Higher-genus characters for the level-two SU(2) WZW model and GKO coset construction

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We obtain the characters for the level-two SU(2) Wess-Zumino-Witten model on the higher-genus Riemann surfaces from the Goddard-Kent-Olive coset construction using the characters for the Ising model. The two-point correlators are constructed by pinching the nonzero-homology cycle and it is easy to obtain the n-point correlators by repeating the pinching limits.

I. INTRODUCTION

The bootstrap program[1] or the sewing procedure[2] provides a useful tool to obtain the characters and correlators on the higher-genus Riemann surfaces. Our previous paper[1] gives the higher-genus characters and correlators for the critical Ising model.

The Wess-Zumino-Witten (WZW) models on the higher-genus Riemann surfaces are described in terms of "character-valued expectation values" and twisted Poincaré series by Bernard,[3] but the explicit construction of the arbitrarily higher-genus characters has not been successful yet. In this paper, we will obtain the characters for SU(2) WZW models at levels one and two on the higher-genus Riemann surfaces. The characters for the Ising model and the level-two SU(2) WZW model have an intimate relation and there is the Goddard-Kent-Olive (GKO) coset construction.[4] We use the characters for the Ising model to find the characters for the level-two SU(2) WZW model on the general Riemann surfaces through the GKO construction. This procedure with the help of the bootstrap program may provide a useful key to finding the characters for some higher-level SU(2) WZW model on the higher-genus Riemann surfaces.

The genus-two characters for the level-two SU(2) WZW model were previously constructed and the corresponding GKO construction is given in Ref. 5. However, these results have to be slightly modified because of the improved form of the characters for the Ising model.[1]

II. LEVEL-ONE SU(2) WZW MODEL

We briefly review the level-one SU(2) WZW model on a torus. The primary fields are 1 (identity operator) and ϕ_α (spin operator) with conformal dimensions 0 and $\frac{1}{4}$, respectively. The fusion rule is just $\phi\times\phi=1$. The $g=1$ characters are listed below:[6]

$$\chi_{0\cdot 1}(z,\tau)=\frac{1}{\eta(\tau)}\theta_{0:1}(z,\tau)\ , \tag{1}$$

$$\chi_{1\cdot 1}(z,\tau)=\frac{1}{\eta(\tau)}\theta_{1:1}(z,\tau)\ , \tag{2}$$

where the indices $n:k$ denote the spin-$n/2$ primary fields of the level-k SU(2) WZW model and

$$\theta_{n\cdot k}(z,\tau)=\theta\begin{bmatrix}\frac{n}{2k}\\0\end{bmatrix}(kz|2k\tau)$$

are the Kac-Moody theta functions at level k in the representation of SU(2).

On the higher-genus Riemann surfaces the fusion rules fix the internal propagators between holes to be the identity operators. Therefore the higher-genus characters can be obtained from the lower-genus ones by a bootstrap program. This procedure gives the genus-g characters which were conjectured previously[7] as

$$\chi_{\mathbf{n}:1}(\mathbf{z},\Omega)=\frac{1}{Z_1^{1/2}}\theta_{\mathbf{n}:1}(\mathbf{z},\Omega)\ , \tag{3}$$

where $\mathbf{n}=(n_1,\ldots,n_g)$, $\mathbf{z}=(z_1,\ldots,z_g)$, and Z_1 is the partition function of the chiral scalar field on the Riemann surface of genus g. The indices n_i represent the primary fields of spin $(n_i/2)(\leq\frac{1}{2})$ circulating around the ith hole.

The two-point correlators can be obtained by pinching the nonzero-homology cycle:[8,9]

$$\chi_{\mathbf{n},1:1}(0,\Omega)\underset{t\to 0}{\sim}2t^{1/4}\frac{1}{Z_1^{1/2}E(a,b)^{1/2}}\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\0\end{bmatrix}(a-b|2\Omega)\ , \tag{4}$$

where the factor 2 appears because two spin (up-down) operators are propagating along the long cylinder under the pinching limit. The two-point correlators of the spin operators are

$$\langle\phi_\alpha(z)\phi_\beta(\omega)\rangle_{\mathbf{n}}=\frac{\epsilon_{\alpha\beta}}{E(z,w)^{1/2}}\frac{1}{Z_1^{1/2}}\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\0\end{bmatrix}(z-w|2\Omega)\ . \tag{5}$$

