Stability of random surfaces

A. Krzywicki*
Physics Department, Brookhaven National Laboratory, Upton, New York 11973
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The interaction of two-dimensional quantum gravity with a set of conformal matter fields is considered. Following an earlier suggestion by Cates, we calculate the free energy of a "spiky" Liouville field configuration at a fixed value of the proper-time cutoff and including lowest-order quantum corrections. This free energy becomes arbitrarily large negative as soon as the central charge of the matter fields is larger than unity.

We shall consider a set of conformal matter fields, with total central charge c coupled in a minimal way to two-dimensional Euclidean quantum gravity. In the original Polyakov's model of random surfaces[1] the fields in question define the embedding of the surface in a d-dimensional (Euclidean) space-time and $c=d$. Since in our discussion only the value of c will matter, we do not need to specify the exact nature of the matter fields. The critical exponents found in Ref. 2 become complex for $1<c<25$, indicating a phase transition at $c=1$ (actually, the fact that a new regime sets in for $c>1$ has been pointed out much earlier[3]). The physics of this phase transition is still obscure. A physical picture has been recently proposed by Cates.[4] He argues that the invariant cutoff introduced to regularize the theory does not prevent the proliferation of gravitational singularities and interprets the transition as due to the condensation of the latter. These arguments are entirely lacking in mathematical rigor but are very suggestive. In view of the importance of the problem for the theory of random surfaces, it is interesting to inquire whether the intuitive picture proposed in Ref. 4 can be supported by a more precise argumentation. This is what we shall try to do in this work. Although we have been unable to find a truly rigorous demonstration, the progress we have made is, we believe, sufficient to warrant a write up.

Let $g_{ab}(\xi)$ be the intrinsic metric of a closed surface. We shall work in the conformal gauge. The Teichmueller deformations of the metric are not relevant for the problem discussed in this paper and will be ignored. Therefore, we set

$$g_{ab}(\xi)=\delta_{ab}e^{\phi(\xi)} . \tag{1}$$

This will be sufficient for our purposes and will simplify writing considerably. As is well known,[1,5] after integrating out the matter fields and the ghosts one gets the so-called Liouville action for the field $\phi(\xi)$:

$$S=[(26-c)/96\pi]\int d^2\xi(-\phi\partial^2\phi+\kappa e^{\phi}) . \tag{2}$$

The measure in the functional integration over $\phi(\xi)$ is to be read from the following conformally invariant definition of the H function-space metric:

$$\|\delta\phi\|^2=\int d^2\xi\, e^{\phi}(\delta\phi)^2 . \tag{3}$$

Consider a field $\phi_0(\xi)$ satisfying the equation

$$-\partial^2\phi_0=2\pi\mu\delta_\alpha(\xi-\xi_0) . \tag{4}$$

Here $\delta_\alpha(\xi-\xi_0)$ is the two-dimensional δ function smeared over a region of linear size α. More explicitly we write

$$\phi_0(\xi)=\frac{-\mu}{2}\ln\{[(\xi-\xi_0)^2+\alpha^2]/\rho^2\} \tag{5}$$

for $|\xi-\xi_0|$ small compared to ρ, and we assume that $\alpha\ll\rho$. Such a field configuration will be called, after Cates, a "spike" at $\xi=\xi_0$. The invariant area of the spike

$$A_{\text{spike}}=\int d^2\xi\, e^{\phi_0}\sim\alpha^{2-\mu}\quad(\mu>2) \tag{6}$$

diverges for $\alpha\to0$ when $\mu\geq2$. In the following we shall assume that $\mu>2$, leaving aside the limiting case $\mu=0$. The parameter α is not a cutoff imposed on the theory, but rather a parameter measuring the size of the spike, a field configuration necessarily involved in the functional integration (provided $\alpha>0$).

Inserting (5) into (2) and using (4) one gets a result diverging like $\ln(1/\alpha)$ when $\alpha\to0$. The question is whether this singular behavior of the action does disappear, or not, when the theory is regularized with an invariant cutoff ϵ, as it ought to be. In flat space the cutoff would be equivalent to a lattice with spacing $\sqrt{\epsilon}$ and on such a lattice a structure living in a space region with liner size $\alpha\ll\sqrt{\epsilon}$ cannot be resolved. In the present case, however, the presence of the spike means that the metric, if nontrivial, and the cutoff should be imposed on the eigenvalues of the *curved* Laplacian.

