S matrix of (1+1)-dimensional QED: Towards the continuum limit on the lattice

Jean François Brière and Helmut Kröger
Département de Physique, Université Laval, Québec, Provence de Québec, Canada G1K 7P4
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We study (1+1)-dimensional QED, which confines single fermions. After introducing Coulomb screening of the interaction, free single fermions can exist asymptotically and we compute the fermion-antifermion scattering. We use the light-cone momentum representation on a lattice and a nonperturbative time-dependent method to compute the S matrix. We present numerical results on the S matrix for fermion-antifermion scattering and investigate the approach towards the continuum limit.

I. INTRODUCTION

Recently, we have presented the first results on the computation of the S matrix of a gauge theory on the lattice.[1] We have studied (1+1)-dimensional QED (QED_{1+1}) as a model for the following reasons: Like QCD, it is a gauge theory and it confines single fermions (see Lowenstein and Swieca,[2] Casher, Kogut, and Susskind[3] as well as Coleman, Jackiw, and Susskind).[4] Moreover, it is a model in one space and one time dimension and hence amenable to computer studies. Scattering in QED_{1+1} can be investigated in two ways. (a) One can scatter composite particles (fermion-antifermion states) which do not feel a confining potential at large distances. That would represent a model for meson-meson scattering in QCD. The first results are given in Ref. 5. (b) One can regularize the interaction (i.e., modify the model) by giving a mass to the gauge particle, which permits single fermions as asymptotic states, and then computing fermion-antifermion scattering. In Ref. 1 we have studied this for small lattices. In this paper we want to discuss results for more extensive calculations searching for a continuum limit for the fermion-antifermion S matrix. In order to perform these calculations we have used two tools: (a) We have used a Hamiltonian formulation on a light-cone momentum lattice; (b) a time-dependent method is employed to compute the S matrix. The choice of our model is motivated also by the recent interest in studying the phase structure of (3+1)-dimensional QED (QED_{3+1}), which has been stirred by the observation of electron-positron peaks in heavy-ion collision at GSI. Phase diagrams have been computed by Dagotto and Wyld,[6] using coordinate-space lattice techniques.

Eller, Pauli, and Brodsky[7] have computed the mass spectrum of QED_{1+1} by introducing a discretization of light-cone momenta (light-cone momentum lattice). The advantage of using light-cone momentum variables in 1+1 dimensions is that they take on positive (nonnegative) values only. In terms of Feynman graphs, this leads to the suppression of certain graphs, compared to graphs using ordinary momentum variables. In a Hamiltonian formulation, using a Fock-space representation corresponding to discretized light-cone momenta, this leads to an upper limit in the occupation number, occurring in a scattering reaction. The reason is that the total light-cone momentum, being the sum of positive single-particle light-cone momenta, is conserved in a scattering reaction. This implies an upper limit on the occupation number and hence an upper limit on the dimension of the Hilbert space. This should be contrasted with discretization in terms of ordinary momenta which does *not* yield an upper limit on the occupation number nor on the dimension of the Hilbert space, but those would have to be introduced as additional parameters.

However, it should be pointed out that the nice properties of positive light-cone momenta do not pertain in three or more dimensions (i.e., they pertain for two dimensions, but the third dimension has to be treated differently.) In this case one has to resort to other means in order to describe physics in terms of a finite number of degrees of freedom. A proposal of how to compute the scattering matrix using stochastic techniques in order to reduce the effective number of degrees of freedom has been made in Ref. 8.

II. HAMILTONIAN ON LIGHT-CONE MOMENTUM LATTICE

The Lagrangian for QED in 1+1 dimensions (massive Schwinger model) is given by

$$\mathcal{L}=\frac{i}{2}(\bar{\psi}\gamma^{\mu}\partial_{\mu}-\partial_{\mu}\bar{\psi}\gamma^{\mu})\psi-m\bar{\psi}\psi-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$
$$-g\bar{\psi}\gamma^{\mu}\psi A_{\mu}\ . \tag{1}$$

Eller, Pauli, and Brodsky[7] have suggested using light-cone variables, and in particular light-cone momenta on a lattice. We use their notation. Starting from one position coordinate x^1 and one time coordinate x^0, the light-cone position x^- and light-cone time x^+ are introduced by $x^{\pm}=x^0\pm x^1$. From the energy-momentum tensor $T^{\mu\nu}$, one obtains the light-cone momentum P^+ and the light-cone Hamiltonian P^- via

$$P^{\pm}=\tfrac{1}{2}\int dx^-T^{+\pm}(x^-,x_0^+)\ . \tag{2}$$

The model is quantized canonically by imposing the anticommutation condition onto one independent field at equal light-cone time x^+. This requires one to fix the gauge. It is convenient to choose the light-cone gague $A^+=0$. Schwinger[9] showed for the massless Schwinger model that fermionic degrees of freedom can be eliminated and the model can be expressed in terms of bosons. Here, the photon degrees of freedom are eliminated and the model is expressed in terms of fermions only. One should note that this property does not pertain in more than two dimensions, because then the transversal photons have to be taken into account. The operators P^+, P^- and the conserved charge Q all commute.