The partition function is given by

$$Z_{k=1}=\sum_{\mathbf{n}}|\chi_{\mathbf{n}:1}(0,\Omega)|^2$$

$$=\frac{1}{|Z_1|}\sum_{\mathbf{n}}\left|\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\0\end{bmatrix}(0|2\Omega)\right|^2\ . \tag{6}$$

The normalized correlator is

$$\langle \phi_\alpha(z,\bar z)\phi_\beta(w,\bar w)\rangle = \epsilon_{\alpha\beta}\frac{1}{Z_{k=1}}\frac{1}{|E(z,w)|}\frac{1}{|Z_1|}\sum_{\mathbf{n}}\left|\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\ 0\end{bmatrix}(z-w|2\Omega)\right|^2 = \epsilon_{\alpha\beta}\frac{1}{|E(z,w)|}\frac{\sum_{\mathbf{n}}\left|\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\ 0\end{bmatrix}(z-w|2\Omega)\right|^2}{\sum_{\mathbf{n}}\left|\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\ 0\end{bmatrix}(0|2\Omega)\right|^2}\ . \tag{7}$$

These are in agreement with Ref. 10. The n-point correlators can be easily obtained by repeating the pinching limit.

III. LEVEL-TWO SU(2) WZW MODEL

The level-two SU(2) WZW model consists of three primary fields: 1 (identity or spin-0 operator), ϕ_α (spin-$\frac{1}{2}$ operator), and χ^a (spin-1 operator) with conformal dimensions 0, $\frac{3}{16}$, and $\frac{1}{2}$, respectively. The fusion rules are similar to those of the Ising model so that the numbers of the characters are the same.

The level-two Kac-Moody characters are expressed in terms of branching coefficients and Kac-Moody theta functions:[6]

$$\chi_{0:2}(z,\tau)=\frac{1}{\eta(\tau)}[\chi_0(\tau)\theta_{0:2}(z,\tau)+\chi_2(\tau)\theta_{2:2}(z,\tau)]\ , \tag{8}$$

$$\chi_{1:2}(z,\tau)=\frac{1}{\eta(\tau)}[\chi_1(\tau)\theta_{1:2}(z,\tau)+\chi_1(\tau)\theta_{-1:2}(z,\tau)]\ , \tag{9}$$

$$\chi_{2:2}(z,\tau)=\frac{1}{\eta(\tau)}[\chi_2(\tau)\theta_{0:2}(z,\tau)+\chi_0(\tau)\theta_{2:2}(z,\tau)]\ , \tag{10}$$

where the indices 0,1,2 represent the primary fields $1,\phi_\alpha,\chi^a$, respectively, and the branching coefficients $\chi_n(\tau)$ are the characters for the Ising model.

We will use the GKO coset construction, not the bootstrap program, in order to obtain the higher-genus level-two Kac-Moody characters. The $g=1$ GKO construction for the first level is

$$\chi_{n_1:1}(z,\tau)\chi_{n_2:1}(z,\tau)=\sum_{d=0,1}{}'\ \chi_n(\tau)\chi_{m:2}(z,\tau)\ , \tag{11}$$

where

$$(n,m)=\begin{cases}(1,1), & n_1+n_2=1\ ,\\ (2d,n_1+n_2+2d), & n_1+n_2\neq 1\end{cases} \tag{12}$$

and the prime means the summation over d only when $n_1+n_2\neq 1$.

We know the level-one Kac-Moody characters and the Ising model characters on the higher-genus Riemann surfaces. Then the extension of the GKO construction leads us to the higher-genus level-two Kac-Moody characters.

