Nuclear-structure dependence of $O(\alpha)$ corrections to Fermi decays and the value of the Kobayashi-Maskawa matrix element V_{ud}

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We calculate nuclear-structure corrections to the ft values of the eight accurately measured superallowed β^+ decays. The statistical fit for the average ft value is very good. The resulting new value for the matrix element of the Kobayashi-Maskawa (KM) matrix is $|V_{ud}|=0.9735(5)$. The error in $|V_{ud}|$ has thus been reduced by 50%. Combining this value for $|V_{ud}|$ with the presently accepted results from kaon-, hyperon-, and B-decay constraints, the unitarity of the KM matrix for three generations of quarks seems to be violated.

I. INTRODUCTION

Considerable interest in the subject of superallowed Fermi β decay has been raised again in recent years, primarily due to the work of Sirlin. It started when Marciano and Sirlin[1] investigated the possibility of a fourth fermion generation based upon what they considered then to be a precision value for the Kobayashi-Maskawa (KM) quark-mixing matrix element V_{ud}; this is derived from the average ft value of the eight accurately measured Fermi transitions. Combined with the kaon-, hyperon-, and B-decay constraints the sum of the squared moduli of the first three elements of the KM matrix $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2$ could be estimated. Assuming only three generations, unitarity requires this sum to be equal to one. The results of Ref. 1 did not exclude the possibility of a fourth generation.

The conditions under which such an analysis had to be performed changed considerably, when Sirlin and Zucchini[2] succeeded in deriving approximate analytical expressions for the radiative corrections of $O(Z\alpha^2)$ to Fermi decays, which have been confirmed by our detailed numerical reanalysis[3] of our earlier work.[4] The new results disagree with our former calculation[4] due to an error in the old computation routine. In addition, finite-nuclear-size contributions to the radiative corrections of $O(Z\alpha^2)$ have been investigated by us[3] and by Sirlin.[5] These new developments have led to information on the $O(Z\alpha^2)$ corrections to superallowed β decays which is now firmly established. We have collected the results for the radiative corrections of $O(\alpha)$, $O(Z\alpha^2)$, and $O(Z^2\alpha^3)$ in Table I. We give first those corrections of $O(\alpha)$, which are sometimes called the outer radiative corrections [see Eq. (1.1) below], averaged over the Fermi spectrum of the emitted positron. We have redone the averaging for the end-point energies as given in Koslowsky *et al.*[6] and using the Fermi function as tabulated by Behrens and Jänecke.[7] In our analysis we shall use the sum of "outer" and "inner" corrections of $O(\alpha)$ including the leading-log terms of higher order [see Eq. (1.4) below], which is a new development due to Marciano and Sirlin.[1] We shall use the results for the corrections of $O(Z\alpha^2)$ and $O(Z^2\alpha^3)$ as given in Ref. 5 without any modification, since the nuclear charge distributions used in Ref. 5 are more realistic than those used by us in Ref. 3.

Another theoretical progress on this subject has been reported by Ormand and Brown,[8] who have presented the results of a refined calculation of the isospin nonconservation correction δ_c, which are considerably different from those given by Towner and Hardy.[9,18]

Furthermore, Szybisz, Silbergleit, and Sidelnik[10] have pointed out disagreements between the results of the usual methods of evaluating the integrated Fermi function due to different methods of calculating the shape factor $C(E)$. We shall consider this problem in the Appendix, where we discuss the connection between the electromag-

TABLE I. Summary of radiative corrections. (1) $\alpha/2\pi\bar{g}(E,E_0)$, historically called outer radiative correction. (2) Δ_S, as defined in Eq. (1.4). (3) δ_2, radiative correction of $O(Z\alpha^2)$. (4) δ_3, radiative correction of $O(Z^2\alpha^3)$. (5) Sum of columns (2)–(4). All corrections in %.

	(1)	(2)	(3)	(4)	(5)
^{14}O	1.29	3.79(8)	0.22	0.01	4.02(8)
^{26}Al^m	1.11	3.60(8)	0.32	0.02	3.94(8)
^{34}Cl	1.01	3.50(8)	0.39	0.03	3.92(8)
^{38}K^m	0.97	3.46(8)	0.41	0.04	3.91(8)
^{42}Sc	0.94	3.42(8)	0.45	0.04	3.91(8)
^{46}V	0.91	3.39(8)	0.47	0.05	3.91(8)
^{50}Mn	0.88	3.36(8)	0.49	0.05	3.90(8)
^{54}Co	0.86	3.34(8)	0.50	0.06	3.90(8)

netic form factor, the isovector form factor, and the shape factor, and their influence on the integrated Fermi function.

In Table II we have listed the ft values obtained by using the half-lives and the end-point energies of Ref. 6. The Fermi function, averaged over the β spectrum, was obtained for a modified Gaussian charge distribution and the corresponding rms charge radius (as discussed in the Appendix) is consistent with the treatment of the remaining nuclear-structure corrections. Listed also are our results for the averaged shape factor, and the values for δ_c from Refs. 8, 9, 18, and 6.

The level of precision in the analysis of the ft values that is required now demands that finite-nuclear-size effects to the radiative corrections are calculated for realistic nuclear charge distributions. This has been demonstrated in Refs. 3 and 5 for the $O(Z\alpha^2)$ corrections. We shall therefore study in this paper an effect of $O(\alpha)$ due to the structure of the nucleus arising from the axial-vector current, which we have mentioned briefly in Ref. 11 and which has never been investigated before.

Radiative corrections of $O(\alpha)$ to superallowed Fermi transition rates are given by Sirlin.[12] They give rise to an overall factor

$$\left[1+\frac{\alpha}{2\pi}\bar{g}(E,E_0)\right] \times\left\{1+\frac{\alpha}{2\pi}\left[4\ln\left[\frac{m_Z}{m_p}\right]+\ln\left[\frac{m_p}{m_A}\right]+2C+\mathcal{A}_g\right]\right\}, \tag{1.1}$$

where E is the energy of the emitted positron and E_0 is its end-point energy. The averaged Sirlin function $\bar{g}(E,E_0)$ (Ref. 13) represents the so-called outer correction and depends upon the nucleus via E_0. The remaining (inner) correction term, the set of curly brackets in Eq. (1.1), consists of three parts: the vector-induced logarithm due to the high-energy behavior of the one-loop diagrams, which dominates, since the ratio of the Z-boson mass to the proton mass is large; a small QCD correction term $\mathcal{A}_g$, which has been calculated by Sirlin;[12] finally, an axial-vector-induced contribution $\ln(m_p/m_A)+2C$, where the logarithm is again the result of short-distance effects, with m_A a low-energy cutoff (presumably roughly equal to the nucleon mass), and where C represents the remaining low-energy contributions. It has been emphasized in Ref. 1 that the value of C could not be calculated reliably as yet, and that this is the main source of uncertainty in the radiative corrections. In the framework of the independent-particle model for the decaying nucleus, C can be easily calculated in Born approximation.[1,14] Starting from the diagrams of Figs. 1(a) and 1(b) and using nucleon electromagnetic and axial-vector form factors one gets

$$C_{\text{Born}}=0.798g_A(\mu_p+\mu_n)=0.885\ , \tag{1.2}$$

where $\mu_p+\mu_n=0.8797$ and $g_A=1.26$. The possible effect of nuclear structure has been guessed roughly in the approximation which treats the nucleus as a structureless scalar particle. This gives $C=0$, since the axial-vector current does not couple to such an object. This uncertainty in the value of C had to be accounted for by a large theoretical error in the analysis of Ref. 1.

