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Brief Reports

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Superposition of the Kerr metric with the generalized Erez-Rosen solution

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An exact asymptotically flat solution of the Einstein vacuum equations is presented. Similar to the well-known Kerr solution it reduces to the Schwarzschild metric in the absence of rotation, and in addition it possesses an arbitrary multipole structure due to the seed Erez-Rosen metric.

In a series of papers[1-3] the Hoenselaers-Kinnersley-Xanthopoulos (HKX) transformations[4] have been used to construct a solution able to describe the exterior gravitational field of a strong rotating deformed mass. This stationary axisymmetric solution has as its static limit the generalized Erez-Rozen metric[5] representing the exterior gravitational field of a nonrotating mass with all mass-multipole moments. Thus, the main physical conjecture in the case of Quevedo's metric[2] is this: the Kerr solution[6] describes the gravitational field of a body whose static limit is the spherically symmetric Schwarzschild metric; all the subsequent generalizations of the Kerr solution describe the gravitational field of a body which is not spherically symmetric in a static limit.

In this paper we give a possible exact solution for another physical problem: assuming that the Kerr metric cannot represent the external field of any (arbitrary) realistic star because of a very special relationship between its multipole moments and angular momentum (see, e.g., Ref. 7), we perform the nonlinear superposition of the Kerr solution with the generalized Erez-Rosen metric in a way different from that of Quevedo; the resulting asymptotically flat solution contains an additional parameter which allows the transition to the Schwarzschild metric in the absence of rotation. Therefore, we used the generalized Erez-Rosen solution not for the description of the field of a deformed mass but for a better description of the gravitational field of a rotating star which is spherically symmetric in a static limit. The seed static metric is used for the only purpose of enriching the multipole structure of the stationary solution.

As is known,[8] a stationary axisymmetric vacuum problem can be reduced to the Ernst equation

$$(\epsilon+\epsilon^*)\Delta\epsilon=2(\nabla\epsilon)^2 \tag{1}$$

for the complex function ϵ, defined by the relations

$$\begin{gathered}\epsilon=f+i\Phi,\quad \epsilon^*=f-i\Phi\,,\\ \Phi_{,x}=k^{-1}(x^2-1)^{-1}f^2\omega_{,y},\quad \Phi_{,y}=k^{-1}(y^2-1)^{-1}f^2\omega_{,x}\,,\end{gathered} \tag{2}$$

where $f(x,y)$ and $\omega(x,y)$ are the unknown coefficients in the Papapetrou line element

$$\begin{aligned}ds^2=k^2f^{-1}&\left[e^{2\gamma}(x^2-y^2)\left[\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right]\right.\\ &\left.+(x^2-1)(1-y^2)d\phi^2\right]\\ &-f(dt-\omega d\phi)^2\end{aligned} \tag{3}$$

written in prolate spheroidal coordinates (x,y), k being a real constant; the operators Δ and ∇ have the form

$$\Delta\equiv k^{-2}(x^2-y^2)^{-1}\{\partial_x[(x^2-1)\partial_x]+\partial_y[(1-y^2)\partial_y]\}, \tag{4}$$

$$\nabla\equiv k^{-1}(x^2-y^2)^{-1/2}[\mathbf{x}_0(x^2-1)^{1/2}\partial_x+\mathbf{y}_0(1-y^2)^{1/2}\partial_y]\,.$$

($\mathbf{x}_0$ and $\mathbf{y}_0$ are unit vectors.) The metric function γ is constructed by quadratures from ϵ.