The left-hand side of the higher-genus version of Eq. (11) is

$$\begin{aligned}\chi_{\mathbf{n}_1:1}(\mathbf{z},\Omega)\chi_{\mathbf{n}_2:1}(\mathbf{z},\Omega)&=\frac{1}{Z_1}\theta\begin{bmatrix}\frac{\mathbf{n}_1}{2}\\ 0\end{bmatrix}(\mathbf{z}|2\Omega)\theta\begin{bmatrix}\frac{\mathbf{n}_2}{2}\\ 0\end{bmatrix}(\mathbf{z}|2\Omega)\\ &=\frac{1}{Z_1}\sum_{\mathbf{d}\in Z^g/2Z^g}\theta\begin{bmatrix}\frac{\mathbf{d}}{2}+\frac{\mathbf{n}_1-\mathbf{n}_2}{4}\\ 0\end{bmatrix}(0|4\Omega)\theta\begin{bmatrix}\frac{\mathbf{d}}{2}+\frac{\mathbf{n}_1+\mathbf{n}_2}{4}\\ 0\end{bmatrix}(2\mathbf{z}|4\Omega)\\ &=\frac{1}{Z_1}\sum_{\mathbf{d}\in Z^g/2Z^g}\theta\begin{bmatrix}\frac{\mathbf{d}}{2}+\frac{\mathbf{n}_1-\mathbf{n}_2}{4}\\ 0\end{bmatrix}(0|4\Omega)\theta_{\mathbf{n}_1+\mathbf{n}_2+2\mathbf{d}:2}(\mathbf{z},\Omega)\ ,\end{aligned} \tag{13}$$

where the second equality is due to the Riemann theta identity.[11] It will be useful to list some identities:

$$\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z}|\Omega)=\theta\begin{bmatrix}\mathbf{a}\\ 0\end{bmatrix}(\mathbf{z}+\mathbf{b}|\Omega)=\sum_{\mathbf{d}\in Z^g/2Z^g}\theta\begin{bmatrix}\frac{\mathbf{d}+\mathbf{a}}{2}\\ 0\end{bmatrix}(2\mathbf{z}+2\mathbf{b}|4\Omega)=e^{2\pi i\mathbf{a}\cdot\mathbf{b}}\sum_{\mathbf{d}\in Z^g/2Z^g}e^{2\pi i\mathbf{d}\cdot\mathbf{b}}\theta\begin{bmatrix}\frac{\mathbf{d}+\mathbf{a}}{2}\\ 0\end{bmatrix}(2\mathbf{z}|4\Omega)\ , \tag{14}$$

$$\sum_{\mathbf{b}\in(1/2)Z^g/Z^g} e^{-2\pi i\mathbf{a}\cdot\mathbf{b}}\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z}|\Omega)=2^g\theta\begin{bmatrix}\dfrac{\mathbf{a}}{2}\\ 0\end{bmatrix}(2\mathbf{z}|4\Omega)\ , \tag{15}$$

$$\delta_{\mathbf{a},\mathbf{b}}=\frac{1}{2^g}\sum_{\mathbf{b}'\in(1/2)Z^g/Z^g} e^{4\pi i(\mathbf{a}-\mathbf{b})\cdot\mathbf{b}'},\quad \mathbf{a},\mathbf{b}\in\tfrac{1}{2}Z^g/Z^g\ . \tag{16}$$

Let us focus attention on the z-independent part of Eq. (13). Applying these identities, it becomes a sum of Riemann theta functions with their argument Ω instead of 4Ω:

$$\frac{1}{Z_1}\theta\begin{bmatrix}\dfrac{\mathbf{d}}{2}+\dfrac{\mathbf{n}_1-\mathbf{n}_2}{4}\\ 0\end{bmatrix}(0|4\Omega)=\frac{1}{2^gZ_1}\sum_{\mathbf{b}\in(1/2)Z^g/Z^g}\exp\left[-2\pi i\left[\mathbf{d}+\frac{\mathbf{n}_1-\mathbf{n}_2}{2}\right]\cdot\mathbf{b}\right]\theta\begin{bmatrix}\dfrac{\mathbf{n}_1-\mathbf{n}_2}{2}\\ \mathbf{b}\end{bmatrix}(0|\Omega)\ , \tag{17}$$

where we used the property of theta functions such that

$$\theta\begin{bmatrix}\mathbf{a}+\mathbf{n}\\ \mathbf{b}\end{bmatrix}(\mathbf{z}|\Omega)=\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z}|\Omega)\quad\text{for } \mathbf{n}\in Z^g\ .$$