By means of a Fourier transformation one goes over from the variables x^+, x^- to k^+, k^-, i.e., light-cone momentum and energy. Then a light-cone momentum discretization (lattice) is introduced via

$$k_n^+ = \frac{2\pi n}{L}, \quad n=1,\ldots,\Lambda . \tag{3}$$

This allows one to express the operators $P^\pm$ in a discretized Fock-space representation. Let $b_n^\dagger$, $d_n^\dagger$, $n=1,\ldots,\Lambda$ denote the fermion and antifermion creation operators, respectively, corresponding to the light-cone momentum k_n^+. They anticommute, except

$$\{b_m, b_n^\dagger\} = \{d_m, d_n^\dagger\} = \delta_{m,n} . \tag{4}$$

Then denoting K via $P^+=(2\pi/L)K$ and H, H_0, V via $P^-=(L/2\pi)H$, $H=m^2H_0+(g^2/\pi)V$, one obtains

$$\begin{aligned} K &= \sum_{n=1}^{\Lambda} n(b_n^\dagger b_n + d_n^\dagger d_n) , \\ H_0 &= \sum_{n=1}^{\Lambda} \frac{1}{n}(b_n^\dagger b_n + d_n^\dagger d_n) . \end{aligned} \tag{5}$$

The interaction takes the form $V=V_N+V_C+V_Q$. The term V_Q is charge dependent and vanishes when we consider the $Q=0$ sector. The term V_C, given by

$$\begin{aligned} V_C &= \sum_{n=1}^{\Lambda} I_n(b_n^\dagger b_n + d_n^\dagger d_n) , \\ I_n &= -\frac{1}{2n^2} + \sum_{m=1}^{n} \frac{1}{m^2} , \end{aligned} \tag{6}$$

commutes with H_0. The term V_N does not commute with H_0:

$$\begin{aligned} V_N = \sum_{k,l,m,n=0}^{\Lambda} \{ & (b_k^\dagger b_l^\dagger b_m b_n + d_k^\dagger d_l^\dagger d_m d_n)[k-n\,|\,l-m]/2 \\ & + b_k^\dagger b_l d_m^\dagger d_n([k+m\,|-l-n]-[k-l\,|\,m-n]) \\ & + (b_k^\dagger d_l^\dagger d_m^\dagger d_n + d_n^\dagger d_m d_l b_k + d_k^\dagger b_l^\dagger b_m^\dagger b_n + b_n^\dagger b_m b_l d_k)[k+m\,|\,l-n]\} , \end{aligned} \tag{7}$$

where $[n|m]=(1/n^2)\delta_{n+m,0}$ if m, n, are both nonzero and vanishes if $n=0$ or $m=0$. We have written the interaction V following the notation of Eller, Pauli, and Brodsky.[7] It has been split into three parts to discuss separately its physical meaning. The term V_Q is explicitly given in Ref. 7. The model is not renormalizable for $Q\neq 0$, but is renormalizable for the $Q=0$ sector. The term V_C, where the coefficients are called moments of inertia, has a similar structure as H_0 and basically leads to a mass renormalization. The proper interaction vertex, which does not commute with H_0, is given by the term V_N.

A particular property of the light-cone representation in $1+1$ dimensions is the fact that P^+ is a positive operator [the eigenvalues $k_n^+=(2\pi n)/L$, $n=1,\ldots,\Lambda$ are positive]. Hence a cutoff in k^+ introduces a cutoff in the occupation number of the Fock space. This leads to finite-dimensional, real and symmetric matrices to represent P^+, P^-. However, Lorentz invariance is broken. The lattice is characterized by the dimensionless parameter Λ and the parameter L of dimension length. In order to go over to the continuum limit, one has to take $L\to\infty$, $\Lambda\to\infty$, $\Lambda/L=(k_{\rm tot}^+/2\pi)=$const, where $k_{\rm tot}^+$ is the total light-cone momentum for a scattering process.