The Ernst potential ϵ describing the superposition of the Kerr solution with an arbitrary static vacuum Weyl field can be written in the form[9]

$$\epsilon=e^{2\psi}\frac{x(1+ab)+iy(b-a)-(1-ia)(1-ib)}{x(1+ab)+iy(b-a)+(1-ia)(1-ib)}\,, \tag{5}$$

where ψ is any solution of the equation $\Delta\psi=0$, and the functions a and b satisfy the following first-order differential equations:

$$\begin{aligned}(x-y)a_{,x}&=2a[(xy-1)\psi_{,x}+(1-y^2)\psi_{,y}]\,,\\ (x-y)a_{,y}&=2a[-(x^2-1)\psi_{,x}+(xy-1)\psi_{,y}]\,,\\ (x+y)b_{,x}&=-2b[(xy+1)\psi_{,x}+(1-y^2)\psi_{,y}]\,,\\ (x+y)b_{,y}&=-2b[-(x^2-1)\psi_{,x}+(xy+1)\psi_{,y}]\,.\end{aligned} \tag{6}$$

The corresponding expressions for f, γ, and ω can be found from (5) with the aid of the technique developed by Yamazaki,[10] Cosgrove,[11] and Dietz and Hoenselaers[12] for the algebraic calculation of the metric functions; the result is

$$f=e^{2\psi}AB^{-1},\quad e^{2\gamma}=K_1(x^2-1)^{-1}e^{2\gamma'}A;$$
$$\omega=2ke^{-2\psi}A^{-1}C+K_2\ ,$$
$$A:=(x^2-1)(1+ab)^2-(1-y^2)(b-a)^2\ , \tag{7}$$
$$B:=[x+1+(x-1)ab]^2+[(1+y)a+(1-y)b]^2\ ,$$
$$C:=(x^2-1)(1+ab)[b-a-y(a+b)]$$
$$+(1-y^2)(b-a)[1+ab+x(1-ab)]\ ,$$

K_1 and K_2 being real constants, and γ' being the γ function of the static metric with $\psi'=\frac{1}{2}\ln[(x-1)/(x+1)]+\psi$.

If we now choose as ψ the generalized Erez-Rosen solution, i.e.,

$$\psi=\alpha\sum_{n=1}^{\infty}(-1)^{n+1}q_nP_n(y)Q_n(x) \tag{8}$$

(we start the summation from $n=1$ to exclude the Schwarzschild solution), where α and q_n are real constants, P_n and Q_n are the Legendre polynomials of the first and second kind, respectively, then, making use of the result obtained in Ref. 2 for the functions $\beta_{n\mp}$, we arrive at the following expressions for a and b satisfying Eqs. (6):

$$a=-\alpha\exp\left[2\alpha\sum_{n=1}^{\infty}(-1)^nq_nD_-\right],$$
$$b=\alpha\exp\left[2\alpha\sum_{n=1}^{\infty}(-1)^nq_nD_+\right], \tag{9}$$
$$D_{\mp}=\tfrac{1}{2}(\pm1)^n\ln\frac{(x\mp y)^2}{x^2-1}-(\pm1)^nQ_1+P_nQ_{n-1}$$
$$-\sum_{k=1}^{n-1}(\pm1)^kP_{n-k}(Q_{n-k+1}-Q_{n-k-1})\ .$$

The function γ' calculated for a static metric defined by the potential

$$\psi'=\alpha\sum_{n=0}^{\infty}(-1)^{n+1}q_nP_n(y)Q(x),\quad q_0\equiv\alpha^{-1} \tag{10}$$

is given by the expression

$$\gamma'=\alpha^2\sum_{m,n=0}^{\infty}(-1)^{m+n}q_mq_n\Gamma^{(mn)}\ , \tag{11}$$

where $\Gamma^{(mn)}$ is determined by rather complicated relations which one may find explicitly in Ref. 5.

Note that the integration constants on the right-hand side of the formulas (9) for a and b (the factors $-\alpha$ and α, respectively) have been chosen in a given form to ensure the asymptotic flatness of the solution under consideration, thus avoiding an additional Ehlers transformation which was used by Quevedo to get rid of the Newman-Unti-Tamburino (NUT) parameter in his solution.

Using the properties of the Legendre polynomials one can show that $a\to-\alpha$, $b\to\alpha$ when $x\to\infty$. Then, with the choice of the constants K_1 and K_2 in the form

$$K_1=(1-\alpha^2)^{-2},\quad K_2=-4k\alpha(1-\alpha^2)^{-1} \tag{12}$$

the asymptotic behavior of f, γ, and ω, at $x\to\infty$, becomes

$$f=1+O(x^{-1}),\quad e^{2\gamma}=1+O(x^{-2})\ ,$$
$$\omega=\text{const}\times(1-y^2)x^{-1}+O(x^{-2})\ ; \tag{13}$$

that is, our solution given by (7)–(12) is indeed asymptotically flat.

