Fractal measures in multiparticle production

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(Received 29 March 1989)

A new set of moments G_q of the multiplicity distribution in a narrow rapidity window is suggested for analysis so that the chaotic fluctuations of the rapidity distribution can be described in the formalism of multifractals. Phenomenological evidence is given to show that there is a fractal structure in the data of multiparticle production at high energy. The fractal dimension and its generalizations can be determined to provide a description of the spectrum of scaling indices. The implication on the emergence of a new approach to the study of multiparticle production in the framework of multifractals in the theory of chaos is discussed.

I. INTRODUCTION

The study of multiparticle production at high energies has recently entered into a new phase. For many years the emphasis has been placed on Koba-Nielsen-Olesen (KNO) scaling and its violation, forward-backward multiplicity correlation, representation of data by negative-binomial distribution, etc. (For recent reviews, see Refs. 1 and 2.) Many models have been suggested to interpret the data; all of them claim impressive successes.[2] More recent experiments on relativistic heavy-ion collisions produced multiplicity (or transverse-energy) distributions that are no more successful than those from hadronic collisions in discriminating among the various models that describe the basic collision process.[3] However, the scenario changed dramatically when data on intermittency appeared last year,[4] showing clear discrepancies from the predictions of some standard models, according to existing Monte Carlo codes. Recent results from high-energy nuclear collisions[5] having even better statistics have reinforced the discrepancy. In the area of multiparticle production we have finally found a disagreement between theory and experiment that cannot be eliminated by the adjustment of some parameters. It is a signal for the discovery of some very basic issues that have escaped notice thus far.

In the past the models were constructed to explain average features in the data. Bialas and Peschanski[6] were the first to raise questions about the significance of sharp spikes in the cosmic-ray data.[7] Having in mind the analogous bursts of turbulence in the theory of chaos,[8] they proposed the investigation of intermittency in multiparticle production. Specifically, they suggested that, if intermittency exists, the normalized factorial moments of multiplicity distribution should exhibit power-law dependences on the width δy of the rapidity window, as $\delta y \to 0$. The most significant aspect of their work is to steer people away from studying average phenomena, and to emphasize the importance of understanding the chaotic structure of the rapidity distribution from event to event.

The point of view of Bialas and Peschanski received a tremendous boost of support when intermittent types of behavior were observed in e^+e^- annihilation,[9] hadron-hadron,[10] hadron-nucleus,[11] and nucleus-nucleus collisions.[5,11] Numerous theoretical models have also been proposed to account for intermittency, or more broadly, the power-law behavior of the factorial moments:[12–17]

$$F_l \propto (\delta y)^{-a_l}, \quad \delta y \to 0 , \tag{1.1}$$

where the exponent a_l increases with the order l of the moment. Indeed, there are proposals, such as that in Ref. 15, which dispute the necessity of invoking intermittency as understood in the theory of deterministic chaos[18] to account for the behavior of (1.1). Given (1.1) as a summary of the phenomenological fact about particle production at small δy, the pressing theoretical question is then the interpretation of the exponents a_l. This paper is directed toward answering that question, but in so doing we have found that F_l are not the optimal moments to investigate. An alternative set of moments G_q are far more direct in exhibiting an order in the seemingly random fluctuations in the rapidity distributions.

The study of deterministic chaos in nonlinear physics is now a well-developed branch of science.[5,18,19] In particular, the behavior of strange attractors in chaotic dynamical systems has been investigated extensively. It has come to be recognized that the characterizations of the attractors, as well as many other similar objects in dissipative systems, can best be done in terms of fractal measures.[20] The most notable property of fractals is their dimensions.[21] A formalism for treating the fractal dimension and its generalization has recently been developed,[21–25] and has been applied effectively to the study of intermittent behavior in turbulent fluids and other transitions to chaos, such as through period doubling and quasiperiodicity. We shall adapt that formalism for the treatment of our problem in multiparticle production.

It should be mentioned that there is a sharp distinction between the nature of problems in deterministic chaos and the one that we encounter in particle production. In the former case the problem is usually set up in the form of a set of coupled, nonlinear differential equations. One attempts to find efficient ways to solve the problem and to present the nature of the solution and its critical properties in some compact and, if possible, universal form.