III. ZERO-MODE PROBLEM

Equations (5)–(7) give the light-cone Hamiltonian and momentum operator after regularization by use of a finite light-cone momentum lattice. The original continuous Hamiltonian has been modified by introducing two light-cone momentum cutoff parameters $\kappa_{\rm low}$ cutting off $k^+<\kappa_{\rm low}$ and $\kappa_{\rm up}$ cutting off $k^+>\kappa_{\rm up}$. Because the total light-cone momentum $k_{\rm tot}^+$ is conserved in a scattering reaction, and all single-particle light-cone momenta k^+ are positive, one can put $\kappa_{\rm up}=k_{\rm tot}^+$. However, there is no corresponding relation for the lower cutoff of light-cone momentum, which corresponds to an upper cutoff in the light-cone energy. In the lattice, described above, one has $\kappa_{\rm low}=2\pi/L$. These modes, in particular the zero mode, are omitted in the standard approach.[10,7] Harindranath and Vary[11] have studied in the $(1+1)$-dimensional Φ^4 $(\Phi^4)_{1+1}$ model the ground state in the broken-symmetry phase and interpreted it as a coherent state of zero modes. The zero mode has been explicitly included in Wittman's treatment,[12] which however leads to complicated constraints.

Here we claim that the time evolution $\exp(iP^-x^+)$ and hence the S matrix converges when $\kappa_{\rm low}\to 0$. That would imply that at most a finite renormalization of the model is required when $\kappa_{\rm low} \to 0$. Let us explain this for the time evolution corresponding to the free light-cone Hamiltonian of the scalar $(\Phi^4)_{1+1}$ model. The free light-cone Hamiltonian is given by

$$P^-_{\rm free} = \int dk^+ \frac{m^2}{k^+} A^\dagger(k^+) A(k^+) , \tag{8}$$

with the commutator

$$[A(k^+), A^\dagger(p^+)] = \delta(k^+ - p^+) . \tag{9}$$

Now we introduce an upper cutoff $\kappa_{\rm up}$ and a lower cutoff $\kappa_{\rm low} = \epsilon$ on the light-cone momenta and study the ϵ dependence of

$$P^-_{\rm free}(\epsilon) = \int_\epsilon^{\kappa_{\rm up}} dk^+ \frac{m^2}{k^+} A^\dagger(k^+) A(k^+) . \tag{10}$$

Let us compute the matrix element of the corresponding light-cone time evolution between one-particle states:

$$\langle F | \exp[iP^-_{\rm free}(\epsilon) x^+] | G \rangle = \int_\epsilon^{\kappa_{\rm up}} dk^+ \exp\left[i \frac{m^2}{k^+} x^+ \right] f^*(k^+) g(k^+) . \tag{11}$$

By a variable transformation $s = 1/k^+$, one has

$$\int_{1/\kappa_{\rm up}}^{1/\epsilon} ds \, \exp(im^2 s x^+) s^{-2} f^*(s^{-1}) g(s^{-1}) . \tag{12}$$

This expression has a finite limit when ϵ tends to zero, provided that $f(k^+)$, $g(k^+)$ are sufficiently regular functions, e.g., test functions. The same argument can be seen easily to hold, if F, G are N-particle states. The problem is much more complicated when interactions are involved. We have found evidence showing that the results still is valid in the $(\Phi^4)_{1+1}$ model, if only that part of the interaction is retained, which conserves the particle number. We believe that this is true more generally, but have been unable to prove or disprove it rigorously. The problem has been studied numerically in this work. Results are presented in Sec. VI.

IV. TIME-DEPENDENT S MATRIX

We have used a time-dependent method to compute the S matrix on the light-cone momentum-space lattice, which has been shown to work for the $(\Phi^4)_{1+1}$ model[13] and the massive Thirring model[14] using a momentum-space lattice. For pedagogical reasons let us first discuss it in terms of ordinary time and Hamiltonians. We want to compute an S-matrix element

$$S_{\rm fi} = \lim_{t \to \infty} \langle \Phi_{\rm fi} | \exp(iH_{\rm as} t) \times \exp(-i2Ht) \exp(iH_{\rm as} t) | \Phi_{\rm in} \rangle , \tag{13}$$

where H and $H_{\rm as}$ denote the full and asymptotic Hamiltonians, respectively, and $\Phi_{\rm fi}$, $\Phi_{\rm in}$ denote the asymptotic states. Let us assume we have constructed finite-dimensional lattice Hamiltonians $H(N)$, $H_{\rm as}(N)$. Then we define the lattice S-matrix element via

$$\bar{S}_{\rm fi}(N,t) = \langle \Phi_{\rm fi} | \exp[iH_{\rm as}(N)t] \times \exp[-i2H(N)t] \times \exp[iH_{\rm as}(N)t] | \Phi_{\rm in} \rangle . \tag{14}$$

The numerical technique to compute the lattice S matrix consists of diagonalizing the finite-dimensional Hamiltonians $H(N)$, $H_{\rm as}(N)$ and compute the exponentials using the corresponding eigenrepresentation. This will be discussed in more detail in Sec. VI.