In this paper we shall calculate explicitly the contribution to C of the nuclear structure due to the diagram of Fig. 1(c) and shall denote this contribution by C_{NS}. The total value of C then is

$$C=C_{\text{Born}}+C_{\text{NS}}\ , \tag{1.3}$$

where C_{Born} is given in Eq. (1.2). The rough guess in Ref. 1 would be equivalent to a value of C_{NS} within the limits $-0.885<C_{\text{NS}}<0$.

Marciano and Sirlin[1] have gone beyond the classical expression Eq. (1.1) for the $O(\alpha)$ radiative corrections, and have approximated the effect of higher orders by summing all leading-log corrections of $O(\alpha^n\ln^n m_z)$, using the renormalization-group method. They found the correction Δ_S defined by

TABLE II. Uncorrected ft values; shape factor and isospin corrections. (1) ft value in sec, calculated by Behrens (private communication) with a uniform nuclear charge distribution corresponding to r_0 as given in column (1) of Table IV including atomic screening and corrected for a modified Gaussian nuclear charge distribution as taken from Ref. 34. The end-point energies and the half-lives used in this calculation are those given in Ref. 6. (2) Averaged shape factor $\overline{C(E)}$ as calculated in the Appendix, corresponding to the correction of column (5) of Table V. (3) Correction due to isospin nonconservation according to OB (Ref. 8). (4) Correction due to isospin nonconservation according to THH (Refs. 6, 9, and 18). Corrections (3) and (4) in %.

	(1)	(2)	(3)	(4)
^{14}O	3037.9(16)	0.9995	−0.19(13)	−0.28(10)
^{26}Al^m	3038.0(12)	0.9984	−0.24(13)	−0.33(11)
^{34}Cl	3057.0(19)	0.9973	−0.48(14)	−0.64(12)
^{38}K^m	3057.0(27)	0.9966	−0.49(17)	−0.70(12)
^{42}Sc	3055.4(20)	0.9951	−0.39(13)	−0.39(12)
^{46}V	3060.5(22)	0.9942	−0.21(13)	−0.45(12)
^{50}Mn	3060.3(39)	0.9932	−0.28(13)	−0.50(13)
^{54}Co	3069.2(23)	0.9925	−0.35(13)	−0.59(12)

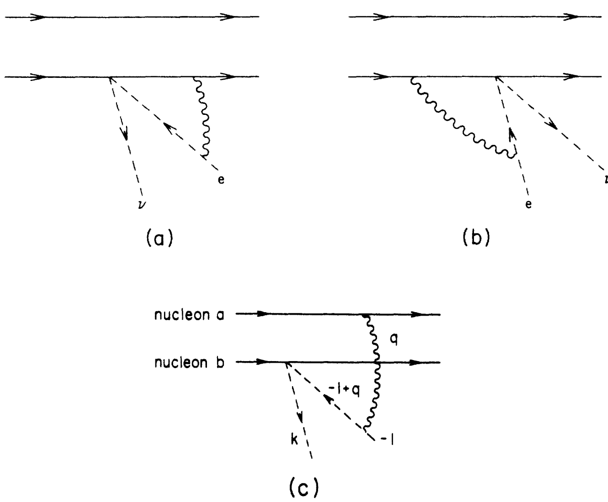

FIG. 1. Feynman diagrams of $O(\alpha)$; l, k, and q are the four-vectors of the outgoing positron, neutrino, and the virtual-photon, respectively.

$$1+\Delta_S\equiv\left\{1+\frac{\alpha}{2\pi}\left[\ln\left[\frac{m_p}{m_A}\right]+2C_{\text{Born}}\right]+\frac{\alpha(m_p)}{2\pi}\left[\bar{g}(E,E_0)+\mathcal{A}_g\right]\right\}S(m_p,m_Z)\ , \tag{1.4}$$

where $S(m_p,m_Z)$ is a QED short-distance enhancement factor. The numerical values are[1]

$$S(m_p,m_Z)=1.02256,\quad \alpha^{-1}(m_p)=133.93\ .$$

The numerical values of Δ_S are given in Table I. The error is due to the uncertainty of the value of the low-energy cutoff m_A, for which Marciano and Sirlin[1] suggested a range 400 MeV $\le m_A\le$ 1600 MeV. This uncertainty will be taken into account only in the averaged ft value.

It is clear from Eq. (1.4) that the former separation of radiative corrections into "outer" and "inner" corrections which are applied in a multiplicative fashion according to Eq. (1.1) is no longer adequate. The proper order of the various corrections should be respected. The fully corrected ft value is then given by

$$\mathcal{F}t=ft\bar{C}(E)\left[1+\Delta_S+\delta_2+\delta_3+\frac{\alpha}{\pi}C_{\text{NS}}\right](1-\delta_c)\ . \tag{1.5}$$

In Sec. II we discuss the matrix element for the process given in Fig. 1(c). The general structure of the nuclear part of the matrix element will be investigated. We use an operator identity to represent it as a sum of four nuclear matrix elements of two-body operators which can be evaluated easily. It is shown that the effect of the nuclear structure can be expressed in terms of a form factor.

For the explicit calculation one needs the wave function for the nuclear states. If it is represented in a harmonic-oscillator shell-model basis the form factor can be calculated analytically. We present a detailed account of our method in Sec. III, using the decay of ^{14}O as an example. The correction C_{NS} is given numerically for different wave functions. For the decays of ^{38}K^m and ^{42}Sc wave functions with almost pure configurations are available also, while for the decays of ^{26}Al^m, ^{34}Cl, ^{46}V, ^{50}Mn, and ^{54}Co we estimate the correction, starting from the somewhat crude approximation that the wave function is a pure configuration. The resulting corrections including our estimate of their errors are given numerically in Sec. IV. The final results are presented and discussed in Sec. V.

II. ANALYSIS AND REDUCTION OF THE TRANSITION MATRIX ELEMENT

We calculate that contribution to the matrix element corresponding to the diagram of Fig. 1(c), which is induced by the axial-vector part of the hadronic current. It is given by

$$M^{(A)}(1(c))=G_FV_{ud}\frac{\alpha}{2\pi^2}\int d^3q\frac{1}{q^2}\frac{1}{q^2-2l\cdot q}L^{\mu\lambda}N^{(A)}_{\mu\lambda}\ , \tag{2.1}$$

where G_F is the Fermi coupling constant. The right-hand side still contains q_0 and we shall return to this point shortly. The leptonic tensor

$$L_{\mu\lambda}=\bar{u}(\mathbf{k})\gamma_\lambda(1-\gamma_5)(-\not{l}+\not{q}+m)\gamma_\mu v(l) \tag{2.2}$$

can easily be shown to be ($\epsilon^{0123}=1$)

$$L_{\mu\lambda}=-g_{\mu\lambda}qL+q_\lambda L_\mu+q_\mu L_\lambda-2l_\mu L_\lambda-i\epsilon_{\mu\lambda\alpha\rho}q^\alpha L^\rho$$

with

$$L_\alpha:=\bar{u}(\mathbf{k})\gamma_\alpha(1-\gamma_5)v(l)\ .$$

The normalization in Eq. (2.1) is such that the unperturbed matrix element for the 0^+-0^+ decays within an isospin triplet is

$$M^{(0)}=G_FV_{ud}L_0\ . \tag{2.3}$$

The weak hadronic current is $J=V+A$ and only the axial-vector part contributes to the nuclear tensor $N^{(A)}$:

$$N^{(A)}_{\mu\lambda}=\frac{1}{\sqrt{2}}\left\langle f\left|\sum_{a\neq b}(j_\mu e^{-i\mathbf{qx}})_a(A_\lambda e^{i(\mathbf{q}-\mathbf{l}-\mathbf{k})\mathbf{x}})_b\right|i\right\rangle\ . \tag{2.4}$$

Here $|i\rangle$ and $|f\rangle$ are the initial and final states of the nucleus with the quantum numbers $J^p=0^+$, $T=1$, respectively, so that, for the positron emitters,

$$|i\rangle=|0^+;T=1,M_T\rangle,\quad |f\rangle=|0^+;T=1,M_T-1\rangle\ . \tag{2.5}$$

The quantities in the parentheses in Eq. (2.4) are the Fourier transforms of the electromagnetic-current operator of nucleon a and of the axial-vector-current operator of nucleon b, respectively. The finite nuclear size introduces a mass scale $\Lambda=\sqrt{6}/r_c$, where r_c is the rms nuclear charge radius. It is sufficient to calculate $M^{(A)}(1c)$ in leading order only and to neglect terms of $O(E_0/\Lambda)$, where E_0 is the positron end-point energy. In this approximation the lepton momenta in the second exponential of Eq. (2.4) can be neglected.