Now we analyze the index vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ in $Z^g/2Z^g$. Assuming the set $J=\{1\le i_1<i_2<\cdots<i_N\le g\}$, the combinations of $\mathbf{n}_1$ and $\mathbf{n}_2$ will be given as

$$(\mathbf{n}_1+\mathbf{n}_2)_i=\begin{cases}1, & i\in J\ ,\\ 0 \text{ or } 2, & i\notin J\ ,\end{cases} \tag{18}$$

$$(\mathbf{n}_1-\mathbf{n}_2)_i=\begin{cases}\epsilon_i, & i\in J\ ,\\ 0, & i\notin J\ ,\end{cases} \tag{19}$$

where $\epsilon_i=+1$ or -1. Then Eq. (17) becomes

$$\frac{1}{2^gZ_1}\sum_{\{b_i\}_{i\notin J}} e^{-2\pi i\sum_{i\notin J}d_ib_i}\sum_{\{b_j\}_{j\in J}}\exp\left[-2\pi i\sum_{j\in J}(d_j+\epsilon_j/2)b_j\right]\theta\begin{bmatrix}\frac{1}{2}\sum_{j\in J}\mathbf{e}_j\\ \mathbf{b}\end{bmatrix}(0|\Omega)$$

$$=\frac{1}{2^gZ_1}\sum_{\{b_i\}_{i\notin J}} e^{-2\pi i\sum_{i\notin J}d_ib_i}\left[\sum_{n=0}^{[N/2]}\sum_{j_1<\cdots<j_{2n}\in J}(-1)^{\sum_{k=1}^{2n}(d_{j_k}+\epsilon_{j_k}/2)}\theta\begin{bmatrix}\frac{1}{2}\sum_{j\in J}\mathbf{e}_j\\ \sum_{i\notin J}b_i\mathbf{e}_i+\frac{1}{2}\sum_{k=1}^{2n}\mathbf{e}_{j_k}\end{bmatrix}(0|\Omega)\right]\ , \tag{20}$$

where $\mathbf{e}_i$ is the unit g vector with the only nonzero ith component and the even theta functions survive in the second summation.

Next, we insert an identity $\sum_{\{b_i'\}_{i\notin J}}\prod_{i\notin J}\delta_{b_i,b_i'}$ into Eq. (20) and change the theta functions into the product of their square roots with characteristics $\mathbf{b}$ and $\mathbf{b}'$ as

$$\theta^{1/2}\begin{bmatrix}\frac{1}{2}\sum_{j\in J}\mathbf{e}_j\\ \sum_{i\notin J}b_i\mathbf{e}_i+\cdots\end{bmatrix}(0|\Omega)\theta^{1/2}\begin{bmatrix}\frac{1}{2}\sum_{j\in J}\mathbf{e}_j\\ \sum_{i\notin J}b_i'\mathbf{e}_i+\cdots\end{bmatrix}(0|\Omega)\ .$$

We rewrite Eq. (20) using the identity (16):

$$\frac{1}{Z_1^{1/2}}\sum_{\{b_i''\}_{i\notin J}}\sum_{n=0}^{[N/2]}\sum_{j_1<\cdots<j_{2n}\in J}(-1)^{\sum_{k=1}^{2n}(d_{j_k}+\epsilon_{j_k}/2)}$$

$$\times\left[\frac{(\sqrt{2})^N}{2^g Z_1^{1/4}}\sum_{\{b_i'\}_{i\notin J}} e^{-4\pi i\sum_{i\notin J}b_i''b_i'}\theta^{1/2}\begin{bmatrix}\frac{1}{2}\sum_{j\in J}\mathbf{e}_j\\ \sum_{i\notin J}b_i'\mathbf{e}_i+\frac{1}{2}\sum_{k=1}^{2n}\mathbf{e}_{j_k}\end{bmatrix}(0|\Omega)\right]$$

$$\times\left[\frac{(\sqrt{2})^N}{2^g Z_1^{1/4}}\sum_{\{b_i'\}_{i\notin J}} e^{\pi i\sum_{i\notin J}(4b_i''-2d_i)b_i}\theta^{1/2}\begin{bmatrix}\frac{1}{2}\sum_{j\in J}\mathbf{e}_j\\ \sum_{i\notin J}b_i\mathbf{e}_i+\frac{1}{2}\sum_{k=1}^{2n}\mathbf{e}_{j_k}\end{bmatrix}(0|\Omega)\right]$$