The problem is essentially deductive. In the case of multiparticle production, on the other hand, the dynamics is not known well enough to render the problem one of deductive mathematics. As mentioned above there are phenomenological hints for intermittent behavior. It is therefore mainly an inductive problem to discover the proper dynamics that can give rise to the chaotic structure. To study the fractal measure of multiparticle production is only the initial stage, in which the data are organized in a manner suitable for inductive exploration.

The first ones to investigate the fractal dimension in hadronic multiparticle production were probably Carruthers and Minh.[26] More recently, Dremin[27] suggested the study of correlation dimension; Lipa and Buschbeck[28] considered other generalized dimensions. In none of those investigations is a formalism developed for a systematic study of the fractal properties. In this paper we make a beginning by first identifying the moments G_q that can be determined from existing and future data on particle multiplicities in narrow rapidity windows. We point out the importance of being careful in the averaging process so that interesting signatures do not get averaged out. From the meager data that are available and relevant to G_q, we draw some interference on the nature of the dimensions D_q that are generalizations of the fractal dimensions for multifractal sets.[22] We then discuss the general properties of the spectrum of scaling indices and indicate how it can provide an effective means of describing a highly nonuniform rapidity distribution.

II. MULTIFRACTAL STRUCTURE IN RAPIDITY DISTRIBUTION

Consider the collision of two systems, each of which may be a lepton, a hadron, or a nucleus. Let $\mathcal{N}(y)$ be the rapidity distributions of particles produced in an event at high energy. We focus our attention on a narrow window of width δ. The averaging over many similar windows is a process that can be considered later for the purpose of increasing the experimental statistics, but should be put aside for the present. Let k denote the number of particles detected in δ in one event. Obviously, k will fluctuate from event to event. Let Q_k be the number of times that the multiplicity k recurs in N events. Note that Q_k is not normalized, so it is not a distribution. Define

$$P_k = Q_k / \mathsf{N} \ , \tag{2.1}$$

where

$$\mathsf{N} = \sum_{k=0}^{\infty} Q_k \ . \tag{2.2}$$

The normalized moments are

$$C_q = \sum_{k=0}^{\infty} k^q P_k \Bigg/ \left[\sum_{k=0}^{\infty} k P_k \right]^q , \quad q > 0 \ . \tag{2.3}$$

These moments have been determined in various experiments for a variety of collisional processes. Their results can be found in some recent reviews.[1,2,29] Care must, however, be exercised in recognizing the averaging processes used in enhancing the statistics.

As δ gets smaller, P_0 increases. Let us exclude $k=0$ in the sum of (2.2) and define

$$N = \sum_{k=1}^{\infty} Q_k \ , \tag{2.4}$$

$$K = \sum_{k=1}^{\infty} k Q_k \ . \tag{2.5}$$

We now suggest the following procedure of data analysis. Let K be fixed at some large value. As δ is decreased, increase the number of events N to keep K constant. Determine, for every δ, a new set of moments

$$G_q = \sum_{k=1}^{\infty} k^q Q_k / K^q \ . \tag{2.6}$$

Note that while C_q is invariant under changes in N, when N is large and δ is fixed, the same is not true for G_q. We shall investigate the dependence of G_q on δ, while K is held fixed. Note also that (2.6) permits q to be negative, a possibility that is denied of (2.3). This will provide a wider range of probing at the properties of Q_k.

To study the characteristics of G_q, let us label the events by an index i, $i=1,2,\ldots,N$, ignoring those events in which $k=0$. Let k_i denote the number of particles detected in δ in the ith event. Defining

$$p_i = k_i / K \tag{2.7}$$

we have

$$\sum_{i=1}^{N} p_i = 1 \ . \tag{2.8}$$

This is another way of expressing (2.4) and (2.5). Evidently, as δ decreases, p_i also decreases, so N must increase to preserve (2.8). Let us now quantify the dependence of p_i on δ. That dependence in turn depends on $\mathcal{N}_i(y)$, the rapidity distribution of the ith event, since

$$k_i = \int_{y_0}^{y_0+\delta} dy \, \mathcal{N}_i(y) \ , \tag{2.9}$$

where y_0 is the position of the left edge of the window. Suppose

$$p_i \propto \delta^{\alpha} \tag{2.10}$$

for some scaling index α, when δ is small. Now collect all the events that have the same scaling property, and call it the set S_r. More precisely, S_r is the set of events in which p_i behaves as in (2.10) with α in the range $\alpha_r \le \alpha \le \alpha_r + d\alpha$ for some α_r. Thus the summation in (2.8) may be divided into two sums:

$$\sum_{i=1}^{N} = \sum_{r} \sum_{i \in S_r} = N \ , \tag{2.11}$$

where the sum over r exhausts all possible values of α. Let the number of elements in S_r be N_r. As δ decreases, N_r generally increases. Let us denote that inverse relationship by

$$N_r \propto \delta^{-f(\alpha_r)} \ . \tag{2.12}$$

The set S_r constitutes a fractal measure, and $f(\alpha_r)$ is the fractal dimension. By breaking up α into many segments $[\alpha_r, \alpha_r + d\alpha]$, we have partitioned N into many fractals.