The matrix element $M^{(A)}(1c)$ depends also on the energy q_0 of the exchanged photon. In principle, q_0 is determined by the procedure which defines a three-dimensional wave function in the framework of a covariant formalism. Such a procedure has been described for the two-nucleon system by Jaus and Woolcock in Ref. 15. In our approximation it is sufficient to know that q_0 is of the order of the single-particle energy differences in the shell model (the oscillator spacing $\hbar\omega$). Retaining q_0 would give terms of $O(q_0/\Lambda)$ compared to the leading term, which can be neglected in our approximation. We can therefore put $q_0=0$ in Eq. (2.2) and in the denominator of Eq. (2.1). We furthermore can neglect l in the denominator of Eq. (2.1), since it only introduces corrections of $O(E/\Lambda)$ compared to the leading term.

In the nonrelativistic limit (τ is the isospin operator)

$$A_0=0, \quad \mathbf{A}=-g_A(\tau_-\boldsymbol{\sigma})$$

so that $N_{\mu 0}^{(A)}=0$. Invariance with respect to rotations and space inversions requires that

$$\left\langle f \left| \sum_{a\neq b} (j_0 e^{-i\mathbf{q}\cdot\mathbf{x}})_a (\tau_-\boldsymbol{\sigma} e^{i\mathbf{q}\cdot\mathbf{x}})_b \right| i \right\rangle = 0 \tag{2.6}$$

so that only the space components of $N^{(A)}$ are different from zero. In the nonrelativistic limit we have

$$(\mathbf{j}e^{-i\mathbf{q}\cdot\mathbf{x}})_a=\frac{1}{2M}[(\mathbf{p}e^{-i\mathbf{q}\cdot\mathbf{x}}+e^{-i\mathbf{q}\cdot\mathbf{x}}\mathbf{p})\hat{e} + i\mathbf{q}\wedge\boldsymbol{\sigma}(\hat{e}+\hat{\mu})e^{-i\mathbf{q}\cdot\mathbf{x}}]_a \tag{2.7}$$

with

$$\hat{e}\equiv\tfrac{1}{2}(1+\tau_z), \quad \hat{\mu}\equiv\tfrac{1}{2}(\mu_S+\mu_V\tau_z) . \tag{2.8}$$

We shall use the free-nucleon values

$$\tfrac{1}{2}(\mu_S-\mu_V)=\mu_n=-1.9131 ,$$

$$\tfrac{1}{2}(\mu_S+\mu_V)+1=\mu_p=2.7928 .$$

The quantities $\mathbf{p}$ and $\mathbf{x}$ are the momentum and position operators of nucleon a; M is the nucleon mass. For our purpose it is sufficient to take the numerical value of the proton mass m_p.

The transformation properties of the nuclear states under rotations and space inversions permits the representation of the nuclear matrix element in terms of a single form factor $\mathcal{N}(q)$ (we shall use the symbol q for $|\mathbf{q}|$ in the remainder of this paper):

$$\begin{aligned} N_{mn}^{(A)} &= -\frac{g_A}{\sqrt{2}}\left\langle f \left| \sum_{a\neq b} (j_m e^{-i\mathbf{q}\cdot\mathbf{x}})_a (\tau_-\sigma_n e^{i\mathbf{q}\cdot\mathbf{x}})_b \right| i \right\rangle \\ &= i g_A \epsilon_{mnk} q_k \mathcal{N}(q) . \end{aligned} \tag{2.9}$$

Because of the antisymmetry of $N_{ml}^{(A)}$ we see that the first three terms in the decomposition Eq. (2.2) do not contribute to $M^{(A)}(1c)$. The fourth term is of $O(E/\Lambda)$ compared to the last term and will be neglected, so that finally the leading term becomes [with Eqs. (2.3) and (2.6)]

$$M^{(A)}(1c)=M^{(0)}\frac{4\alpha}{\pi}g_A\int_0^\infty \mathcal{N}(q)dq . \tag{2.10}$$

With Eq. (2.1) we can express $C_{\rm NS}$, defined in Eqs. (1.1) and (1.3), in terms of $\mathcal{N}(q)$:

$$C_{\rm NS}=8g_A\int_0^\infty \mathcal{N}(q)dq . \tag{2.11}$$

The analysis of the nuclear matrix element is simplified considerably by using the operator identity (from now on $\mathbf{l}=\mathbf{x}\wedge\mathbf{p}$ stands for the orbital angular momentum operator of a nucleon)

$$\begin{aligned} \mathbf{p}e^{-i\mathbf{q}\cdot\mathbf{x}}+e^{-i\mathbf{q}\cdot\mathbf{x}}\mathbf{p} \equiv i&\left[\mathbf{p}^2,\mathbf{x}\int_0^1 e^{-i\mathbf{q}\cdot\mathbf{x}s}ds\right] \\ &+i\mathbf{q}\wedge\mathbf{l}\int_0^1 s e^{-i\mathbf{q}\cdot\mathbf{x}s}ds \\ &+i\int_0^1 s e^{-i\mathbf{q}\cdot\mathbf{x}s}\mathbf{q}\wedge\mathbf{l}\, ds . \end{aligned} \tag{2.12}$$

We shall assume that the nuclear wave function is a linear superposition of antisymmetrized one-particle harmonic-oscillator eigenfunctions.

The one-particle Hamiltonian then has the form

$$H=\frac{\mathbf{p}^2}{2M}+V(\mathbf{x})$$

and satisfies the operator identity

$$\left[\frac{\mathbf{p}^2}{2M},\mathbf{x}e^{-i\mathbf{q}\cdot\mathbf{x}s}\right]=\left[H,\mathbf{x}e^{-i\mathbf{q}\cdot\mathbf{x}s}\right] . \tag{2.13}$$

Inserting Eq. (2.12) into Eq. (2.7) and using Eq. (2.13) we see with Eq. (2.9) that $N_{mn}^{(A)}$ can be expressed by the four nuclear matrix elements N_{mn}^{x} and N_{jn}^{1}, $\tilde{N}_{jn}^{1}$, N_{jn}^{σ} defined by

$$\mathcal{N}_{mn}^{x}:=\left\langle f \left| \sum_{a\neq b} ([H,x_m e^{-i\mathbf{q}\cdot\mathbf{x}s}]\hat{e})_a (\tau_-\sigma_n e^{i\mathbf{q}\cdot\mathbf{x}})_b \right| i \right\rangle , \tag{2.14}$$

$$\mathcal{N}_{jn}^{l}:=\left\langle f \left| \sum_{a\neq b} (l_j e^{-i\mathbf{q}\cdot\mathbf{x}s}\hat{e})_a (\tau_-\sigma_n e^{i\mathbf{q}\cdot\mathbf{x}})_b \right| i \right\rangle , \tag{2.15}$$

$$\tilde{\mathcal{N}}_{jn}^{l}:=\left\langle f \left| \sum_{a\neq b} (e^{-i\mathbf{q}\cdot\mathbf{x}s}l_j\hat{e})_a (\tau_-\sigma_n e^{i\mathbf{q}\cdot\mathbf{x}})_b \right| i \right\rangle , \tag{2.16}$$

$$\mathcal{N}_{jn}^{\sigma}:=\left\langle f \left| \sum_{a\neq b} [\sigma_j e^{-i\mathbf{q}\cdot\mathbf{x}}(\hat{e}+\hat{\mu})]_a [\tau_-\sigma_n e^{i\mathbf{q}\cdot\mathbf{x}}]_b \right| i \right\rangle . \tag{2.17}$$