In order to formulate the correct argument we must give a precise meaning to the Liouville action in the presence of an invariant cutoff. Thus, we replace the first term in the Liouville action by the manifestly invariant expression[1]

$$E(\phi)=\int d^2\xi\, d^2\xi'\sqrt{g}\; R(\xi)\Delta^{-1}(\xi,\xi',\epsilon)\sqrt{g}\; R(\xi') , \tag{7}$$

where Δ^{-1} is the Green's function of the curved Laplacian Δ,

$$\Delta\Delta^{-1}(\xi,\xi')=\delta(\xi-\xi')/\sqrt{g} , \tag{8}$$

regularized with a proper-time cutoff ϵ and R is the scalar curvature. In the conformal gauge (1),

$$\sqrt{g}\, R=-\partial^2\phi \ , \tag{9}$$

$$\Delta=-e^{-\phi}\partial^2 \ , \tag{10}$$

and one easily checks that formally

$$E(\phi)\to -\int d^2\xi\, \phi\partial^2\phi \tag{11}$$

for $\epsilon\to 0$. We keep ϵ fixed and replace $\phi\to\phi_0$ in (7). Using (4), one finds, for $\alpha \ll \sqrt{\epsilon}$,

$$E(\phi_0)=(2\pi\mu)^2\Delta^{-1}(\xi_0',\xi_0',\epsilon) \ , \tag{12}$$

where $\xi_0'=\xi_0+O(\alpha)$. We observe, that the singular part of $\Delta^{-1}(\xi,\xi')$ is $-(1/4\pi)\ln(\xi-\xi')^2$ for $\xi\neq\xi'$, and therefore[1,5]

$$\Delta^{-1}(\xi,\xi',\epsilon)\to -(1/4\pi)\ln\Lambda(\xi) \quad \text{for} \quad \xi,\xi'\to\xi \ , \tag{13}$$

where $\Lambda(\xi)=\epsilon e^{-\phi(\xi)}$ is the cutoff in parameter space. Hence

$$\Delta^{-1}(\xi,\xi,\epsilon)=(1/4\pi)[\ln(1/\epsilon)+\phi(\xi)] \tag{14}$$

and (12) becomes

$$E(\phi_0)=\pi\mu^3\ln(1/\alpha)+(\text{finite when } \alpha\to 0) \ . \tag{15}$$

A more rigorous derivation of (14) (see, e.g., Ref. 6) uses the short-time behavior of the heat kernel (see, e.g., Ref. 7):

$$\langle\xi|e^{-\epsilon\Delta}|\xi\rangle=1/4\pi\epsilon+R(\xi)/24\pi+O(\sqrt{\epsilon}) \ . \tag{16}$$

The term proportional to ϕ in (14) comes from the first term on the right-hand side in (16). It is easy to check that when $\phi=\phi_0$ and $\xi=\xi_0+O(\alpha)$ one has

$$R(\xi)=\text{const}\times\rho^{-2}(\alpha/\rho)^{\mu-2} \ . \tag{17}$$

Therefore, the second term in (16) is negligible compared to the first one, provided $\mu>2$ and α/ρ is small enough, and the derivation of (14) goes through as usual.

The $E(\phi_0)$ given by (15) is indeed logarithmically divergent for $\alpha\to 0$. Although only modes with eigenvalues $\lambda_n\le\epsilon$ contribute effectively to the expansion

$$\Delta^{-1}(\xi,\xi',\epsilon)=\sum_n \chi_n(\xi)\chi_n(\xi')e^{-\epsilon\lambda_n}/\lambda_n \ , \tag{18}$$

the spike is resolved because the eigenfunctions $\chi_n(\xi)$ oscillate violently near $\xi=\xi_0$ and because the number of modes with $\lambda_n\le\epsilon$ increases indefinitely when $\alpha\to 0$. Notice that the number of active modes is estimated by the trace of the heat kernel,

$$\sum_n e^{-\epsilon\lambda_n}=\int d^2\xi\, e^{\phi_0}\langle\xi|e^{-\epsilon\Delta}|\xi\rangle$$
$$\sim(1/4\pi\epsilon)\alpha^{\mu-2} \ , \tag{19}$$

and is therefore proportional to the invariant area of the spike, as expected.