The concept of substituting the continuous Hamiltonians by finite-dimensional ones necessitates also replacing the time limit $t \to \infty$ by a large but finite-time parameter T. The question raises how to choose this parameter. This has been studied for the scalar $(\Phi^4)_{1+1}$ model[13] and the massive Thirring model.[14] The following behavior of the lattice S matrix as a function of the time parameter is observed in general: For small values of time, the lattice S matrix has a linear behavior. Then there is some intermediate region of time where the lattice S matrix is stable. For large values of time, the lattice S matrix has an oscillatory behavior. If the lattice S matrix is to have some physical content, the only possibility is to consider it in the region of stability. It has been verified for different models, where the continuum S matrix is known either analytically or from perturbation theory, that the value of the lattice S matrix at the region of stability converges to the value of the continuum S matrix, when the lattice increases. From the practical point of view, one could hence determine T by searching the minimum of $|\partial S(N,t)/\partial t|$. However, there is some physical criterion available for the determination of the time parameter T, in the following called lattice scattering time. We define

$$\begin{aligned} \Delta_{\langle E \rangle}(N,t) &= |[\langle \Omega(N,t)\Phi^{\rm in} | H(N) | \Omega(N,t)\Phi^{\rm in} \rangle / \langle \Phi^{\rm in} | H_{\rm as}(N) | \Phi^{\rm in} \rangle] - 1| , \\ \Omega(N,t) &= \exp[iH(N)t] \exp[-iH_{\rm as}(N)t] . \end{aligned} \tag{15}$$

The function $\Delta_{\langle E \rangle}(N,t)$ gives the relative deviation of the expectation value of the full Hamiltonian in the scattering state from the expectation value of the asymptotic Hamiltonian in the asymptotic state. Hence we can define the lattice scattering time T as the value of t, where the violation of energy conservation on the lattice becomes minimal. It has been observed for models, where the exact S matrix is known, that the scattering time obtained by the latter definition, the region of stability of the lattice S matrix, and the region of the smallest relative error between the lattice S matrix and the exact S matrix, do coincide numerically.

Now we translate this into the language of light-cone momentum space. Then we have to substitute $t \to x^+$, $H \to P^-$, and $H_{\rm as} \to P^-_{\rm as}$, and on the lattice $H(N) \to P^-(L)$, $H_{\rm as}(N) \to P^-_{\rm as}(L)$ [where the light-cone lattice is characterized by the parameter L; see Eq. (3)]. Then the light-cone lattice S matrix reads

$$S_{\rm fi}(L,x^+) = \langle \Phi_{\rm fi} | \exp[iP^-_{\rm as}(L)x^+] \exp[-i2P^-(L)x^+] \exp[iP^-_{\rm as}(L)x^+] | \Phi_{\rm in} \rangle \; . \tag{16}$$

Accordingly, we define $\Delta_{\langle E \rangle}(L,x^+)$, the energy violation on the light-cone lattice:

$$\begin{aligned} &\Delta_{\langle E \rangle}(L,x^+) = | [\langle \Omega(L,x^+)\Phi^{\rm in} | P^-(L) | \Omega(L,x^+)\Phi^{\rm in} \rangle / \langle \Phi^{\rm in} | P^-_{\rm as}(L) | \Phi^{\rm in} \rangle] - 1 | \; , \\ &\Omega(L,x^+) = \exp[iP^-(L)x^+] \exp[-iP^-_{\rm as}(L)x^+] \; . \end{aligned} \tag{17}$$

The asymptotic Hamiltonian will be taken as $P^-_{\rm as} = (L/2\pi)m^2 H_0$, describing noninteracting fermions. Those states are admissible as asymptotic states only after the long-range behavior of the interaction has been modified (see Sec. V). The expression for the light-cone momentum lattice S matrix, given by Eq. (16), has been computed numerically in this work. It has been shown by Chang[15] in perturbation theory to arbitrary order that the S matrix obtained by quantizing at equal light-cone times agrees with the S matrix obtained by quantizing at equal ordinary times.

V. RENORMALIZATION

In the calculation of the mass spectrum of this model, Eller, Pauli, and Brodsky[7] have adjusted the bare parameters g, m of the model such that the lowest eigenvalue of the invariant-mass operator is 1. Also the computation of the S matrix requires one to renormalize the bare parameters of the model. Stationary scattering theory evaluated in perturbation theory in terms of Feynman graphs generates singularities which appear as infinities of pole type, logarithmic type, etc. On the other hand, the singularities of the time-dependent lattice S matrix can only appear as oscillatory behavior, because the lattice S matrix is exactly unitary for any finite value of the light-cone time parameter. Renormalization means first to get rid of the singularities such that physical observables (masses, coupling constants, etc.) become finite and, second, to arrange that the physical observables take on physical values. So let us first describe the steps which are necessary to get rid of the singularities.