Since the set S_r is associated with a particular incremental segment of α values, the sum in r is equivalent to an integration in α:

$$\sum_r = \int d\alpha\, h(\alpha) , \tag{2.13}$$

where $h(\alpha)$ is some continuous function, the exact nature of which is not important. From (2.11)–(2.13) we have

$$\sum_{i=1}^{N} \propto \int d\alpha\, h(\alpha)\delta^{-f(\alpha)} . \tag{2.14}$$

On the other hand, from (2.4), (2.6), and (2.7) we have

$$G_q = \sum_{i=1}^{N} p_i^q . \tag{2.15}$$

Combining (2.14) and (2.15), and using (2.10), we get

$$G_q \propto \int d\alpha\, h(\alpha)\delta^{-f(\alpha)+q\alpha} , \tag{2.16}$$

when δ is small. Let the resultant δ dependence of G_q be

$$G_q \propto \delta^{\tau(q)} , \tag{2.17}$$

where the properties of $\tau(q)$ will be discussed in Sec. IV. $\tau(q)$ is related to the dimension D_q, introduced by Hentschel and Procaccia[22] for all q, by

$$\tau(q) = (q-1)D_q . \tag{2.18}$$

D_0 is the fractal dimension, D_1 the information dimension, and D_2 the correlation dimension.[21,23] We note that the moments G_q themselves cannot easily be compared among the experiments, since they depend on the number of events N of the sample and on δ. However, it is the δ dependence of G_q that is the real goal of studying G_q; in particular, experimental analysis should aim to extract the dimensions D_q, where

$$D_q = (q-1)^{-1} \lim_{\delta\to 0} (\ln G_q / \ln\delta) . \tag{2.19}$$

Some comments are in order concerning the limit $\delta \to 0$ in (2.19). It should be noted that the behavior of (2.10), (2.12), (2.16), and (2.17) are all for small δ, but not in the exact mathematical limit of $\delta \to 0$. It is in that same sense that (2.19) should be understood. The reason is that neither p_i nor G_q vanish in the mathematical limit $\delta \to 0$ when K is held fixed. That is because at fixed K not all events can have $k=0$, but as δ becomes ever smaller, the only nonzero-k events are $k=1$ events. Thus the minimum p_i is $1/K$, which does not vanish. If an experiment has good resolution so that the bin size can be reduced below the level necessary to separate individual particles, then the minimum value of p_i must be taken into account, so (2.10) should be modified to become

$$p_i = \beta\delta^{\alpha} + K^{-1} . \tag{2.20}$$

The fractal analysis presented here is useful only if the K^{-1} term is smaller than the $\beta\delta^{\alpha}$. That means that the dynamical system must exhibit self-similarity at scales greater than the absolute minimum, which is not an unreasonably severe restriction. If we did not require K to be fixed, but let N, say, be fixed, then p_i would behave as N^{-1} for a much wider range of small δ [a range in which K^{-1} would have been negligible in (2.20) for fixed K] because for fixed N, K would decrease with δ proportionally. In that case, fractal behavior will not emerge from an investigation of p_i, which is the key link with the conventional multifractal analysis.

In recognition of the existence of a minimum value of p_i at small δ, a phenomenological formula for G_q that can provide a more meaningful way of extracting $\tau(q)$ from high-resolution experiments would be

$$G_q = \gamma_q(K)\delta^{\tau(q)} + K^{1-q} \tag{2.21}$$

instead of (2.17). Since K, δ, and q are within experimental control, it should be possible to determine $\tau(q)$ by using (2.21) to fit the experimental moments G_q. For $q>1$, one should plot $\ln G_q^{-1}$ vs $\ln\delta^{-1}$; a linear portion before a flat asymptote is reached gives the (positive) slope $\tau(q)$. For $q<0$, one should plot $\ln G_q$ vs $\ln\delta^{-1}$; a rising linear portion is also expected before a plateau is reached, yielding a negative value for $\tau(q)$.