From (2), (7), (11), and (15) we finally get the classical action of a spike:

$$S_{\text{spike}}=[(26-c)/96][\mu^3\ln(1/\alpha)+(\kappa/\pi)A_{\text{spike}}]\cdot \tag{20}$$

up to terms finite when $\alpha\to 0$.

Let us now integrate over the collective coordinate ξ_0 (the position of the spike) and over small quantum fluctuations around ϕ_0. Write

$$e^{\phi}=e^{\phi_0}(1+\chi) \tag{21}$$

or

$$\phi=\phi_0+\ln(1+\chi) \ . \tag{22}$$

We assume that χ is orthogonal to $\phi_{0a}\equiv\partial\phi_0/\partial\xi_0^a$ and to the zero mode of the Laplacian $\chi=\text{const}$, always present on a compact surface. Let $\{f_n\}$ be an orthonormal set

$$(f_n,f_m)=\int d^2\xi\, e^{\phi_0}f_nf_m=\delta_{nm} \tag{23}$$

such that $f_0=\text{const}$ and $f_{1,2}$ are linear combinations of $\phi_{0a}, a=1,2$. We expand

$$\chi=\sum_{n>2} b_nf_n \tag{24}$$

and calculate a small variation of $e^{\phi/2}$:

$$\delta e^{\phi/2}=e^{\phi_0/2}\left[\left[\tfrac{1}{2}\phi_{0a}(1+\chi)^{1/2}+\tfrac{1}{2}(1+\chi)^{-1/2}\sum_{n>2} b_n\partial f_n/\partial\xi_0^a\right]\delta\xi_0^a+\tfrac{1}{2}(1+\chi)^{-1/2}\sum_{n>2}\delta b_nf_n\right] \ . \tag{25}$$

The functional integration measure $[D\phi]$ is found from the quadratic form obtained by inserting (25) into (3):

$$[D\phi]=J(\chi)d^2\xi_0\prod_{n>2} db_n \ . \tag{26}$$

The Jacobian $J(\chi)$ can easily be calculated for $\chi=0$ only, but this is sufficient if one works to the lowest order of perturbation theory. Thus, $J(0)$ is read from the quadratic form

$$\|\delta\phi\|_0^2=\int d^2\xi\, e^{\phi_0}\phi_{0a}\phi_{0b}\delta\xi_0^a\delta\xi_0^b+\sum_{n>2}(\delta b_n)^2 \ . \tag{27}$$

We have

$$J(0)=[\det(\phi_{0a},\phi_{0b})]^{1/2} \ . \tag{28}$$

Replace

$$\prod_{n>2} db_n\to\delta(b_1)\delta(b_2)\prod_{n>2} db_n \ , \tag{29}$$

where $b_{1,2}$ are now defined by $\chi=\sum_{n>0}b_nf_n$. Hence

$$\delta(b_1)\delta(b_2)=\delta[(f_1,\chi)]\delta[(f_2,\chi)] \ . \tag{30}$$

However $f_{1,2}$ are linear combinations of $\phi_{0a}/(\phi_{0a},\phi_{0a})^{1/2}$ $(a=1,2)$ and since the latter are linearly independent and normalized whatever α is, the transformation $f_{1,2}\to\phi_{0a}/(\phi_{0a},\phi_{0a})^{1/2}$ $(a=1,2)$ is nonsingular when $\alpha\to 0$. Thus up to a regular factor

$$[D\phi]=J(0)\prod_{a=1,2}\{(\phi_{0a},\phi_{0a})^{1/2}\delta[(\chi,\phi_{0a})]d\xi_0^a\}\prod_{n>0}db_n\ . \tag{31}$$