(a) Regularization of the interaction. This model allows only solutions in the charge 0 sector,[7] and does not admit single fermions and/or antifermions as asymptotic states.[4] The latter property is familar from the long-range Coulomb interaction in electrodynamics (in three dimensions) or from confinement of quarks in strong-interaction dynamics. In QED_{1+1}, the interaction V_N contains the term $b^\dagger_k b_l d^\dagger_m d_n [k-l|m-n]$, which behaves on the lattice as $1/(k-l)^2$, or expressed in terms of continuous light-cone momenta $1/(k_1^+ - k_2^+)^2$. For coinciding light-cone momenta, there is a singularity which looks like the singularity of the Coulomb potential. However, in 1+1 dimensions the measure of the space is different. Thus by Fourier transformation one obtains a linear confining potential. One expects and we have verified by numerical calculations that any expectation value $\langle V_N \rangle$ computed on the lattice does not have a continuum limit due to this singularity. This proliferates to the S matrix. Hence in order to have a stable continuum limit for the S matrix, one has to regularize this singularity. We regularize the interaction by introducing a nonzero screening mass $m_{\rm scr}$:

$$\frac{1}{(k_1^+ - k_2^+)^2} \to \frac{1}{(k_1^+ - k_2^+)^2 + m^2_{\rm scr}} \; . \tag{18}$$

This modifies the interaction.

(b) Renormalization of the energy scale. The regularization introduced in (a) is not sufficient to get rid of the singularities of the S matrix. There are other terms in the interaction, which still generate an oscillatory behavior. In the interaction V_C, Eq. (6), the coefficient I_n has the behavior

$$I_n \underset{n \to \infty}{\sim} \frac{\pi^2}{6} - \frac{1}{n} + O\left[\frac{1}{n^3}\right] \; . \tag{19}$$

When going to the continuum ($L \to \infty$), the leading term gives a contribution to P^- increasing like $(L/2\pi)(\pi^2/6)$, and hence produces in the exponentials a behavior of rapid oscillations, i.e., a singularity in the limit. We remove the above singularity by subtracting $\pi^2/6$ from I_n. This is analogous to dropping the constant $\frac{1}{2}$ from the harmonic-oscillator Hamiltonian, and means a renormalization of the energy scale.

(c) Mass renormalization. As mentioned above, there is a term in the interaction, V_C, which commutes with H_0. This also leads in the S matrix to an oscillatory, i.e., singular behavior. This behavior can be understood by an example from non-relativistic quantum mechanics. Assume we want to describe two-particle scattering, corresponding to a full Hamiltonian $H = H_0 + W$, where the asymptotic dynamics are described by a free Hamiltonian H_0. Suppose the interaction can be split $W = V + H_0'$, where V stands for a short-range two-body potential, e.g., the Yukawa potential for NN scattering, and H_0' is such that it commutes with H_0, e.g., $H_0' = \lambda H_0$. Then the operator

$$\Omega(t) = \exp(iHt)\exp(-iH_0 t) \tag{20}$$

does not have a limit s $-$ $\lim_{t \to \mp\infty} \Omega(t)$, but tends weakly to zero. This can be seen as follows. We write

$H_0''=H_0+H_0'$, $H=H_0''+V$, and

$$\Omega''(t)=\exp(iHt)\exp(-iH_0''t) \; . \tag{21}$$

Then $\Omega''(t)$ has a limit

$$\Omega''_{\pm}=s-\lim_{t\to\mp\infty}\Omega''(t) \; , \tag{22}$$

as a result of standard scattering theory. Then the right-hand side of Eq. (19) behaves as

$$\Omega(t) \underset{t\to\mp\infty}{\sim} \Omega''_{\pm}\exp(iH_0't) \; , \tag{23}$$

which shows that $\Omega(t)$ oscillates, i.e., tends weakly to zero. In our case the problem of oscillatory behavior can be cured by modifying the free Hamiltonian and the interaction Hamiltonian:

$$\begin{aligned} P^-_{\text{free}} &= \frac{L}{2\pi}m^2H_0 \rightarrow \frac{L}{2\pi}\left[m^2H_0+\frac{g^2}{\pi}V_C\right] \; , \\ P^-_{\text{int}} &= \frac{L}{2\pi}\frac{g^2}{\pi}(V_C+V_N) \rightarrow \frac{L}{2\pi}\frac{g^2}{\pi}V_N \; . \end{aligned} \tag{24}$$

However, the full Hamiltonian remains the same:

$$P^-=P^-_{\text{free}}+P^-_{\text{int}} \; . \tag{25}$$

This means a mass renormalization $m^2 \rightarrow m^2-g^2/\pi$.

