}V_\mu V^\rho - \delta_\mu^\rho m^2 - 2F_\mu^\rho \tag{A15}$$

$$\begin{aligned}(G_\mu^{(2)\rho}) = -\tfrac{1}{2}(\hat{P}_{\mu\nu}G^{(1)\nu\rho}) = {} & \tfrac{1}{6}(\nabla^2\hat{P}_{\mu\alpha}G^{(0)\alpha\rho}) - \tfrac{1}{6}V^\nu(\nabla_\mu\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) \\ & + \tfrac{1}{12}(-4\nabla_\mu V^\nu - V_\mu V^\nu + 6m^2\delta_\mu^\nu + 12F_\mu^\nu)(\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) . \end{aligned} \tag{A16}$$

Taking further derivatives we find

$$\begin{aligned}(\nabla_\delta\nabla_\gamma\nabla_\beta G_\mu^{(0)\rho}) = {} & -\tfrac{1}{3}(\nabla_\gamma F_{\beta\delta} + \nabla_\delta F_{\beta\gamma})\delta_\mu^\rho - \tfrac{1}{6}(g_{\delta\mu}F_{\beta\gamma} + g_{\gamma\mu}F_{\beta\delta} + g_{\beta\mu}F_{\gamma\delta})V^\rho - \tfrac{1}{12}V^\rho(g_{\delta\mu}F_{\beta\gamma} + g_{\gamma\mu}F_{\beta\delta} + g_{\beta\mu}F_{\gamma\delta}) \\ & + \tfrac{1}{6}(g_{\delta\mu}\nabla_\gamma\nabla_\beta V^\rho + g_{\gamma\mu}\nabla_\delta\nabla_\beta V^\rho + g_{\beta\mu}\nabla_\delta\nabla_\gamma V^\rho) + \chi^\chi_{\beta\gamma\delta,\mu} , \end{aligned} \tag{A17}$$

where

$$\chi^\rho_{\beta\gamma\delta,\mu} = \tfrac{1}{12}g_{\delta\mu}(\nabla_\gamma V_\beta)V^\rho + \tfrac{1}{24}g_{\delta\mu}V_\gamma\nabla_\beta V^\rho + \tfrac{1}{48}g_{\delta\mu}V_\gamma V_\beta V^\rho + (\beta,\gamma,\delta 5\ \text{perm}) \tag{A18}$$

and

$$\begin{aligned}(\nabla_\delta\nabla^2 G_\mu^{(0)\rho}) = {} & \tfrac{1}{3}m^2 V_\delta\delta_\mu^\rho - \tfrac{1}{3}F_{\mu\delta}V^\rho - \tfrac{1}{6}V^\rho F_{\mu\delta} + \tfrac{1}{6}(g_{\delta\mu}\nabla^2 V^\rho + 2\nabla_\delta\nabla_\mu V^\rho) + \tfrac{1}{12}(g_{\delta\mu}V^\beta\nabla_\beta V^\rho + V_\mu\nabla_\delta V^\rho + V_\delta\nabla_\mu V^\rho) \\ & + \tfrac{1}{6}(\nabla_\delta V_\mu + \nabla_\mu V_\delta)V^\rho + \tfrac{1}{24}(g_{\delta\mu}V^\beta V_\beta + V_\mu V_\delta + V_\delta V_\mu)V^\rho , \end{aligned} \tag{A19}$$

$$(\nabla_\kappa\nabla^2 G_\mu^{(1)\rho}) = -\tfrac{1}{2}F_{\beta\kappa}(\nabla^\beta G_\mu^{(1)\rho}) + \frac{m^2}{4}V_\kappa(G_\mu^{(1)\rho}) + \tfrac{1}{8}g_{\mu\kappa}[\nabla^2(V_\nu G^{(1)\nu\rho})] + \tfrac{1}{4}[\nabla_\kappa\nabla_\mu(V_\nu G^{(1)\nu\rho})] - \tfrac{1}{4}(\nabla_\kappa\nabla^2\hat{P}_{\mu\nu}G^{(0)\nu\rho}) , \tag{A20}$$

which gives

$$\begin{aligned}(\nabla_\delta G_\mu^{(1)\rho}) & = -\tfrac{1}{4}g_{\delta\mu}V^\nu(\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) - \tfrac{1}{2}(\nabla_\delta\hat{P}_{\mu\nu}G^{(0)\nu\rho}) \\ & = -\frac{m^2}{2}g_{\delta\mu}V^\rho - \frac{m^2}{6}V_\delta\delta_\mu^\rho - \tfrac{1}{2}g_{\delta\mu}V_\nu F^{\nu\rho} - \tfrac{1}{6}V^\rho F_{\mu\delta} - \tfrac{1}{3}F_{\mu\delta}V^\rho - \nabla_\delta F_\mu^\rho \\ & \quad + \tfrac{1}{3}\nabla_\delta\nabla_\mu V^\rho - \tfrac{1}{12}g_{\delta\mu}\nabla^2 V^\rho + \tfrac{1}{6}(\nabla_\delta V_\mu + \nabla_\mu V_\delta)V^\rho \\ & \quad + \tfrac{1}{12}(g_{\delta\mu}V^\beta\nabla_\beta + V_\mu\nabla_\delta V^\rho + V_\delta\nabla_\mu V^\rho) + \tfrac{1}{24}(g_{\delta\mu}V^\beta V_\beta + V_\mu V_\delta + V_\delta V_\mu)V^\rho , \end{aligned} \tag{A21}$$