The result is

$$\begin{aligned} N_{mn}^{(A)}=-i\frac{g_A}{\sqrt{2}}&\left[\int_0^1 ds[\mathcal{N}_{mn}^{x}+\epsilon_{mkj}q_k(\mathcal{N}_{jn}^{l}+\tilde{\mathcal{N}}_{jn}^{l})s]\right. \\ &\left.+\epsilon_{mkj}q_k\mathcal{N}_{jn}^{\sigma}\right] . \end{aligned} \tag{2.18}$$

Again, based on the transformation properties of the nuclear states the nuclear matrix elements, Eqs. (2.14)–(2.17) can be represented in the following way:

$$\mathcal{N}_{mn}^{x}=\mathcal{N}^{x}\epsilon_{mnk}\frac{q_k}{q} , \tag{2.19}$$

$$\mathcal{N}_{jn}^{l}=\mathcal{N}^{l}\delta_{jn}+\mathcal{M}^{l}q_j q_n , \tag{2.20}$$

$$\tilde{\mathcal{N}}^{l}_{jn}=\tilde{\mathcal{N}}^{l}\delta_{jn}+\tilde{\mathcal{M}}^{l}q_jq_n\ , \tag{2.21}$$

$$\mathcal{N}^{\sigma}_{jn}=\mathcal{N}^{\sigma}\delta_{jn}+\mathcal{M}^{\sigma}q_jq_n\ . \tag{2.22}$$

Here $\mathcal{N}^x$, $\mathcal{N}^l$, $\tilde{\mathcal{N}}_l$, and $\mathcal{N}^\sigma$ are scalar functions of q and s. The decomposition of the form factor $\mathcal{N}(q)$ corresponding to Eq. (2.18) is given by

$$\mathcal{N}(q)=\int_0^1\left\{-\frac{1}{\sqrt{2}}\frac{1}{q}\mathcal{N}^x(q,s)+\frac{1}{2M}\frac{1}{\sqrt{2}}[\mathcal{N}^l(q,s)+\tilde{\mathcal{N}}^l(q,s)]s\right\}ds+\frac{1}{2M}\frac{1}{\sqrt{2}}\mathcal{N}^\sigma(q)\ . \tag{2.23}$$

We see that for our purpose we do not need the quantities $\mathcal{M}$ appearing in Eqs. (2.20)–(2.22).

After this reduction it is the main purpose of the rest of the paper to analyze the nuclear matrix elements of Eqs. (2.14)–(2.17) for some superallowed transitions and to separate the scalars $\mathcal{N}^x$, $\mathcal{N}^l$, $\tilde{\mathcal{N}}^l$, and $\mathcal{N}^\sigma$. Inserting these into Eq. (2.23) gives the form factor, and the nuclear-structure correction can be calculated by means of Eq. (2.11). We will show that this program can be performed analytically if the nuclear wave function is known in the harmonic-oscillator shell-model basis.

III. THE CORRECTIONS FOR THE DECAY OF ^{14}O

We take ^{14}O as the example to demonstrate the evaluation of the form factor $\mathcal{N}(q)$ in more detail. For this purpose we shall rely upon a shell-model description of the isospin triplet ^{14}O, ^{14}N, ^{14}C given by True.[16] In Ref. 16 a model space is assumed where ^{12}C is taken as an inert core of closed $(1s\frac{1}{2})^4(1p\frac{3}{2})^8$ shells. The remaining two protons are assumed to occupy only the next higher orbits $(1p\frac{1}{2})$, $(2s\frac{1}{2})$, $(1d\frac{5}{2})$, and $(1d\frac{3}{2})$. The components of the resulting wave function are antisymmetrized products of harmonic-oscillator eigenfunctions. The result of Ref. 16 in obvious notation (including the i^l in the harmonic-oscillator eigenfunctions to conform with the convention of Towner,[17] which we use in the following) is

$$|i\rangle=a_0|(1p\tfrac{1}{2})^2,M_T\rangle+a_1|(2s\tfrac{1}{2})^2,M_T\rangle+a_2|(1d\tfrac{5}{2})^2,M_T\rangle+a_3|(1d\tfrac{3}{2})^2,M_T\rangle \tag{3.1}$$

with

$$a_0=0.9501,\quad a_1=0.1219\ ,\qquad a_2=0.2635,\quad a_3=0.1139\ . \tag{3.2}$$

Here $M_T=1$ for the initial state $|i\rangle$ and $M_T=0$ for the final state $|f\rangle$ in the β^+ decay of ^{14}O.

For a typical operator Q occurring in Eqs. (2.14)–(2.17) we get [neglecting terms $O(a_i^2)$, $i\geq 1$]

$$\langle f|Q|i\rangle=a_0^2\langle(1p\tfrac{1}{2})^2,M_T'|Q|(1p\tfrac{1}{2})^2,M_T\rangle+2a_0[a_1\langle(2s\tfrac{1}{2})^2,M_T'|Q|(1p\tfrac{1}{2})^2,M_T\rangle+a_2\langle(1d\tfrac{5}{2})^2,M_T'|Q|(1p\tfrac{1}{2})^2,M_T\rangle+a_3\langle(1d\tfrac{3}{2})^2,M_T'|Q|(1p\tfrac{1}{2})^2,M_T\rangle]\ . \tag{3.3}$$

It turns out that the interference terms cannot be neglected and we have to calculate the matrix elements between the relevant 0^+ core-plus-two-particle states. Applying the usual methods we can express these two-particle matrix elements in terms of one-particle matrix elements. We perform the isospin decomposition using $M_T'=M_T-1$ and suppress M_T in the notation of the matrix element. We write $[j]$ as shorthand for (n,l,j). Furthermore we abbreviate $\hat{j}\equiv\sqrt{2j+1}$. The result then is

$$\begin{aligned}\langle[j']^2|Q|[j]^2\rangle=&\sqrt{2}\frac{1}{\hat{j}}\frac{1}{\hat{j}'}\sum_{m,m'}(-1)^{j+j'+m+m'}\langle[j']-m'|g_2|[j]-m\rangle\langle[j']m'|g_1|[j]m\rangle[\xi+\nu(2M_T-1)]\\&+\delta_{jj'}\sum_{[j_1]\in C}\sum_{m,m_i}\frac{\sqrt{2}}{\hat{j}^2}[2\xi\langle[j_i]m_i|g_1|[j_i]m_i\rangle\langle[j]m|g_2|[j]m\rangle\\&-(\xi-\nu)\langle[j_i]m_i|g_2|[j]m\rangle\langle[j]m|g_1|[j_i]m_i\rangle\\&-(\xi+\nu)\langle[j_i]m_i|g_1|[j]m\rangle\langle[j]m|g_2|[j_i]m_i\rangle]\ .\end{aligned} \tag{3.4}$$

Here $g_2=\sigma_n e^{i\mathbf{q}\cdot\mathbf{x}}$ and

$g_1=[H,x_m e^{-i\mathbf{q}\cdot\mathbf{x}s}]=(E'-E)x_m e^{-i\mathbf{q}\cdot\mathbf{x}s}$,