Finally, we perform an orthogonal transformation of coordinates in space orthogonal to the zero mode of the Laplacian $\Delta_0=-e^{-\phi_0}\partial^2$ so as to choose the coordinates along the eigenfunctions χ_n of Δ_0. Writing

$$\chi=\sum_{n>0}c_n\chi_n \tag{32}$$

we can replace $\prod_{n>0}db_n\to\prod_{n>0}dc_n$ in $[D\phi]$. This completes the construction of the measure:

$$[D\phi]=[\det(\phi_{0a},\phi_{0b})]^{1/2}\prod_{a=1,2}\{(\phi_{0a},\phi_{0a})^{1/2}\delta[(\chi,\phi_{0a})]d\xi_0^a\}\prod_{n>0}dc_n\ . \tag{33}$$

It is easy to check that

$$[\det(\phi_{0a},\phi_{0b})]^{1/2}[(\phi_{01},\phi_{01})(\phi_{02},\phi_{02})]^{1/2}\sim\alpha^{-2\mu}\ . \tag{34}$$

We insert (22) into the action and use (4) to get

$$\int d^2\xi\ln(1+\chi)\partial^2\phi_0=-2\pi\mu\ln[1+\chi(\xi_0')]\ , \tag{35}$$

so that this cross term yields in the integrand a factor $[1+\chi(\xi_0')]^{\text{const}}$ which can, to the lowest order, be replaced by unity. Furthermore

$$-\int d^2\xi\ln(1+\chi)\partial^2\ln(1+\chi)=(\chi,\Delta_0\chi)+O(\chi^3)\ . \tag{36}$$

Thus to the lowest order of perturbation theory,

$$Z\sim e^{-S_{\text{spike}}}\alpha^{-2\mu}\int d^2\xi_0\int\prod_{n>0}dc_n\delta[(\chi,\phi_{01})]\delta[(\chi,\phi_{02})]\exp\{-[(26-c)/96\pi](\chi,\Delta_0\chi)\}\ . \tag{37}$$

The last integration is easily done, once the δ functions in the integrand are replaced by their Fourier transforms, and yields the factor

$$(\text{Det}\Delta_0)^{-1/2}[\det(\phi_{0a},\Delta_0^{-1}\phi_{0b})]^{-1/2}\ . \tag{38}$$

Using the well-known[5,7] expressions for the DetΔ one finds that S_{spike} is renormalized: $(26-c)$ is replaced by $(25-c)$ and an $O(\epsilon^{-1})$ term is added to κ. Since $\phi_{0a}=-\partial\phi_0/\partial\xi^a$, one has

$$(\phi_{0a},\Delta_0^{-1}\phi_{0b})=\int d^2\xi\,d^2\xi' e^{\phi_0(\xi)}G_{ab}(\xi,\xi')e^{\phi_0(\xi')}\ , \tag{39}$$

where

$$G_{ab}(\xi,\xi')=\partial^2\Delta_0^{-1}(\xi,\xi',\epsilon)/\partial\xi^a\partial\xi'^b\ . \tag{40}$$

For $\mu>2$ and $\alpha\to0$, $\alpha^{\mu-2}e^{\phi_0}$ acts as a (smeared) δ function. Hence

$$(\phi_{0a},\Delta_0^{-1}\phi_{0b})\sim\alpha^{2(2-\mu)}G_{ab}(\xi_0',\xi_0')\ . \tag{41}$$

Since $G_{ab}(\xi_0',\xi_0')\sim\Lambda^{-1}(\xi_0')\sim\alpha^{-\mu}$ we find

$$[\det(\phi_{0a},\Delta_0^{-1}\phi_{ab})]^{-1/2}\sim\alpha^{\mu-2(2-\mu)}\ . \tag{42}$$

Collecting the results obtained above we finally obtain the following expression for the free energy of a spike:

$$F_{\text{spike}}=\mu\ln(1/\alpha)[(25-c)\mu^2/96-1-2(2-\mu)/\mu] + \kappa_{\text{ren}}\times\text{const}\,A_{\text{spike}}\ . \tag{43}$$