Data on multiparticle productions at high energies have not so far been analyzed to yield G_q directly. In Sec. III we shall examine some features of G_q in an indirect way. From those features we can infer some properties of D_q. But before so doing, let us consider the problem, when an unlimited supply of events is not available.

In cosmic-ray experiments there are only a few events to study, although the number of particles produced can be huge. To study the structure of rapidity distribution for one event, let us follow the convenient nomenclature of Ref. 15, and refer to it as "horizontal" analysis, in contradistinction from the consideration given above, which shall be called "vertical" analysis, i.e., the study of multiplicities in a fixed rapidity interval δ for many events.

In a horizontal analysis it is actually easier to visualize the fractal structure in one event, assuming that a large number of particles are produced in that event so that the rapidity distribution $\mathcal{N}(y)$ is rich in peaks and valleys. Select a sizable rapidity interval Y in the central region, and divide Y into M bins, each of width δ. As δ is decreased, M increases as Y/δ. Let Q_k now denote the number of bins in which k particles are detected. We ignore the bins with $k=0$. Define

$$M = \sum_{k=1}^{\infty} Q_k \tag{2.22}$$

so that M is the number of bins with at least one particle in each bin. Let K retain the same definition as in (2.5), but now it denotes the total number of particles in Y. With these changes in interpretation, the procedure specified above for vertical analysis can now be carried over to horizontal analysis with essentially no change in notation, except for $\mathsf{N} \to \mathsf{M}$, $N \to M$, and $N_r \to M_r$. Thus, if we use j to label a bin, then p_j is the probability of finding a particle in bin j. The scaling behavior (2.10) then associates a small value of α with a bin in which

$\mathcal{N}_j(y)$ has a peak, while a valley is associated with a high value of α. It is clear then that if $\mathcal{N}(y)$ were approximately constant for all y in Y, then the spectrum of α would be sharply peaked at 1, and there would be only one set S_r of interest, which includes all bins, $j=1,\ldots,M$. In that case we have $f(\alpha)=1$, which is the topological dimension of the rapidity space. Evidently, the interesting part of a rapidity distribution that has spikes and troughs would reveal itself in a nontrivial spectrum $f(\alpha)$, the determination of which is the objective in the search for universal features in multiparticle production processes.

Since the moments determined in the two procedures are different, let us, in distinguishing the two, use the notation $G_q^{(v)}$ and $G_q^{(h)}$ for the moments obtained from vertical and horizontal analysis, respectively. Physically, they are different because they involve different collisional circumstances. Vertical analysis involve collisions with different impact parameters, while in a horizontal analysis there is only one collision with one particular impact parameter and one specific process of particle production. However, at laboratory energies where the multiplicity is not high, it is necessary at small δ to enhance the statistics by supplementing the vertical analysis with horizontal averaging, or vice versa. If we use the superscript v to also denote a bin label, then the horizontal averaging of $G_q^{(v)}$ is

$$\langle G_q^{(v)}\rangle \equiv \frac{1}{\mathsf{M}}\sum_{v=1}^{\mathsf{M}} G_q^{(v)} . \tag{2.23}$$

On the other hand, vertical averaging of horizontal moments is

$$\langle G_q^{(h)}\rangle \equiv \frac{1}{\mathsf{N}}\sum_{h=1}^{\mathsf{N}} G_q^{(h)} , \tag{2.24}$$

where h runs over all events. Except for $q=1$, we have, in general,

$$\langle G_q^{(v)}\rangle \neq \langle G_q^{(h)}\rangle . \tag{2.25}$$

This can most easily be seen by an example. Consider the hypothetical situation that there is one event in which the multiplicity of particles produced is vastly larger than the average multiplicity, say ten times. Then $\langle G_q^{(v)}\rangle$ at high q would be dominated by the characteristics of that event. But $G_q^{(h)}$ for that event would not particularly stand out from those of other events, since p_i for horizontal analysis is normalized by the total multiplicity of an event. Thus $\langle G_q^{(h)}\rangle$ is likely to retain very little of the details about that super event. An extreme case would be one where $\mathcal{N}(y)$ is perfectly smooth for every event, but the height of $\mathcal{N}(y)$ can vary from event to event. In that case $G_q^{(h)}$ would have nothing interesting, but $G_q^{(v)}$ would contain the effects of multiplicity fluctuation. From this example, it is clear the $G_q^{(v)}$ and $\langle G_q^{(v)}\rangle$ are generally the quantities to investigate, if rare events of special features are not to be missed. On the other hand, $G_q^{(h)}$ for the special events, would contain the maximum information about those events, before it is washed out by the averaging process in getting $\langle G_q^{(h)}\rangle$.