$\xi=\nu=\frac{1}{2}$ for the contribution to N^x_{mn} ,

$g_1=l_j e^{-i\mathbf{q}\cdot\mathbf{x}s}$,

$\xi=\nu=\frac{1}{2}$ for the contribution to N^l_{jn} ,

$g_1=e^{-i\mathbf{q}\cdot\mathbf{x}s}l_j$,

$\xi=\nu=\frac{1}{2}$ for the contribution to $\tilde{N}^{l}_{jn}$,

$g_1=\sigma_j e^{-i\mathbf{q}\cdot\mathbf{x}}$,

$\xi=\frac{1}{2}(1+\mu_S)$, $\nu=\frac{1}{2}(1+\mu_V)$, for the contribution to N^σ_{jn} . (3.5)

Applying the usual methods of angular momentum algebra we can split off the contribution of Eq. (3.4) to the scalars $\mathcal{N}^x$, $\mathcal{N}^l$, $\tilde{\mathcal{N}}^l$, $\mathcal{N}^\sigma$ defined in Eqs. (2.19)–(2.22). All matrix elements can be calculated explicitly using harmonic-oscillator eigenfunctions. We give first the result for the diagonal contribution $[j]=[j']=(1p\frac{1}{2})$ in ^{14}O:

$$\mathcal{N}^x=\frac{\hbar\omega}{q}\frac{2}{3}\sqrt{2}ze^{-z(1+s^2)},$$
$$\mathcal{N}^l+\tilde{\mathcal{N}}^l=\tfrac{4}{3}\sqrt{2}[1-z(1+s^2)+z^2s^2+\tfrac{1}{2}zs]e^{-z(1+s^2)}, \tag{3.6}$$
$$\mathcal{N}^\sigma=-\tfrac{1}{9}\sqrt{2}[(1-4z+4z^2)\mu_p+(8-2z+8z^2)(\mu_p+\mu_n)],$$

where $z=\frac{1}{4}q^2r_p^2$ and the oscillator parameter r_p is given in the Appendix and $r_p^2=\hbar/(m_p\omega)$. For the nondiagonal case $[j]\neq[j']$ we find the following contributions (the core part vanishes in that case).

$[j']=(2s\frac{1}{2})$:

$$\mathcal{N}^x=-\frac{\hbar\omega}{q}\tfrac{4}{9}\sqrt{2}[z-z^2(1+s^2)+z^3s^2]e^{-z(1+s^2)},$$
$$\mathcal{N}^l+\tilde{\mathcal{N}}^l=\tfrac{4}{9}\sqrt{2}[zs-z^2s(1+s^2)+z^3s^3]e^{-z(1+s^2)},$$
$$\mathcal{N}^\sigma=-\tfrac{4}{9}\sqrt{2}\mu_pz(1-z)^2e^{-2z}.$$

$[j']=(1d\frac{5}{2})$:

$$\mathcal{N}^x=-\frac{\hbar\omega}{q}\frac{16}{45}\sqrt{6}s^2z^3e^{-z(1+s^2)},$$
$$\mathcal{N}^l+\tilde{\mathcal{N}}^l=\tfrac{8}{45}\sqrt{6}[3zs-z^2s(1+2s^2)]e^{-z(1+s^2)},$$
$$\mathcal{N}^\sigma=-\tfrac{8}{45}\sqrt{6}\mu_p(3z-2z^2+z^3)e^{-2z}.$$

$[j']=(1d\frac{3}{2})$:

$$\mathcal{N}^x=\frac{\hbar\omega}{q}\frac{2}{45}[25z-10z^2(1+s^2)+4z^3s^2]e^{-z(1+s^2)},$$
$$\mathcal{N}^l+\tilde{\mathcal{N}}^l=\tfrac{2}{45}(59zs-38z^2s-26z^2s^3+20z^3s^3)e^{-z(1+s^2)},$$
$$\mathcal{N}^\sigma=-\tfrac{4}{45}\mu_p(7z-8z^2+4z^3)e^{-2z}.$$

In order to derive the form factor $\mathcal{N}$ an integration over the variable s is required according to Eq. (2.23). However, it is easier to calculate instead the correction $C_{\rm NS}$, Eq. (2.11), since doing the integrations in the reverse order involves only elementary functions.

It is convenient to decompose $C_{\rm NS}$ in an obvious notation corresponding to Eq. (3.3):

$$C_{\rm NS}=a_0^2C_0(1p\tfrac{1}{2})+2a_0[a_1C_1(2s\tfrac{1}{2})+a_2C_2(1d\tfrac{5}{2})+a_3C_3(1d\tfrac{3}{2})]. \tag{3.7}$$

Numerically we get

$$C_0=-0.71,\quad C_1=0.03,\quad C_2=-0.33,\quad C_3=0.39.$$

Using Eqs. (3.7) and (3.2) gives

$$C_{\rm NS}=-0.88. \tag{3.8}$$

Another wave function for ^{14}O has been given by Zamick in Ref. 19. Here it is assumed that the triplet can be described as a two-hole state of the ^{16}O core. The configuration-mixed wave function is approximated by

$$|i\rangle=b_0|(1p\tfrac{1}{2})^{-2},M_T\rangle+b_1|(1p\tfrac{3}{2})^{-2},M_T\rangle, \tag{3.9}$$
$$b_0=0.975,\quad b_1=0.220. \tag{3.10}$$

Doing for the two-hole states the same calculation as before for the two-particle states, we get, with an obvious modification of the notation used before,

$$C_0[(1p\tfrac{1}{2})^{-1}]\equiv C_0(1p\tfrac{1}{2})=-0.71,\quad C_1[(1p\tfrac{3}{2})^{-1}]=-0.86. \tag{3.11}$$

From the analogue of Eq. (3.7) we get, with Eqs. (3.10) and (3.11) numerically,

$$C_{\rm NS}=-1.04. \tag{3.12}$$

Cohen and Kurath[20] have given a wave function for ^{14}O by taking into account the same configurations as in Ref. 19. Instead of Eq. (3.10) they get

$$b_0=0.91505,\quad b_1=0.40380.$$

Calculating the corresponding $C_{\rm NS}$ one also has to take into account the terms $O(b_1^2)$. The result is

$$C_{\rm NS}=-1.46. \tag{3.13}$$

The agreement between Eqs. (3.8), (3.12), and (3.13) is satisfactory in view of the different models used in Refs. 16, 19, and 20. For our final correction we have taken an unweighted average of these values with the corresponding error

$$C_{\rm NS}=-1.13(17)\ \text{for}\ ^{14}\text{O}. \tag{3.14}$$

Kühner *et al.*[21] announce a phenomenological wave function for ^{14}O with

$$b_0=0.46969,\quad b_1=0.88283.$$

It leads to $C_{\rm NS}=-2.02$, which seems to us to be too big in absolute value. It is therefore not included in our final analysis.

IV. THE NUMERICAL CORRECTIONS FOR THE DECAYS OF ^{42}Sc, ^{38}K^m, ^{26}Al^m, ^{34}Cl, ^{46}V, ^{50}Mn, and ^{54}Co

All wave functions discussed in the following are again represented in a harmonic-oscillator shell-model basis. The nucleus ^{42}Sc is treated in Ref. 22 under the assumption, that ^{40}Ca acts as an inert core. The configuration mixed wave function is then approximated by a linear superposition with two additional particles in $(1f\frac{7}{2})$, $(2p\frac{3}{2})$, $(1f\frac{5}{2})$, $(2p\frac{1}{2})$, or $(1g\frac{9}{2})$ orbitals. In the notation of Eq. (3.1) the result of Ref. 22 is

$$|i\rangle=a_0|(1f\tfrac{7}{2})^2,M_T\rangle+a_1|(2p\tfrac{3}{2})^2,M_T\rangle+a_2|(1f\tfrac{5}{2})^2,M_T\rangle+a_3|(2p\tfrac{1}{2})^2,M_T\rangle+a_4|(1g\tfrac{9}{2})^2,M_T\rangle$$

with $a_0=0.93$, $a_1=0.20$, $a_2=0.20$, $a_3=0.11$, and $a_4=0.22$. Here $M_T=0$ for the initial state and $M_T=-1$ for the final state $|f\rangle$ in the β^+ decay of ^{42}Sc.