The well-known particular cases arising from the general solution are the following.

(a) Putting in (7)–(12) $\alpha=0$ one comes to the Schwarzschild metric.

(b) Assuming $q_n=0$, $n\geq1$, one arrives at the Kerr solution

$$f_K=\frac{p^2x^2+q^2x^2-1}{(px+1)^2+q^2y^2},\quad e^{2\gamma_K}=\frac{p^2x^2+q^2y^2-1}{p^2(x^2-y^2)}\ ,$$
$$\omega_K=-\frac{2kq(1-y^2)(px+1)}{p(p^2x^2+q^2y^2-1)} \tag{14}$$

with

$$p=\frac{1-\alpha^2}{1+\alpha^2},\quad q=\frac{2\alpha}{1+\alpha^2},\quad p^2+q^2=1\ . \tag{15}$$

The coordinate transformation

$$kx=r-m,\quad k=mp,\quad a_0=mq,\quad k^2=m^2-a_0^2 \tag{16}$$

reduces the Kerr metric to its usual form, the parameter m being the total mass, and the angular momentum being defined by $J=ma_0$.

It should be mentioned that one cannot obtain the Kerr-NUT solution from Eqs. (7)–(12) because our metric is already asymptotically flat [the choice of the functions a and b in the form (9) made the NUT parameter vanish].

If $q_n\neq0$, $n\geq1$, then the formulas (7)–(12) describe the superposition of the Kerr solution with the generalized Erez-Rosen metric. Because of an arbitrary multipole structure of the latter metric our stationary solution possesses an arbitrary infinite set of the relativistic multipole moments M_i or J_i, describing, respectively, the mass distribution and the angular momentum distribution, which can be calculated using the invariant definition of the multipole moments proposed by Geroch[13] and Hansen.[14] The first three moments, with account of (15) and (16), are found to be

$$M_0=m,\quad M_1=-\tfrac{1}{3}\alpha(m^2-a_0^2)q_1\ ,$$
$$M_2=-ma_0^2+\tfrac{2}{15}\alpha q_2(m^2-a_0^2)^{3/2}\ , \tag{17}$$
$$J_0=0,\quad J_1=ma_0,\quad J_2=-\tfrac{1}{3}\alpha q_1a_0(m^2-a_0^2)\ ,$$

whence one can see that q_1 can be put equal to zero (i.e., the reference system is placed into the center of mass), so that $M_1=0$, $J_2=0$.

From (17) one can also see that the angular momentum is defined by the same expression as that of the Kerr solution, a_0 being the Kerr parameter, and m being the total mass of the source. In the expression for the mass quadrupole moment M_2 an additional term appears due to the Erez-Rosen solution, and this result up to the constant factor α (which is physically very important) in the additional term agrees with the expression for the quadrupole moment obtained by Quevedo[3] [it should be mentioned that in the formula (33) of Ref. 3 there occurred a misprint: the expression $(1-m^2/a^2)$ should be raised to the power $\frac{3}{2}$ instead of 2]. On the other hand, the quantities J_i, $i \geq 2$ in our solution turn out to be completely different from those following from Quevedo's solution for a stationary deformed mass: our J_i involve the parameters q_{i-1}, whereas the Quevedo's J_i involve already the parameters q_i which appear due to the Ehlers transformation.

Apparently, in a static limit with $q_n \neq 0$ only M_0 survives in case of our solution, whereas in the solution for a deformed mass one has the whole set of the mass multipole moments due to the Erez-Rosen metric.

Although a more detailed examination of the physical properties still remains a task for the future, the above analysis makes us hope that the reported exact solution possibly represents the exterior gravitational field of a rotating star and as such can exhibit interest for astrophysicists.

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