VI. NUMERICAL METHODS AND DISCUSSION OF RESULTS

We have computed S-matrix elements, given by Eq. (16). As asymptotic states we have chosen $f\bar{f}$ (one-fermion–one-antifermion) states. Each fermion and/or antifermion has a wave-packet distribution, given by

$$\Phi(k^+)=1-\cos\left[2\pi\frac{k^+-k^+_{\text{low}}}{k^+_{\text{up}}-k^+_{\text{low}}}\right]$$
$$\text{if } k^+_{\text{low}} \le k^+ \le k^+_{\text{up}} \; , \quad 0 \text{ elsewhere} \; . \tag{26}$$

This is a bell-shaped wave packet, with a half-width $(k^+_{\text{up}}-k^+_{\text{low}})/2$ and a peak at $(k^+_{\text{up}}+k^+_{\text{low}})/2$. Thus an asymptotic fermion-antifermion state has contributions to total light-cone momentum, which range from $k^+_{\text{tot},low}=k^+_{f,\text{low}}+k^+_{\bar{f},\text{low}}$ to $k^+_{\text{tot,up}}=k^+_{f,\text{up}}+k^+_{\bar{f},\text{up}}$. Using the light-cone momentum discretization, given by Eq. (3) with the parameters L and Λ, and respecting fermion antisymmetrization, one can build the subspace of Fock space to fixed total light-cone momentum, say, k^+_{tot}. This is a finite-dimensional space, denoted by $\mathcal{F}_{k^+_{\text{tot}}}$. Because the light-cone momentum operator P^+ commutes with the light-cone Hamiltonian operator P^-, the subspace $\mathcal{F}_{k^+_{\text{tot}}}$ is invariant under P^-. We have used another space, denoted by $\mathcal{F}^{\text{red}}_{k^+_{\text{tot}}}$, which is the subspace restricted to the 1-fermion–1-antifermion sector. Numerically we have computed the light-cone time evolution $\exp[iP^-(L)x^+]$ and also the asymptotic light-cone time evolution $\exp[iP^-_{\text{as}}(L)x^+]$ via algebraic diagonalization:

$$P^-(L)|e_\alpha\rangle=\epsilon^-_\alpha|e_\alpha\rangle, \quad \alpha=1,\ldots,\Omega \; , \tag{27}$$

where Ω is the dimension of $\mathcal{F}_{k^+_{\text{tot}}}$. Then the light-cone time evolution is diagonalized by

$$\exp[iP^-(L)x^+]|e_\alpha\rangle=\exp[i\epsilon^-_\alpha x^+]|e_\alpha\rangle \; , \quad \alpha=1,\ldots,\Omega \; , \tag{28}$$

and hence the S matrix can be expressed as

$$\begin{aligned} \sum_{\alpha,\beta,\gamma=1}^{\Omega} & |e_\alpha(as)\rangle\exp[i\epsilon_\alpha(as)x^+]\langle e_\alpha(as)|e_\beta\rangle \\ & \times\exp(-i2\epsilon^-_\beta x^+)\langle e_\beta|e_\gamma(as)\rangle \\ & \times\exp[i\epsilon^-_\gamma(as)x^+]\langle e_\gamma(as)| \; . \end{aligned} \tag{29}$$

This yields an S-matrix element for a given total light-cone momentum. Finally, in order to obtain the S matrix for scattering of the fermion-antifermion wave packet, Eq. (26), this has to be repeated for all total light-cone momenta k^+_{tot}, lying in the interval between $k^+_{\text{tot,low}}$ and $k^+_{\text{tot,up}}$, and being compatible with the lattice discretization. The wave-packet S matrix is obtained by summing those S-matrix elements.

To get an idea of the dimension of the matrices, we consider the largest case corresponding to a total light-cone momentum of 3.5 MeV, corresponding to $q^f_{\text{up}}=0.5$ MeV and $q^{\bar{f}}_{\text{up}}=1.0$ MeV, in terms of ordinary momenta. Then the number Λ of light-cone momentum nodes on the lattice, parametrized by L, is given by Eq. (3). One has, e.g., $\Lambda=56$, 112, 168 for $L=100$, 200, 300 MeV^{-1}, respectively. That is in this case we have a lattice of 168 nodes maximally. In the case of the fermion-antifermion reduced space, this number is also the dimension of the reduced space. This corresponds to Fig. 1. For the full space, one has $n_{\text{dim}}=125$, 550, 2000 for $L=30$, 40, 50 MeV^{-1}, respectively. Here we have used a lattice of 25 nodes maximally. In the latter case the dimension increases nearly exponentially. For the case of $n_{\text{dim}}=2000$, this requires one to store matrix elements of 4×10^6 words. The calculations with large matrices were performed at the Höchstleistungsrechenzentrum at Jülich on a Cray XMP 48. The runs took CPU time in the order of several hours.