III. PHENOMENOLOGY OF D_q

Although the moments G_q have not yet been determined by direct analysis of the experimental data, we can extract some aspects of their properties by examining the usual normalized moments C_q. From (2.3) we have, for vertical analysis,

$$\begin{aligned} C_q^{(v)} &= \left[\sum_{k=0}^{\infty} k^q Q_k \Big/ \left[\sum_{k=0}^{\infty} kQ_k \right]^q \right] \mathsf{N}^{q-1} \\ &= G_q^{(v)}\mathsf{N}^{q-1}, \quad q>0 , \end{aligned} \tag{3.1}$$

where the quantity in the square brackets can be identified with $G_q^{(v)}$ only if q is positive so that the $k=0$ term is immaterial. We assume that N is large enough so that $G_q^{(v)}$ can be defined for some large K. Since Q_0 increases with the decrease of δ, we can expect a scaling behavior

$$N/\mathsf{N} \propto \delta^{d_0} \tag{3.2}$$

for some positive exponent d_0. From (2.4), (2.17), and (2.18), we have, in the limit of small δ,

$$G_0^{(v)} = N \propto \delta^{\tau(0)} = \delta^{-D_0} . \tag{3.3}$$

Consequently, if we denote the δ dependence of N by

$$\mathsf{N} \propto \delta^{-\mathsf{D}_0} , \tag{3.4}$$

then we have

$$\mathsf{D}_0 = D_0 + d_0 . \tag{3.5}$$

Let us now consider (3.1) in the limit $\delta\to 0$, $N\to\infty$, with K fixed. Since it also means $\mathsf{N}\to\infty$, under which $C_q^{(v)}$ is invariant, we obtain

$$C_q^{(v)} \propto \delta^{\tau(q)-(q-1)\mathsf{D}_0} = \delta^{(q-1)(D_q-\mathsf{D}_0)}, \quad q>0 . \tag{3.6}$$

Thus the exponents can be determined from a log-log plot of $C_q^{(v)}$ vs δ for various values of $q>1$. However, unless D_0 is known independently, the value of D_q cannot be determined from (3.6) without an unknown constant shift.

The available data on the slopes of the log-log plots are not for C_q, but for the normalized factorial moments:

$$\langle F_l^{(h)}\rangle = \langle k\rangle^{-l} \left\langle \frac{1}{\mathsf{M}} \sum_{j=1}^{\mathsf{M}} k_j(k_j-1)\cdots(k_j-l+1) \right\rangle , \tag{3.7}$$

where $\langle k\rangle$ is the grand average (horizontal and vertical) multiplicity per bin, with $k=0$ taken into account in the normalization of the multiplicity distribution. From the power-law behavior of the δ dependences of those moments, one cannot infer the exponents for $\langle C_q^{(h)}\rangle$. Thus we cannot make use of, for example, either the results of the EHS/NA22 Collaboration[10] on π^+p and K^+p collisions at 250 GeV/c, or those of the Krakow-Louisiana-Minnesota Collaboration[11] on nuclear emulsion, both of which have observed power-law behavior of the factorial moments. The UA5 Collaboration[30] has published data on $C_q^{(v)}$ for various δ, but unfortunately their minimum value of δ is only 0.4, which is not small enough for our own purpose. The same is true with the data of the EHS/NA22 Collaboration,[31] whose minimum is δ is 0.5. I have not seen the data of WA80 or NA35 on C_q, although both groups claim to have found power-law behavior of F_l (Ref. 5).