$$=\frac{1}{Z_1^{1/2}}\sum_{\{b_i''\}_{i\notin J}}\sum_{n=0}^{[N/2]}\sum_{j_1<\cdots<j_{2n}\in J}(-1)^{\sum_{k=1}^{2n}(d_{j_k}+\epsilon_{j_k}/2)}\chi^{(2\sum_{k=1}^{n}\sum_{j=j_{2k-1}}^{j_{2k}-1}\mathbf{e}_j)}_{\sum_{i\notin J}4b_i''\mathbf{e}_i+\sum_{j\in J}\mathbf{e}_j}(\Omega)\,\chi^{(2\sum_{k=1}^{n}\sum_{j=j_{2k-1}}^{j_{2k}-1}\mathbf{e}_j)}_{\sum_{i\notin J}(4b_i''-2d_i)\mathbf{e}_i+\sum_{j\in J}\mathbf{e}_j}(\Omega)\;, \tag{21}$$

where we used the following expression of the higher-genus characters for the Ising model:[1]

$$\chi_{\mathbf{n}}^{(\mathbf{m})}(\Omega)=\frac{1}{2^g Z_1^{1/4}}N(\mathbf{n})\sum_{\{b_i\}_{n_i\neq 1}}(-1)^{c(\mathbf{m},\mathbf{n})}\theta^{1/2}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(0|\Omega) \tag{22}$$

with

$$N(\mathbf{n})=(\sqrt{2})^{\sum_i\delta_{n_i,1}},\quad c(\mathbf{m},\mathbf{n})=\sum_{i(n_i\neq 1)}n_ib_i,\quad a_i=\tfrac{1}{2}\delta_{n_i,1},\quad b_i=\begin{cases}\frac{1}{2}(1-\delta_{m_{i-1},m_i}), & n_i=1\;,\\ b_i\in\{0,\frac{1}{2}\}, & n_i\neq 1\;.\end{cases}$$

Substitution of Eq. (21) into Eq. (13) leads to the formula

$$\chi_{\mathbf{n}_1:1}(\mathbf{z},\Omega)\chi_{\mathbf{n}_2:1}(\mathbf{z},\Omega)$$

$$=\frac{1}{Z_1^{1/2}}\sum_{\{b_i\}_{i\notin J}}\sum_{n=0}^{[N/2]}\sum_{j_1<\cdots<j_{2n}\in J}(-1)^{\sum_{k=1}^{2n}\epsilon_{j_k}/2}\chi^{(2\sum_{k=1}^{n}\sum_{j=j_{2k-1}}^{j_{2k}-1}\mathbf{e}_j)}_{\sum_{i\notin J}4b_i\mathbf{e}_i+\sum_{j\in J}\mathbf{e}_j}(\Omega)$$

$$\times\sum_{\mathbf{d}\in Z^g/2Z^g}(-1)^{\sum_{k=1}^{2n}d_{j_k}}\chi^{(2\sum_{k=1}^{n}\sum_{j=j_{2k-1}}^{j_{2k}-1}\mathbf{e}_j)}_{\sum_{i\notin J}(4b_i-2d_i)\mathbf{e}_i+\sum_{j\in J}\mathbf{e}_j}(\Omega)\theta_{\mathbf{n}_1+\mathbf{n}_2+2\mathbf{d}:2}(\mathbf{z},\Omega)\;. \tag{23}$$

Interpreting the last sum in Eq. (23) to be the characters for the level-two SU(2) WZW model, we can rewrite Eq. (23) as the formula of the GKO coset construction on the higher-genus Riemann surfaces:

$$\chi_{\mathbf{n}_1:1}(\mathbf{z},\Omega)\chi_{\mathbf{n}_2:1}(\mathbf{z},\Omega)=\sum_{\{d_i\}\{l_i\}d_i,l_i\in Z/2Z}(-1)^{c(\mathbf{m},\mathbf{n}_1,\mathbf{n}_2)}\chi_{\mathbf{n}}^{(\mathbf{m})}(\Omega)\chi^{(\mathbf{m})}_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}':2}(\mathbf{z},\Omega)\;, \tag{24}$$

where

$$(n_i,n_i',m_i)=\begin{cases}(1,0,2l_i), & (\mathbf{n}_1+\mathbf{n}_2)_i=1\;,\\ (2d_i,2d_i,m_{i-1}) & \text{otherwise}\;,\end{cases}$$

$$c(\mathbf{m},\mathbf{n}_1,\mathbf{n}_2)=\tfrac{1}{2}\sum_{i=1}^{g}(\mathbf{n}_1-\mathbf{n}_2)_i\delta_{(\mathbf{n}_1+\mathbf{n}_2)_i,1}(1-\delta_{m_{i-1},m_i})\;.$$