The results are presented in Figs. 1 and 2. Below we have expressed the light-cone momentum parameters of the wave packets k^+_{low}, k^+_{up} in terms of ordinary momenta q_{up}, q_{low}. Before discussing those results, let us recall the results from Ref. 1 on the relation between the region of stability of the S matrix and the position of the minimum of the violation of light-cone energy on the lattice. For all three cases investigated, i.e., (i) $f\bar{f}$ scattering in a $f\bar{f}$ reduced Fock space, (ii) $f\bar{f}$ scattering in the full Fock space, and (iii) $2f2\bar{f}$ scattering in a $2f2\bar{f}$ reduced Fock space, we have found a region of stability of the S matrix as a function of the light-cone time and the minimum of $\Delta_{\langle E\rangle}(L,x^+)$ located within this region. Thus the lattice scattering light-cone time can be defined unambiguously.

Now let us address the following questions: What happens if we increase the lattice? Is it possible to see the continuum limit? As pointed out above, increasing the

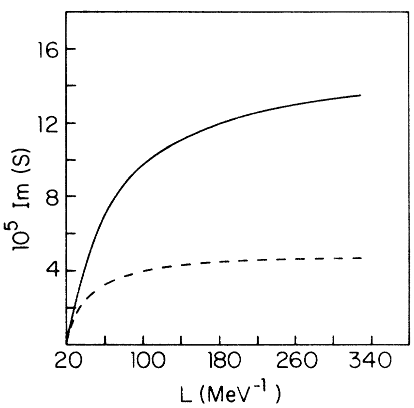

FIG. 1 Imaginary part of the S-matrix element for elastic $f\bar{f}$ scattering computed in $f\bar{f}$ reduced Fock space. The dashed curve corresponds to a fixed light-cone time $x^+=10$ MeV^{-1}. The solid curve corresponds to choosing x^+ such that the energy violation $\Delta_{\langle E\rangle}$ is minimized. It increases with an increased lattice (L) and tends to infinity in the continuum limit.

lattice means that the upper cutoff for the light-cone momentum is given by the total light-cone momentum and remains fixed. The lattice parameters Λ, L go to infinity, but $\Lambda/L=$const. But increasing L means that we increase the number of latice nodes and that the light-cone momentum resolution $\Lambda k^+=2\pi/L$ becomes finer, and also the lower cutoff $\kappa_{\rm low}=2\pi/L$ goes to zero. But that implies that the upper bound of the eigenvalue spectrum of the light-cone Hamiltonian increases. For example, the upper bound of P_0^- (corresponding to H_0) in the 1-fermion–1-antifermion sector of Fock space is $(L/2\pi)m^2$. Moreover, increasing L means that the fermion-antifermion number increases (exponentially).[7] Thus we have studied the approach towards the contunuum in two ways: (a) Λ, L increase, but the fermion-antifermion number is fixed. (b) Λ, L increase, and the fermion-antifermion number increases accordingly.

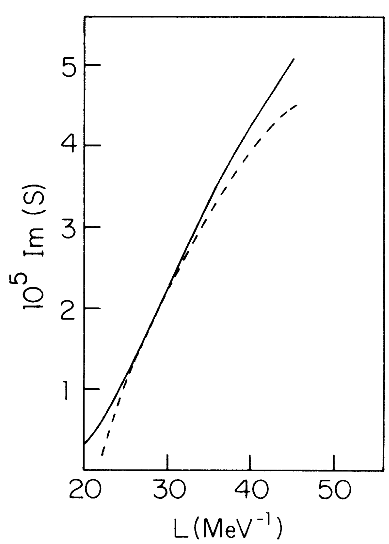

FIG. 2. Same as Fig. 1, but using the full Fock space.

If we carry out the numerical calculations, taking into account the renormalization steps described in Sec. V, we get an unphysical S matrix. In order to get the physical S matrix, one has to adjust the parameters g, m, $m_{\rm scr}$ of the bare model for each lattice, i.e., Λ, L. In non-Abelian gauge theories one can compute the relation between the bare coupling g and the lattice spacing a in the asymptotic regime,[16] which yields

$$a=\Lambda_{\rm latt}^{-1}(g^2\beta_0)^{-\beta_1/2\beta_0^2}\exp(-1/2\beta_0 g^2) \tag{30}$$

for a pure SU(N) gauge theory at the two-loop level. In our case one has to determine this dependence numerically. This has been done by Eller, Pauli, and Brodsky[7] in their calculation of the mass spectrum, by adjusting the bare parameters g/m, such that the lowest mass of the mass spectrum is kept fixed.

In this work we have not carried out a systematical numerical search for the adjusted parameters. But the numerical calculations show that we are in a regime where renormalization effects are small. This is indicated by the facts that (a) the imaginary part of the S matrix is small compared to unity (see Figs. 1 and 2), (b) contributions to $f\bar{f}$ scattering from higher virtual particle number states are small (see comparison between Figs. 1 and 2), and (c) we have computed the eigenvalues of the mass operator in the $f\bar{f}$ sector and the lowest one gives $m_{\rm phys}=1.021$ MeV, which differs very little from the bare mass of the constituents.