Fortunately, the Buffalo group[32] has data from their emulsion experiment at the CERN SPS, in which they have determined $C_q^{(v)}$ for a wide range of δ. However, contrary to the usual definition of C_q as given in (2.3), the event used in Ref. 32 for the determination of C_q exclude those that have no particles detected in the narrow rapidity window;[33] i.e., each event has at least one particle in δ. Since they are not the usual C_q, let us refer to them as C_q^{Buff}, which is then related to $G_q^{(v)}$ by

$$C_q^{\text{Buff}}=G_q^{(v)}N^{q-1}, \quad q>0 \tag{3.8}$$

instead of (3.1). It then follows from (2.17), (2.18), and (3.3) that

$$D_0-D_q=\frac{1}{q-1}\frac{\Delta \ln C_q^{\text{Buff}}}{\Delta \ln(1/\delta)} . \tag{3.9}$$

The data on C_q^{Buff} for ^{16}O-emulsion collisions are shown in Fig. 1 in log-log plot for 60 and 200 GeV/n. Because of the uncertainty at very small δ, we have used wedges to fit the low-δ regions. The resultant values of D_0-D_q vs q are shown in Fig. 2. Despite the large error bars, we can identify the following features. Smooth extrapolations of the data points in Fig. 2 to $q=0$ are consistent with zero within the error bars, as D_0-D_q should become. In the absence of an independent determination of D_0, we can only conclude that D_q is a function that decreases monotonically from some value (probably $\lesssim 1$), as

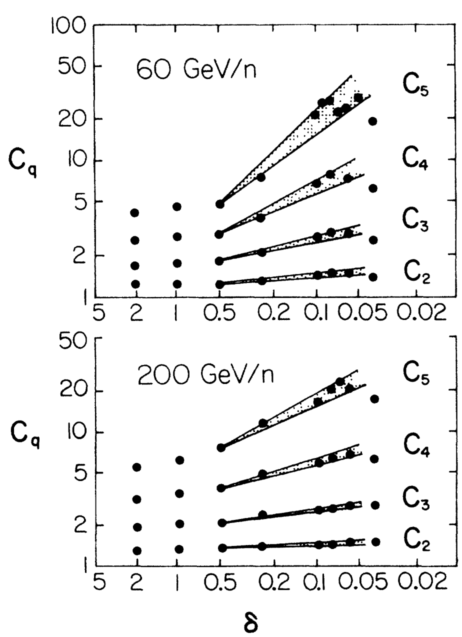

FIG. 1. Normalized moments C_q^{Buff} of Ref. 32 are denoted by C_q here for brevity. The data are for ^{16}O-emulsion collisions at 60 GeV/nucleon and 200 GeV/nucleon.

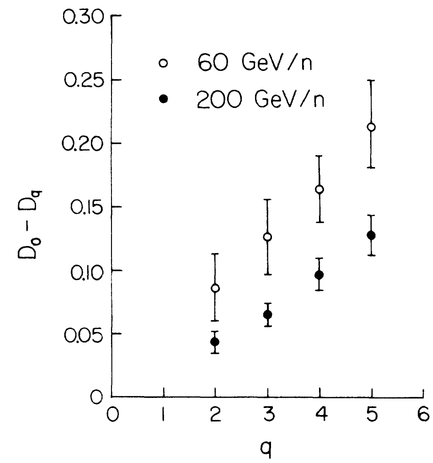

FIG. 2. Data points for D_0-D_q vs q, as determined from Fig. 1.

q increases from 0. While this rough feature of D_q is sufficient for our use in the following section to deduce some qualitative properties of $f(\alpha)$, we urge the experimentalists to establish directly the exact nature of D_q simply by determining the moments G_q for small values of δ and applying (2.19).

IV. THE SPECTRUM $f(\alpha)$

From the phenomenological analysis done in the preceding section, we have obtained a rough indication of the nature of D_q. A direct evaluation of G_q can not only lead to a determination of the absolute normalization of D_q, but also extend the range of q to negative values. After a firm determination of D_q over a wide and continuous range of q is obtained, the properties of the spectrum $f(\alpha)$ can then be derived, as we now indicate.

By the saddle-point method one can obtain the conditions under which (2.17) follows from (2.16) (Refs. 21, 24, and 25). When δ is small, the dominant part of the integral in (2.16) arises from the region where the exponent is at the minimum. For every value of q, the corresponding value of α (call it α_q) where the minimum occurs must be such that the following conditions are satisfied:

$$\frac{d}{d\alpha}[q\alpha-f(\alpha)]|_{\alpha=\alpha_q}=0 , \tag{4.1}$$

$$\frac{d^2}{d\alpha^2}[q\alpha-f(\alpha)]|_{\alpha=\alpha_q}>0 . \tag{4.2}$$

Thus by evaluating the integrand of (2.16) at α_q where

$$\frac{d}{d\alpha_q}f(\alpha_q)=q , \tag{4.3}$$

$$\frac{d^2}{d\alpha_q^2}f(\alpha_q)<0 \tag{4.4}$$

we get (2.17) with

$$\tau(q)=q\alpha_q-f(\alpha_q) \tag{4.5}$$

in the limit of small δ. From (4.5) we also have

$$\alpha_q = \frac{d}{dq}\tau(q) \ . \tag{4.6}$$

Consequently, once $\tau(q)$ is determined from (2.19), we can obtain $f(\alpha)$ from

$$f(\alpha_q) = q\frac{d\tau(q)}{dq} - \tau(q) \ . \tag{4.7}$$

We now see the importance of the experimental determination of the values of $\tau(q)$ or D_q for small incremental changes of q, since they provide the phenomenological avenue to the spectrum of indices $f(\alpha)$.

It follows from (2.18) and (4.5) that

$$f(\alpha_0) = D_0 \tag{4.8}$$

and from (4.3) that

$$f'(\alpha_0) = 0 \ . \tag{4.9}$$

Thus, $f(\alpha)$ is a function that has a maximum at α_0, and because of (4.4) $f(\alpha)$ is concave downward everywhere. At $q=1$ we have

$$f(\alpha_1) = \alpha_1, \quad f'(\alpha_1) = 1 \ . \tag{4.10}$$

A positive slope means that $\alpha_1 < \alpha_0$. Indeed, with D_q expected to decrease with increasing q, it follows from (4.6) that α_q also decreases with increasing q. On the other hand, the region $\alpha > \alpha_0$ corresponds to negative q, for which we have no hint from the phenomenological analysis of Sec. III.

For the purpose of visualization only and without any claim to accuracy, a sketch of $f(\alpha)$ is shown in Fig. 3. Different values of q probe different regions of this spectrum. Large positive q emphasizes the high k of Q_k in (2.6) and therefore selects the low-α side of the spectrum, which describes the peaks in $\mathcal{N}(y)$, according to (2.9) and (2.10). The reverse is true for negative q. That $f(\alpha)$ is not sharply peaked at α_0 reveals the fact that $\mathcal{N}(y)$ is not smooth in y from event to event.

As energy increases, we see from Fig. 2 that D_q increases for all values of q. It then follows from (4.6) and (4.8) that both α_0 and $f(\alpha_0)$ increase also. We surmise that the whole spectrum $f(\alpha)$ increases and broadens. It implies that the rapidity distribution $\mathcal{N}(y)$ becomes more jagged and irregular, with increasing occurrence of sharp peaks and deep valleys. What is most interesting about the analysis considered here is that the highly chaotic behavior of $\mathcal{N}(y)$ can be described by a smooth function $f(\alpha)$. Note that if we average $\mathcal{N}(y)$ over all events, we get a smooth rapidity distribution dN/dy, which is devoid of any useful information about fluctuations and intermittency. Note also that if the experimental resolution is not good enough to reveal the sharp peaks and deep valleys, the consequence on $f(\alpha)$ is that it would appear narrower than it would otherwise be.

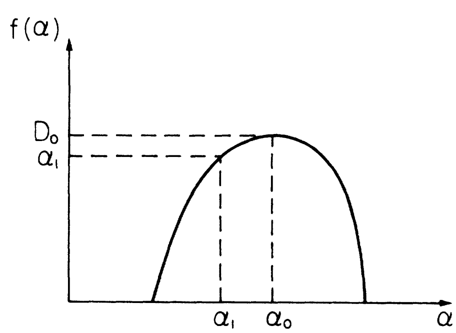

FIG. 3. A schematic sketch of $f(\alpha)$.

V. CONCLUSION

To analyze the properties of fluctuation in multiparticle production in the multifractal formalism will no doubt open up a new area of investigation in high-energy particle and nuclear physics. It is regrettable that the active era of hadronic experiments on the subject seems to have passed, but fortunately the experiments in relativistic heavy-ion collisions can still be designed to take advantage of this new development.

What we have done here is only the beginning. We have identified the moments G_q that should be measured. From those moments the dimensions D_q can directly be extracted, and then from D_q the spectrum $f(\alpha)$ can be determined. $f(\alpha)$ is the smooth function that characterizes the fluctuation of the rapidity distribution $\mathcal{N}(y)$. Phenomenologically, we need to determine all aspects of $f(\alpha)$: its dependences on energy, particle and nuclear types, impact-parameter selection, fragmentation versus central region, etc.