We decompose C_{NS} again as in Eq. (3.7) and find, using the methods of Sec. III,

$$C_0=0.38,\quad C_1=0.07,\quad C_2=0.67$$

$$C_3=0.06,\quad C_4=0.18\ .$$

This gives

$$C_{NS}=0.69 \quad \text{for } ^{42}\text{Sc}\ . \tag{4.1}$$

The interference terms here all have the same sign as the main term coming from the pure configuration and contribute 50% to the result.

Another wave function for ^{42}Sc has been given in Ref. 23; it does not include the mixing of the configuration $(1g\frac{9}{2})$. The result for the amplitudes is

$$a_0=0.989,\quad a_1=0.102,\quad a_2=0.083,\quad a_3=0.041\ .$$

This gives

$$C_{NS}=0.50 \quad \text{for } ^{42}\text{Sc}\ . \tag{4.2}$$

The agreement between Eqs. (4.1) and (4.2) is as good as can be expected and for our final correction we take an average with the corresponding error:

$$C_{NS}=0.60(10) \quad \text{for } ^{42}\text{Sc}\ . \tag{4.3}$$

^{38}K^m belongs to the class of *s-d* shell nuclei, which has been intensively investigated by Wildenthal and collaborators.[25] It is assumed that ^{16}O acts as an inert core. The configuration mixed wave function then is approximated by a superposition of two hole in $(1d\frac{3}{2})$, $(2s\frac{1}{2})$, or $(1d\frac{5}{2})$ orbitals of a ^{40}Ca core:

$$|i\rangle=a_0|(1d\tfrac{3}{2})^{-2},M_T\rangle+a_1|(2s\tfrac{1}{2})^{-2},M_T\rangle$$
$$+a_2|(1d\tfrac{5}{2})^{-2},M_T\rangle$$

with[25]

$$a_0=0.970,\quad a_1=0.153,\quad a_2=0.177\ .$$

Again $M_T=0$ for the initial state and $M_T=-1$ for the final state in the β^+ decay of ^{38}K^m. Performing the calculation analogous to the one described in Sec. III, this time for two-hole–two-hole matrix elements, we get

$$C_0=-0.50,\quad C_1=0.09,\quad c_2=0.89\ .$$

This gives

$$C_{NS}=-0.14 \quad \text{for } ^{38}\text{K}^m\ . \tag{4.4}$$

An earlier wave function given in Ref. 26 results in $C_{NS}=-0.04$. The two wave functions given in Ref. 27 result in $C_{NS}=-0.03$ and $C_{NS}=-0.13$, respectively. In any case the correction for ^{38}K^m is small, since the sign of the interference terms is opposite to the sign of the main term coming from the pure configuration and the two contributions essentially cancel. For our final analysis we have taken the value of Eq. (4.4) without an error.

It seems that the three nuclear configurations treated above—two-particle or two-hole states in a doubly closed shell—can be handled rather reliably. Even if different nuclear wave functions are available there is only a small change in the resulting correction.

For open-shell nuclei strong configuration mixing occurs and the calculation of the correction is considerably more complicated and uncertain. We therefore present only a preliminary analysis of the correction for the remaining five nuclear transitions which uses the value C_0 of the correction for a pure configuration. We take the correction to be one half of this value. This crude approach makes it necessary to give an estimate of the range, within which the results of a calculation with more realistic wave functions should lie. We assume the corresponding error to be also one-half of the correction for the pure configuration. This procedure is based on the observation that strong configuration mixing weakens the amplitude of the pure configuration, and it is consistent with our observation that configuration mixing tends either to enhance or to cancel the correction due to the pure configuration.

We believe that this approximation is a sufficient estimate, since it certainly gives an indication of the sign and of the order of magnitude of the correction, and also shows the relative effect of the shell structure in comparing different nuclear transitions. Numerically we get for

$$^{26}\text{Al}^m\text{:}\quad C_0=0.52,\quad C_{NS}=0.26(26)\ ,$$

$$^{34}\text{Cl:}\quad C_0=-0.52,\quad C_{NS}=-0.26(26)\ ,$$

$$^{46}\text{V:}\quad C_0=0.63,\quad C_{NS}=0.32(32)\ ,$$

$$^{50}\text{Mn:}\quad C_0=0.63,\quad C_{NS}=0.32(32)\ ,$$

$$^{54}\text{Co:}\quad C_0=0.36,\quad C_{NS}=0.18(18)\ .$$

We will present an improved and refined investigation of this problem in the future.

V. CONCLUSIONS

We have calculated and discussed the effect of nuclear structure on the $O(\alpha)$ corrections for eight superallowed β^+ decays. The emitted positron is affected by the structure of the daughter nucleus, as shown in lowest order in the diagram of Fig. 1(c). If the vector part of the weak current mediates the decay of the nucleon, then this diagram is completely accounted for by the Fermi function and the shape factor as discussed in the Appendix. The contribution of this diagram due to the axial-vector part of the weak current leads to the correction we have calculated for the first time and collected in column (1) of Table III.

Considering the high level of experimental precision for the *ft* value of ^{14}O, our correction for this transition is remarkable. The error indicates how the correction changes if different nuclear wave functions are used for its evaluation, as discussed extensively in Sec. III.

The influence of nuclear structure is less pronounced in the transitions of the heavier nuclei. Though we have used in most cases only a somewhat crude approximation

TABLE III. Nuclear-structure corrections, recoil and atomic-excitation corrections, and corrected ft values. (1) Nuclear-structure correction $\alpha/\pi C_{\rm NS}$ in %. (2) Recoil correction in sec as given on page 488 of Ref. 28. (3) Atomic-excitation corrections in sec from Ref. 29. (4) ft values from Table II column (1), corrected according to Eq. (1.5), with $1-\delta_c$ from Table II column (3) and corrected for column (2) and (3), in sec. (5) ft values from Table II column (1), corrected according to Eq. (1.5), with $1-\delta_c$ from Table II column (4) and corrected for column (2) and (3), in sec. The systematic uncertainty of Δ_s is not included in the errors of columns (4) and (5).

	(1)	(2)	(3)	(4)	(5)
^{14}O	−0.26(4)	−0.8	−1.6	3142.2(46)	3139.5(36)
^{26}Al^m	+0.06(6)	−0.7	−0.7	3145.5(42)	3142.6(41)
^{34}Cl	−0.06(6)	−0.7	−0.7	3149.9(51)	3145.0(47)
^{38}K^m	−0.03(0)	−0.7	−0.6	3147.9(62)	3141.3(45)
^{42}Sc	+0.14(2)	−0.7	−0.6	3150.2(45)	3150.2(43)
^{46}V	+0.07(7)	−0.7	−0.6	3156.0(52)	3148.4(49)
^{50}Mn	+0.07(7)	−0.7	−0.5	3150.1(61)	3143.3(61)
^{54}Co	+0.04(4)	−0.7	−0.5	3153.9(49)	3146.4(47)
	Avg			3149.1(18)	3144.2(16)
	χ^2			5.9	5.2

for the nuclear wave function, we think that a reliable estimate of the sign and the order of magnitude of the nuclear-structure effect has been obtained. The resulting corrections lie within the errors of the corresponding ft values.