Now let us discuss our numerical results. In Figs. 1 and 2 we show the behavior of the imaginary part of the S matrix for elastic $f\bar{f}$ scattering as a function of the lattice parameter L, but with $\Lambda/L=$const. Figure 1 corresponds to the calculation in a $f\bar{f}$ reduced Fock space, while Fig. 2 corresponds to the calculation in the full Fock space. Figures 1 and 2 correspond to the following model parameters: $m=0.511$ MeV, $g/m=0.01$ and the screening mass $m_{\rm scr}$ = 1.0 MeV. The asymptotic states are given by $q_{\rm low}^f=0.0$ MeV, $q_{\rm up}^f=0.5$ MeV, $q_{\rm low}^{\bar{f}}=0.5$ MeV, $q_{\rm up}^{\bar{f}}=1.0$ MeV. The dashed curve corresponds to a fixed value of the light-cone time $x^+=10$ MeV^{-1}. The solid line corresponds to the S matrix with x^+ given by the lattice scattering light-cone time (minimum of $\Delta_{\langle E\rangle}$). Let us first look at the dashed curve of Fig. 1. This is the only curve, where convergence is clearly seen. This is numerical evidence supporting our claim that the time-evolution operator for a fixed time converges, when approaching to $k^+=0$ (see Sec. III). Looking at the solid curve in Fig. 1, one observes a tendency of convergence, if one is optimistic. The reason for the much slower convergence when compared with the dashed curve is the fact that lattice scattering light-cone time depends on the lattice. It increases when the lattice approaches the continuum. This is necessary because in the continuum limit the lattice scattering time is infinite (the energy violation is zero only at infinite time). By extrapolation of L beyond $L=340$ MeV^{-1} one could expect some stabilization in the region of $L=500-1000$ MeV^{-1}. In Fig. 2 we are confined to much smaller values of L, because for the same L the Fock space is much larger. The results show much similarity with the results of the $f\bar{f}$ reduced Fock space. This confirms the observation of Eller, Pauli, and Brodsky[7] that the mass spectrum computed in a reduced Fock space and in the full Fock space does not differ much in the small coupling regime. This is also in agreement with results on the S matrix of the scalar $(\Phi^4)_{1+1}$ model,[13] where the same behavior was found. Under the hypothesis that the curves in Fig. 2 show a similar behavior as those of Fig. 1 also for $L > 45$ MeV^{-1} one can estimate the distance from the regime of stability of the S matrix computed in the full Fock space, which may be reached with a computer of the next generation. Similar results have been obtained for a smaller screening mass $m_{\mathrm{scr}} = 0.1$ MeV.

VII. CONCLUSION

We have studied fermion-antifermion scattering in QED_{1+1} with a modified large-distance behavior. We have used a light-cone momentum lattice. The numerical results show indication for a stable S matrix, when approaching the continuum limit. We have approached this limit in two ways. First, we kept the light-cone time x^+ fixed and increased the lattice. This gives fast convergence. Second, we have chosen x^+ according to a minimum of the energy violation. This value of x^+ increases when the lattice increases and hence the convergence of the S matrix is much slower. Nevertheless some indication of convergence is seen. These results have been obtained for large lattices (up to 170 nodes) but a reduced Fock space (fermion-antifermion). The corresponding calculations using the full Fock space with the same light-cone momentum resolution Δk^+, i.e., the same parameter L, require a much larger dimension of Fock space. Because of storage limitations, we were forced to use smaller lattices of only up to 25 nodes. The results of the latter calculations nearly agree with the results from the reduced Fock space. The reason for this is that we are working in the regime of a small coupling constant. The same observation has been made in the calculations of the mass spectrum by Eller, Pauli, and Brodsky.[7] From this one would predict the behavior for the full Fock-space S matrix, when extrapolating in the parameter L, to show a similar convergence behavior as the reduced Fock-space S matrix. However, actually performing a numerical calculation using the full Fock space, for lattices with $L \geq 45$ MeV^{-1}, is prohibited by the exponential increase of the dimension of Fock space. As mentioned above, the situation becomes worse if one considers the model in $2+1$ or $3+1$ dimensions. This clearly sets limits for the technique as used in this study. However, progress has been made to reduce the effective number of degrees of freedom when describing a scattering reaction based on a Hamiltonian formulation. A stochastic approach has been suggested in Ref. 8, which is not limited by a particular number of dimensions nor species of particles.

The value of this study lies in the experience gained for future computations of composite particle scattering in confining non-Abelian gauge theories. First preliminary results on composite particle scattering in QED_{1+1} without modifying the fermion-confining large-distance behavior are presented in Ref. 5.

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