The level-two Kac-Moody characters are given by

$$\chi^{(\mathbf{m})}_{\mathbf{n}:2}(\mathbf{z},\Omega)=\frac{1}{Z_1^{1/2}}\sum_{\mathbf{d}\in Z^g/2Z^g}(-1)^{c(\mathbf{m},\mathbf{n},\mathbf{d})}\chi_{\mathbf{n}'}^{(\mathbf{m})}(\Omega)\theta_{\mathbf{n}+2\mathbf{d}:2}(\mathbf{z},\Omega)\;, \tag{25}$$

where

$$n_i'=\begin{cases}1, & n_i=1\;,\\ 2d_i & \text{otherwise}\;,\end{cases} \tag{26}$$

$$c(\mathbf{m},\mathbf{n},\mathbf{d})=\sum_{i=1}^{g} d_i\delta_{n_i,1}(1-\delta_{m_{i-1},m_i}) \ . \tag{27}$$

Here we set $m_0=m_g=0$.

We find that the GKO construction (24) has an extra sign factor unexpected in the $g=1$ case. It classifies some representations in the same $\mathbf{n}_1+\mathbf{n}_2$ label. An example will illustrate:

$$\chi_{10:1}(\mathbf{z},\Omega)\chi_{01:1}(\mathbf{z},\Omega)=\chi^{(0)}_{11}(\Omega)\chi^{(0)}_{11:2}(\mathbf{z},\Omega)+\chi^{(2)}_{11}(\Omega)\chi^{(2)}_{11:2}(\mathbf{z},\Omega) \ ,$$

$$\chi_{11:1}(\mathbf{z},\Omega)\chi_{00:1}(\mathbf{z},\Omega)=\chi^{(0)}_{11}(\Omega)\chi^{(0)}_{11:2}(\mathbf{z},\Omega)-\chi^{(2)}_{11}(\Omega)\chi^{(2)}_{11:2}(\mathbf{z},\Omega) \ ,$$

where $\mathbf{n}_1+\mathbf{n}_2=(1,1)$ in both cases but their GKO formulas are different.

The combination of the GKO construction and the characters for the Ising model through the Riemann theta identities gives us the explicit formulas of the higher-genus level-two Kac-Moody characters, which agree with the known results in the $g=1$ case.

One can obtain two-point correlators from the pinching limit of a nonzero-homology cycle. When pinching the $(g+1)$th a cycle with $n_{g+1}=2$(χ^a circulating), one gets the two-point correlator $\langle\chi^a(z)\chi^a(\omega)\rangle$ on the genus g Riemann surface. In this pinching limit, the level-two Kac-Moody characters become

$$\begin{aligned}
\chi^{(\mathbf{m},0)}_{\mathbf{n},2:2}(0,\Omega)
&=\frac{1}{Z_1^{1/2}}\sum_{\mathbf{d}\in Z^g/2Z^g}(-1)^{c(\mathbf{m},\mathbf{n},\mathbf{d})}[\chi^{(\mathbf{m},0)}_{\mathbf{n}',0}(\Omega)\theta_{\mathbf{n}+2\mathbf{d},2;2}(0,\Omega)+\chi^{(\mathbf{m},0)}_{\mathbf{n}',2}(\Omega)\theta_{\mathbf{n}+2\mathbf{d},0:2}(0,\Omega)]\\
&\to\frac{t^{1/2}}{E(a,b)}\frac{1}{2^g(Z_1^{3/4})_g}N(\mathbf{n}')\sum_{\mathbf{b}}{}'\sum_{\mathbf{d}}(-1)^{c(\mathbf{m},\mathbf{n},\mathbf{d})+c(\mathbf{m},\mathbf{n})}\\
&\quad\times\left[\theta^{1/2}[\boldsymbol{\alpha}](0|\Omega)\left\{\theta\begin{bmatrix}\frac{\mathbf{d}}{2}+\frac{\mathbf{n}}{4}\\ 0\end{bmatrix}(2(a-b)|4\Omega)+\theta\begin{bmatrix}\frac{\mathbf{d}}{2}+\frac{\mathbf{n}}{4}\\ 0\end{bmatrix}(-2(a-b)|4\Omega)\right\}\right.\\
&\quad\left.+\frac{\theta[\alpha](a-b|\Omega)}{\theta^{1/2}[\boldsymbol{\alpha}](0|\Omega)}\theta\begin{bmatrix}\frac{\mathbf{d}}{2}+\frac{\mathbf{n}}{4}\\ 0\end{bmatrix}(0|4\Omega)\right] \ ,
\end{aligned}\tag{28}$$