For the determination of our final ft values (last two columns of Table III) we have included also the small corrections due to the recoil of the daughter nucleus[28] and due to atomic excitation.[29]

The errors of the ft values for the individual transitions in the last two columns of Table III is the combination in quadrature of the experimental errors [column (1) of Table II] and the theoretical uncertainties of δ_c (Table II) and $\delta_{\rm NS}$ (Table III). The relatively smaller error in our nuclear-structure correction hardly affects the final error.

To get the averaged ft value we have separately averaged the ft values of column (4) and (5), respectively, of Table III. The statistical fit for both cases is good, but the two averages are not compatible. For the further discussion we have taken the average of column (4) of Table III as our nucleus-independent ft value (called Ft):

$$Ft = 3149.1(31)\ \text{sec}\ . \tag{5.1}$$

The error assigned to Ft is the combination of two independent uncertainties in quadrature: (a) The statistical uncertainty of the averaged ft value quoted in Table III; (b) the systematic uncertainty of the radiative correction Δ_S (Table I) due to the poorly known value for the low-energy cutoff m_A.

$|V_{ud}|$ is then given by

$$|V_{ud}|^2 = \frac{\pi^3 \ln 2}{Ft} \frac{\hbar^7}{G_F^2 m^5 c^4}\ .$$

Taking the numerical values of mc^2, $\hbar$, and the Fermi coupling constant from Ref. 30, we get

$$|V_{ud}| = \frac{54.630(1)}{\sqrt{Ft}}\ . \tag{5.2}$$

From Eqs. (5.1) and (5.2) we find

$$|V_{ud}| = 0.9735(5)\ . \tag{5.3}$$

Since our calculation of C has reduced the theoretical uncertainty in the structure-dependent part of the $O(\alpha)$ corrections arising from the axial-vector current, the error in $|V_{ud}|$ has been lowered by a factor 2 compared to the error quoted before.[8,30]

The result Eq. (5.3) can now be combined with the element V_{us} of the KM matrix as determined from an analysis of kaon and hyperon decays,[31,30]

$$|V_{us}| = 0.2197(19)\ , \tag{5.4}$$

and the bound from B decay[32,30] for $|V_{ub}|$,

$$|V_{ub}| \le 0.01\ , \tag{5.5}$$

to give

$$|V|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9960(13)\ . \tag{5.6}$$

The error given in Eq. (5.6) is the propagated error of Eqs. (5.3) and (5.4). It is smaller than the error given in Ref. 8 by nearly a factor 2.

It seems difficult to reconcile Eq. (5.6) with the unitarity condition for the KM matrix in the standard model with three generations.

We therefore have listed for comparison in Table IV the values for $|V_{ud}|$ and $|V|^2$ if the higher $O(\alpha^n \ln^n m_z)$ of Ref. 1 as discussed in Sec. I are omitted. Though the effect of these higher-order radiative corrections is small, yet they increase the deviation from unitarity by an additional one standard error. In order to decide the important question of the unitarity of the KM matrix, it is obvious that a level of precision is required such that even radiative corrections of higher order, as estimated by Marciano and Sirlin[1] for the first time, must be known.

We have listed also the corresponding results if the analysis is based upon the ft values of column (5) of Table III, which are corrected for isospin nonconserva-

TABLE IV. Values for $|V_{ud}|$ and $|V|^2$. (1) isospin-impurity correction according to OB (Ref. 8), renormalization-group treatment included. (2) isospin-impurity correction according to OB (Ref. 8), without renormalization-group treatment. (3) isospin-impurity correction according to THH (Refs. 6, 9, and 18), renormalization-group treatment included. (4) isospin-impurity correction according to THH (Refs. 6, 9, and 18), without renormalization-group treatment.

	(1)	(2)	(3)	(4)
$\|V_{ud}\|$	0.9735(5)	0.9743(5)	0.9743(5)	0.9751(5)
$\|V\|^2$	0.9960(13)	0.9975(13)	0.9975(13)	0.9991(13)

tion according to Towner, Hardy, and Harvey (THH).[9,18,6] These results are closer to the unitarity limit.

We wish to emphasize that the nuclear-structure correction calculated in this work and the refinements in the analysis of the ft values which we have discussed did not essentially reduce the spread in the ft values of the eight superallowed transitions, but reduces the uncertainty in the radiative corrections substantially. The theoretical error in $|V_{ud}|$, as given in Eq. (5.3) and in the sum of the squared moduli Eq. (5.6), is now dominated by the uncertainty in Δ_S due to the generous range admitted for the low-energy cutoff m_A in Ref. 1. The error of δ_c only dominates the uncertainties of the individual ft values given in Table III, while the error of the averaged ft value in Table III is quite small.

An alternative approach of estimating the uncertainty of δ_c has been proposed in Ref. 6, where the errors quoted by the authors (Table II, columns 3 and 4) are essentially replaced by the differences between the results obtained with the methods of OB and THH. This procedure would lead to results which are the average of the corresponding results given in Table IV with slightly larger errors. This approach basically assumes that the two quite different methods are equally good approximations, which however leads to an effective increase of the uncertainty of δ_c. It would certainly be more satisfactory to find arguments which enable a clear preference for one calculation or the other. Since our statistical analysis gives equally good results for both cases, the present data do not permit such a distinction as yet. But Ormand and Brown have argued in Ref. 8 that their more realistic treatment of the Coulomb potential and the nuclear mean fields essentially leads to a much better overlap between the converted proton and the corresponding neutron than the estimate of THH (Refs. 9 and 18). We rely upon the quoted arguments of Ormand and Brown[8] to present the values given in Eqs. (5.4) and (5.6) as our preferred result.

While this result indicates a violation of the unitarity condition for three generations, which might be evidence for new physics, such as the existence of a fourth fermion generation, we emphasize once more that the remaining discrepancies between different ft values might signal that additional theoretical uncertainties are present. The determination of δ_c seems to be the crucial issue, and a further refinement of the corresponding nuclear structure calculation, which could reduce the large errors involved, would be of great interest.

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APPENDIX: CONNECTION BETWEEN THE ELECTROMAGNETIC FORM FACTOR, THE ISOVECTOR FORM FACTOR, AND THE SHAPE FACTOR

The isovector form factor $G_V(Q^2)$ is defined by the matrix element of the vector current between states of an isospin triplet, with momenta $\mathbf{P}'$ and $\mathbf{P}''$, respectively. With $Q \equiv P'' - P'$ we have

$$\langle \mathbf{P}''|V_0(0)|\mathbf{P}'\rangle = \frac{P_0'+P_0''}{\sqrt{2P_0' 2P_0''}}\sqrt{2}G_V(Q^2) \simeq \sqrt{2}G_V(\mathbf{Q}^2) \ . \tag{A1}$$

The Fermi function, and therefore f, is usually calculated in terms of the positron spinor $\psi_c(\mathbf{x})$, which is the solution of the Dirac equation in the electric field of a static nuclear charge distribution. It is convenient to define the Fermi function $F(Z,E)$ in terms of the product $\psi_c^*(\mathbf{x})\psi_c(\mathbf{x})$ evaluated at the origin $\mathbf{x}=\mathbf{0}$. This approximation is equivalent to a pointlike isovector charge distribution, which means $G_V(\mathbf{Q}^2)\equiv 1$. Tables of β-decay functions using this procedure with a uniform charge distribution of finite radius, including atomic screening, are given in Ref. 7.