We shall assume that the average area of the surface is larger than the area of the spike:

$$\kappa_{\text{ren}}^{-1}>A_{\text{spike}}\ . \tag{44}$$

Setting $\mu=2+\eta$ we observe that F_{spike} is large positive for $c<1$ but becomes large negative for $c>1+O(\eta)$. Since η is arbitrary and α can be as small as one wishes, we recover the conclusion of Ref. 4: *the creation of spikes on a nearly critical surface is energetically favored as soon as the central charge of the matter fields becomes larger than unity.*

An extension of this calculation to higher orders seems very difficult. One should remark, that our calculation rests heavily on the choice of the conformal gauge. It would be interesting to check whether the same physical conclusion can be reached in a different gauge. It would be extremely interesting to see what happens in a superstring theory. All this goes beyond the scope of this paper.

The surface with spikes is no longer in a smooth phase, but is not really a branched polymer. The demonstration that the transition at $c=1$ produces branched polymers requires a discussion of the behavior of the *interacting gas* of spikes, a very tough problem. However, the identification of the $c>1$ phase with the phase of branched polymers is very plausible.

It is amusing to note that the quantization of two-dimensional gravity seems to imply a limit on the number of "generations" ($c=1$ corresponds to two species of Majorana fermions). It is so because the central charge counts the matter fields and also controls the short-distance behavior of the energy-momentum tensor. The constraint can perhaps be avoided if the matter fields obey a gauge symmetry, so that one has extra ghosts whose negative central charge compensates the central charge of matter fields.

We would like to end this paper with a few comments on the discrete version of the theory, which has been extensively studied numerically (for a review see, e.g., Ref. 8). In the numerical simulations one observed that the triangulated surfaces do degenerate into branched polymers, but only when c increases above $c=10$, or so. The transition from the Liouville to the branched polymer phase looks smooth, there is no sign of any accident occurring near $c=1$ and, if the data are taken literally, the existence of an intermediate scaling phase is suggested. A trivial interpretation of these experimental findings is that the simulated surfaces have been actually too small (in spite of the fact that the number of triangles has been as large as 3000 in certain simulations). A less trivial possibility is that for $c>1$ the discrete model belongs to a different universality class than the Liouville theory. The equivalence between the integration over metrics and the summation over abstract triangulations is demonstrated, for $c<1$, by the identity between critical exponents found in Refs. 2 and 9 for the continuum theory and those derived for the solvable examples of the discrete model. The equivalence could break down for $c>1$.

Another comment: in Ref. 10 we have studied Ising spins (equivalent to Majorana fermions) living on a dynamically triangulated random surface. When the number of species of fermions equals d, the integration over the embedding and the integration over fermion fields yield, for a given triangulation, a contribution to the partition function proportional to $(\mathrm{Det}D/\mathrm{Det}\Delta)^{d/2}$ (D denotes the discrete Dirac operator). We noticed in Ref. 10 that the presence of fermions does not stabilize the surface, the variation of Det$D$ failing to compensate that of DetΔ, but while completing the paper we did not achieve a full understanding of this issue. In the continuum and for massless fermions Det$D$ behaves like $[\mathrm{Det}\Delta]^{-1/2}$ when the metric of the surface is varied,[11] the behavior of the determinants being completely determined by the central charge of the corresponding fields.[5] Using Ising spins to represent fermions one finds[12] in the discrete theory $\langle \mathrm{Det}D\rangle \sim (\mathrm{Det}\Delta)^{-\gamma}$, with $\gamma>0$, in qualitative agreement with the continuum calculation (although now, for every given triangulation, the mass of the fermions—the fermion bare mass— is not vanishing). The presence of naively added fermion fields only adds to the instability of the surface.

Summarizing, our calculation confirms the intuitive arguments of Ref. 4. The stability of a surface without extrinsic curvature term in the action is controlled by the value of the total central charge of the matter fields interacting with the intrinsic metric of the surface. For $c>1$, one finds, keeping fixed the proper-time cutoff and in the conformal gauge, that the proliferation of gravitational singularities causes a breakdown of the theory.

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*Permanent address: LPTHE, Bâtiment 211, Université Paris-Sud, F-91405 Orsay, France.

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