where $\boldsymbol{\alpha}$ is the characteristic corresponding to $\mathbf{n}'$ in the Ising model characters and $\sum'$ means the summation over $\mathbf{b}$-characteristic components b_i whenever $n_i'\neq 1$. Since $(-1)^{c(\mathbf{m},\mathbf{n},\mathbf{d})+c(\mathbf{m},\mathbf{n})}=e^{2\pi i\mathbf{b}\cdot\mathbf{d}}$ where $\mathbf{b}$ is the $\mathbf{b}$ characteristic corresponding to $\mathbf{n}'$, the Riemann theta identity (14) is applied to give

$$\begin{aligned}
\chi^{(\mathbf{m},0)}_{\mathbf{n},2:2}(0,\Omega)&\to\frac{t^{1/2}}{E(a,b)}\frac{1}{2^g(Z_1^{3/4})_g}N(\mathbf{n})\sum_{\mathbf{b}}{}'e^{-i\pi\mathbf{n}\cdot\mathbf{b}}\left[\theta^{1/2}[\boldsymbol{\alpha}](0|\Omega)\left\{\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\ \mathbf{b}\end{bmatrix}(a-b|\Omega)+\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\ \mathbf{b}\end{bmatrix}(-(a-b)|\Omega)\right\}\right.\\
&\quad\left.+\frac{\theta[\boldsymbol{\alpha}](a-b|\Omega)}{\theta^{1/2}[\alpha](0|\Omega)}\theta\begin{bmatrix}\frac{\mathbf{n}}{2}\\ \mathbf{b}\end{bmatrix}(0|\Omega)\right]\\
&=3\frac{t^{1/2}}{E(a,b)}\frac{1}{2^g(Z_1^{3/4})_g}N(\mathbf{n})(-1)^{\tilde{N}(\mathbf{m},\mathbf{n})}\sum_{\mathbf{b}}{}'(-1)^{c(\mathbf{m},\mathbf{n})}\theta[\boldsymbol{\alpha}](a-b|\Omega)\theta^{1/2}[\boldsymbol{\alpha}](0|\Omega) \ ,
\end{aligned}\tag{29}$$

where the last equality is due to the fact that the characteristic $\boldsymbol{\alpha}$ is the same as

$$\begin{bmatrix}\mathbf{n}/2\\ \mathbf{b}\end{bmatrix}$$

and $\tilde{N}(\mathbf{m},\mathbf{n})=\frac{1}{2}\sum_{i=1}^{g}\delta_{n_i,1}(1-\delta_{m_{i-1},m_i})$ is a non-negative integer. The overall factor 3 is due to the three modes of χ^a propagating internally along the long cylinder under the pinching limit. Thus we have the (unnormalized) correlators of χ^a:

$$\langle\chi^a(z)\chi^b(w)\rangle^{(\mathbf{m})}_{\mathbf{n}}=\delta_{ab}\frac{1}{2^gZ_1^{3/4}}\frac{1}{E(z,w)}N(\mathbf{n})\sum_{\mathbf{b}}{}'(-1)^{c(\mathbf{m},\mathbf{n})}\theta[\boldsymbol{\alpha}](z-w|\Omega)\theta^{1/2}[\boldsymbol{\alpha}](0|\Omega) \ . \tag{30}$$