To take the effect of the finite isovector charge distribution into account an expansion of $\psi_c(\mathbf{x})$ in powers of αZ can be used to calculate an energy-dependent correction factor $C(E)$, the so-called shape factor.[24] The integrated Fermi function is then given by

$$f\bar{C}(E) = \frac{1}{m^5}\int_m^{E_0} dE\, S(E)F(Z,E)C(E) \ , \tag{A2}$$

where $S(E)$ is the statistical positron spectrum of the decaying nucleus of charge $Z+1$.

The power series for $C(E)$ is well approximated by the first three terms, which are given in Ref. 24. With

$$C(E)=1+\Delta_0(E)+\Delta_1(E)+\Delta_2(E)+\cdots , \tag{A3}$$

we have

$$\Delta_0(E)=\frac{1}{4\pi}\int d\Omega_\nu[G_V(\mathbf{Q}^2)-1] ,$$

$$\Delta_1(E)=-\frac{Z\alpha}{\pi^2}\frac{1}{4\pi}\int d\Omega_\nu \int d^3q\, F(q^2) \times\{G_V[(\mathbf{q}-\mathbf{Q})^2]-G_V(\mathbf{Q}^2)\} \times\frac{2E-\mathbf{l}\mathbf{q}/E-\mathbf{k}\mathbf{q}/E_\nu}{\mathbf{q}^2(\mathbf{q}^2-2\mathbf{l}\mathbf{q})} , \tag{A4}$$

$$\Delta_2(E)=2\left[\frac{Z\alpha}{2\pi^2}\right]^2\int d^3q\, d^3q' F(\mathbf{q}^2)F(\mathbf{q}'^2) \times\{G_V[(\mathbf{q}+\mathbf{q}')^2]-1\} \times\frac{\mathbf{q}'^2+\mathbf{q}\mathbf{q}'}{\mathbf{q}'^4\mathbf{q}^2(\mathbf{q}+\mathbf{q}')^2} .$$

Here $\mathbf{Q}=\mathbf{l}+\mathbf{k}$ and $E=(m^2+l^2)^{1/2}$ ($\equiv l_0$) are defined as before and $E_\nu=E_0-E$ ($\equiv k_0$). $F(\mathbf{q}^2)$ is the electromagnetic form factor of the nucleus. In the integrand for $\Delta_2(E)$ the lepton momenta have been neglected, since they contribute effectively to higher order in αZ.

The electromagnetic form factor of each member of the isospin triplet, which we approximate by a pure shell-model configuration, is then given by

$$F(q^2)=\frac{1}{Z}\sum_i R_{l_i0l_i}(q) , \tag{A5}$$

where the sum extends over all protons. The radial integral

$$R_{lLl}(q)=\int_0^\infty dr\, r^2|R_{nl}(r)|^2 j_L(qr)$$

is evaluated by means of harmonic-oscillator radial wave functions

$$R_{nl}(r)=\left[\frac{2(n-1)!}{\Gamma(n+l+\frac{1}{2})r_0^3}\right]^{1/2}\rho^l e^{-\rho^2/2}L_{n-1}^{l+1/2}(\rho^2)$$

with

$$\rho=r/r_p \quad\text{and}\quad r_p^2=\hbar/(m_p\omega) .$$

Here r_p^2 is the oscillator parameter corresponding to protons. To be completely consistent, we take for r_p those numerical values which reproduce [via the charge form factor $F(\mathbf{q}^2)$ of the shell model] the rms charge radius $r_c=\sqrt{3/5}r_0A^{1/3}$, as given numerically in column (2) of Table V. We have listed these values for r_p in column (3) of Table V. The values for r_0 are taken from Ref. 33 and are given in column (1).

The isovector form factor $G_V(\mathbf{q}^2)$ is then obviously given by the radial integral of the valence orbital (with orbital angular momentum l):

$$G_V(\mathbf{q}^2)=R_{l0l}(q) . \tag{A6}$$

Our results for the averaged shape-factor correction are given in column (5) of Table V. The results of Behrens and Bühring[34] and Hardy and Towner[35] as quoted and discussed in Ref. 10 are given in columns (6) and (7) for comparison.

It has been noted in Ref. 10 that there are considerable differences between the shape factors calculated according to the methods of Refs. 34 and 35, respectively. These differences are even more pronounced between our results and those of Ref. 34. The reason for this disagreement is easily understood, since we can reproduce the results of Ref. 34 starting from Eq. (A4) and putting $G_V(\mathbf{q}^2)\equiv F(\mathbf{q}^2)$. For example, in Ref. 24 we used a modified Gaussian charge distribution (with $A=2$ as defined in Ref. 34) for $G_V(\mathbf{q}^2)$ and $F(\mathbf{q}^2)$ and derived results which are identical with those of Ref. 34. It is self-evident that in comparing different methods the radii that are used must be consistent.

The differences between the various methods are largest for ^{54}Co; using the approach of Ref. 24 and $r_c=3.718$ as given in Table V for ^{54}Co, we find the values -0.56% and -0.61% for $\overline{C(E)}-1$ using a uniform and a modified Gaussian ($A=2$) charge distribution, respec-

TABLE V. Details concerning the shape factor. (1) r_0 in fm, as used for the rms radius $\sqrt{3/5}r_0A^{1/3}$ of a uniform charge distribution of the daughter nucleus in the calculation of the integrated Fermi function as taken from Ref. 33. (2) rms charge radius r_c in fm corresponding to r_0. (3) Harmonic-oscillator parameter r_p in fm for protons, corresponding to r_c. (4) rms isovector charge radius r_V in fm, corresponding to r_c. (5) Correction due to the shape factor, as calculated in the Appendix, in %. (6) Correction due to the shape factor according to the method of BB (Ref. 34), as quoted in Ref. 10, in %. (7) Correction due to the shape factor according to the method of TH (Ref. 35), as quoted in Ref. 10, in %.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
^{14}O	1.37	2.558	1.719	2.718	−0.049	−0.043	−0.048
^{26}Al^m	1.34	3.075	1.883	3.523	−0.161	−0.123	−0.153
^{34}Cl	1.33	3.337	1.968	3.682	−0.273	−0.212	−0.249
^{38}K^m	1.31	3.411	1.988	3.719	−0.338	−0.268	−0.307
^{42}Sc	1.31	3.527	2.036	4.319	−0.492	−0.333	−0.451
^{46}V	1.29	3.580	2.021	4.287	−0.583	−0.401	−0.532
^{50}Mn	1.28	3.653	2.026	4.298	−0.680	−0.474	−0.618
^{54}Co	1.27	3.718	2.033	4.312	−0.750	−0.553	−0.709

tively. These results should be compared with the correction, −0.75%, given in column (5) of Table V and derived on the basis that the electromagnetic and the isovector charge form factors are given by Eqs. (A5) and (A6), respectively.

The method of Ref. 35 has been simulated in Ref. 10 by taking into account only the major shell filled by the valence nucleons. The corresponding results lie between ours and those of Ref. 34 [see column (7) of Table V].

This discussion shows that the form and the shape of the charge distributions must be accounted for as carefully as possible. We note that also the Fermi function, which enters the integrated Fermi function according to Eq. (A2), depends upon the shape of the nuclear charge distribution.[34] We use the values of f (in Ref. 34 the symbol $\bar{f}$ is used instead) as calculated by Behrens[36] according to the procedure in Ref. 7, with the end-point energies taken from Ref. 6 and the radius of the uniform charge distribution $r_c=\sqrt{3/5}r_0A^{1/3}$. This f value we have corrected for a modified Gaussian nuclear charge distribution as taken from Ref. 34 (the parameter A in Ref. 34 is taken to be $A=2$). The reason to correct for this specific form is that its effect is rather close to the nuclear shell-model form factor Eq. (A5).

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