$$\begin{aligned}(\nabla_\kappa\nabla^2 G_\mu^{(1)\rho}) = {} & \Bigg[-\frac{m^2}{4}\delta_\mu^\nu V_\kappa + \tfrac{1}{8}F_{\mu\kappa}V^\nu + \tfrac{1}{12}V^\nu F_{\mu\kappa} - \tfrac{1}{8}g_{\mu\kappa}\nabla^2 V^\nu - \tfrac{1}{4}\nabla_\kappa\nabla_\mu V^\nu \\ & - \tfrac{1}{16}(\nabla_\mu\nabla_\kappa + \nabla_\kappa V_\mu)V^\nu - \tfrac{1}{24}(g_{\mu\kappa}V^\beta\nabla_\beta V^\nu + V_\kappa\nabla_\mu V^\nu + V_\mu\nabla_\kappa V^\nu) \\ & - \tfrac{1}{96}(g_{\mu\kappa}V^\beta V_\beta + V_\kappa V_\mu + V_\mu V_\kappa)V^\nu \Bigg] (\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) \\ & + [\tfrac{1}{4}F_{\beta\kappa}\delta_\mu^\nu - \tfrac{1}{8}(g_{\mu\kappa}\nabla_\beta V^\nu + g_{\beta\kappa}\nabla_\mu V^\nu + g_{\beta\mu}\nabla_\kappa V^\nu) - \tfrac{1}{48}(g_{\mu\kappa}V^\beta V^\nu + \delta_\mu^\beta V_\kappa V^\nu + \delta_\kappa^\beta V^\nu)](\nabla^\beta\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) \\ & - \tfrac{1}{24}g_{\mu\kappa}V^\nu(\nabla^2\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) - \tfrac{1}{12}V^\nu(\nabla_\kappa\nabla_\mu\hat{P}_{\nu\alpha}G^{(0)\alpha\rho}) - \tfrac{1}{4}(\nabla_\kappa\nabla^2\hat{P}_{\mu\nu}G^{(0)\nu\rho}) , \end{aligned} \tag{A22}$$

$$\begin{aligned}(\nabla_\epsilon\nabla_\delta\nabla^2 G_\mu^{(0)\rho}) = {} & \Bigg[-\tfrac{1}{2}[\nabla_\epsilon(F_{\beta\delta}\nabla^\beta G_\mu^{(0)\rho})] + \frac{m^2}{4}[\nabla_\epsilon(V_\delta G_\mu^{(0)\rho})] + \tfrac{1}{8}g_{\epsilon\mu}[\nabla_\delta\nabla^2(V_\nu G^{(0)\nu\rho})] + (\delta\leftrightarrow\epsilon) \Bigg] \\ & - \tfrac{1}{4}F_{\delta\epsilon}(\nabla^2 G_\mu^{(0)\rho}) + \tfrac{1}{4}[\nabla_\epsilon\nabla_\delta\nabla_\mu(V_\nu G^{(0)\nu\rho})] , \end{aligned} \tag{A23}$$

$$\begin{aligned}[\nabla_\kappa(\nabla^2)^2 G_\mu^{(0)\rho}] = {} & -\tfrac{2}{5}[\nabla^2(F_{\beta\kappa}\nabla^\beta G_\mu^{(0)\rho})] - \tfrac{2}{5}F_{\beta\kappa}(\nabla^\beta\nabla^2 G_\mu^{(0)\rho}) \\ & + \frac{m^2}{5}[\nabla^2(V_\kappa G_\mu^{(0)\rho})] + \frac{m^2}{5}V_\kappa(\nabla^2 G_\mu^{(0)\rho}) - \frac{2m^2}{5}[\nabla_\kappa(V^\delta\nabla_\delta G_\mu^{(0)\rho})] \\ & + \tfrac{4}{5}[\nabla_\kappa(F^{\beta\delta}F_{\beta\delta}G_\mu^{(0)\rho})] + \tfrac{1}{10}g_{\kappa\mu}[(\nabla^2)^2(V_\nu G^{(0)\nu\rho})] + \tfrac{1}{5}[\nabla_\kappa\nabla_\mu\nabla^2(V_\nu G^{(0)\nu\rho})] + \tfrac{1}{5}[\nabla_\kappa\nabla^2\nabla_\mu(V_\nu G^{(0)\nu\rho})] . \end{aligned} \tag{A24}$$

[1]M. Veltman, Nucl. Phys. **B7**, 637 (1968); H. Reiff and M. Veltman, *ibid.* **B13**, 545 (1969).
[2]D. Boulware, Ann. Phys. (N.Y.) **56**, 140 (1970).
[3]E. S. Fradkin and I. V. Tyutin, Phys. Lett. **30B**, 562 (1969); A. Slavnov and L. Faddeev, Teor. Mat. Fiz. **3**, 18 (1970) [Theor. Math. Phys. **3**, 312 (1970)].
[4]K. Shizuya, Nucl. Phys. **B87**, 255 (1975).
[5]E. C. G. Steuckelberg, Helv. Phys. Acta **11**, 225 (1938).