Likewise one can achieve the pinching limit of the $(g+1)$th a cycle with $n_{g+1}=1$ (ϕ_α circulating):

$$\chi^{(\mathbf{m},m)}_{\mathbf{n},1:2}(0,\Omega)\to 2\frac{t^{3/16}}{E(a,b)^{3/8}}\frac{1}{2^g(Z_1^{3/4})_g}N(\mathbf{n})e^{3\pi i b/2}(-1)^{\tilde{N}(\mathbf{m},m;\mathbf{n},1)}\sum_{\mathbf{b}}{}'(-1)^{c(\mathbf{m},\mathbf{n})}\theta^{3/2}[\boldsymbol{\alpha}](\tfrac{1}{2}(a-b)|\Omega) \ , \tag{31}$$

where $\tilde{N}(\mathbf{m},m;\mathbf{n},1)=\frac{1}{2}\sum_{i=1}^{g+1}\delta_{n_i,1}(1-\delta_{m_{i-1},m_i})$ is a non-negative integer and $2b=\frac{1}{2}m$. The overall factor 2 means that two modes of ϕ_α are internally propagating along the long cylinder. Therefore two-point correlators of ϕ_α are given by

$$\langle \phi_\alpha(z)\phi_\beta(w)\rangle_{\mathbf{n}}^{(\mathbf{m},m)}=\epsilon_{\alpha\beta}\frac{1}{2^g Z_1^{3/4}}\frac{1}{E(z,w)^{3/8}}N(\mathbf{n})\sum_{\mathbf{b}}{}'(-1)^{c(\mathbf{m},\mathbf{n})}\theta^{3/2}[\boldsymbol{\alpha}](\tfrac{1}{2}(z-w)|\Omega)\ , \tag{32}$$

where the characteristic $\boldsymbol{\alpha}$ is even or odd when $m=0$ or 2.

The partition function of the level-two SU(2) WZW model is the sum of its characters:

$$Z_{k=2}=\sum_{\mathbf{m},\mathbf{n}}|\chi_{\mathbf{n}:2}^{(\mathbf{m})}(0,\Omega)|^2=\frac{1}{2^g|Z_1^{3/2}|}\sum_{\text{even}}|\theta[\boldsymbol{\alpha}](0|\Omega)|^3\ , \tag{33}$$

which agrees with the fact that the level-two SU(2) WZW model is equivalent to three Majorana fermions.

The normalized two-point correlators on the general Riemann surfaces are

$$\begin{aligned}\langle \phi_\alpha(z,\bar z)\phi_\beta(w,\bar w)\rangle &=\epsilon_{\alpha\beta}\frac{1}{Z_{k=2}}\frac{1}{2^g|Z_1^{3/2}|}\frac{1}{|E(z,w)|^{3/4}}\sum_{\boldsymbol{\alpha}}|\theta[\boldsymbol{\alpha}](\tfrac{1}{2}(z-w)|\Omega)|^3\\ &=\epsilon_{\alpha\beta}\frac{1}{|E(z,w)|^{3/4}}\frac{\sum_{\boldsymbol{\alpha}}|\theta[\boldsymbol{\alpha}](\frac{1}{2}(z-w)|\Omega)|^3}{\sum_{\text{even}}|\theta[\boldsymbol{\alpha}](0|\Omega)|^3}\ ,\end{aligned} \tag{34}$$

$$\begin{aligned}\langle \chi^a(z,\bar z)\chi^b(w,\bar w)\rangle &=\delta^{ab}\frac{1}{Z_{k=2}}\frac{1}{2^g|Z_1^{3/2}|}\frac{1}{|E(z,w)|^2}\sum_{\text{even}}|\theta^2[\boldsymbol{\alpha}](z-w|\Omega)\theta[\boldsymbol{\alpha}](0|\Omega)|\\ &=\delta^{ab}\frac{1}{|E(z,w)|^2}\frac{\sum_{\text{even}}|\theta^2[\boldsymbol{\alpha}](z-w|\Omega)\theta[\boldsymbol{\alpha}](0|\Omega)|}{\sum_{\text{even}}|\theta[\boldsymbol{\alpha}](0|\Omega)|^3}\ ,\end{aligned} \tag{35}$$

which are in agreement with the special case of genus one given in Ref. 10. It is easy to obtain n-point correlators by repeating the pinching limits.

IV. CONCLUSION

We have constructed the characters and correlators for the level-two SU(2) WZW model on the higher-genus Riemann surfaces. In addition to the bootstrap program used for the Ising model characters, we apply the GKO coset construction for the level-two Kac-Moody characters. Through the Riemann theta identities, we explicitly realize the GKO construction in the higher-genus representations. We hope that some higher-level SU(2) WZW models will be described by extending the above procedure, the bootstrap program, and the GKO construction.

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