Theoretically, there is a vast and uncharted territory to explore. We need to relate $f(\alpha)$ that describes the inhomogeneity in rapidity to temporal intermittency. In particle physics one usually does not consider the temporal development in the particle production process. In relativistic heavy-ion collisions the evolution of the dense hadronic or quark matter is treated in hydrodynamics, but since only average quantities are considered in that approach, the intricacy of the chaotic behavior of strong interaction is not the subject to scrutinize, but to ignore. Indeed, one would be forced by the experimental determination of $f(\alpha)$ to consider the temporal evolution of the particle production process at the parton level in order to establish a dynamical connection between parton interaction and the onset of chaos. The spectrum of $f(\alpha)$ and its temporal counterpart in intermittency constitute the goal that a successive model must aim at and be able to explain.

The relationship between multifractal objects discussed here and the intermittency considered in Ref. 6 is not at all clear. The similarity between (1.1) and (2.17) is interesting, but largely superficial at this point. In what sense is the power-law behavior of (1.1) an indication of intermittency is not well understood. In fact, even the definition of intermittency is not unique, especially in this subject in which the dynamics is not well specified, nor commonly agreed upon. Nevertheless, the impact that Ref. 6 has made is in leading to the discovery of an area where experimental data can rule out certain theoretical models. Apparently, the power-law behavior is a natural consequence of processes involving multiplicative random samplings, such as in a cascade or branching model.[14,17] In such models, temporal evolution is an essential part of the dynamics. Thus it is important to check whether they can give rise to temporal intermittency as well as reproduce the spectrum $f(\alpha)$ that characterize the rapidity fluctuation.

It would not be too far fetched to conjecture here that this line of investigation is likely to be very fruitful in identifying signatures for phase transition between quark-gluon plasma and hadron gas. When latent heat is converted to kinetic energy, the probability of having turbulent motion is greatly enhanced. Although the onset of turbulence is unpredictable, the properties of the system that can support such random bursts of strong chaoticity are precisely what one learns by studying intermittency. The concomitant effect on the rapidity distribution will undoubtedly be such as to render it more chaotic and therefore to broaden $f(\alpha)$. Since phase transitions may take place in one collision but not the next (even if the total number of particles produced are the same), one must be very cautious in averaging over events. The types of analyses that we have described in Sec. II are therefore well suited to deal with the problem. The basic points of this whole approach, as far as the interplay between experiment and theory is concerned, are to find ways to register rare processes that can be important, to characterize chaotic features in an analytically manageable formalism, and to infer from those features the dynamical process that is responsible for them. It seems that all these attributes are the necessary components of an effective program to identify the unique signatures of phase transitions. Furthermore, regardless of whether phase transition actually occurs, this approach offers a very promising method to learn at greater depth the hadronization process in high-energy collisions.

ACKNOWLEDGMENTS

I am grateful to K. Fialkowski for a critical communication. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG06-85ER-4022.

[1] *Hadronic Multiparticle Production,* edited by P. Carruthers (Directions in High Energy Physics, Vol. 2) (World Scientific, Sinagpore, 1988).

[2] *Multiparticle Production,* Proceedings of the Shandong Workshop, Jinan, China, 1987, edited by R. C. Hwa and Q.-B. Xie (World Scientific, Singapore, 1988).

[3] *Quark Matter '88,* proceedings of the Seventh International Conference on Ultrarelativistic Nucleus-Nucleus Collisions, Lenox, Massachusetts, 1988, edited by G. Baym, P. Braun-Munzinger, and S. Nagamiya [Nucl. Phys. **A498** (1989)].

[4] W. Kittel, in *Proceedings of the 24th International Conference on High Energy Physics,* Munich, West Germany, 1988, edited by R. Kotthaus and J. H. Kuhn (Springer, Berlin, 1989), p. 625.

[5] I. Derado, in *Hadronic Matter in Collision,* Tucson, Arizona, 1988, edited by P. Carruthers and J. Rafelski (World Scientific, Singapore, 1989), p. 222.

[6] A. Bialas and R. Peschanski, Nucl. Phys. **B273**, 703 (1986).

[7] Japanese-American Cooperative Emulsion Experiment, T. H. Burnett *et al.* Phys. Rev. Lett. **50**, 2062 (1983).

[8] B.-L. Hao, *Chaos* (World Scientific, Singapore, 1984).

[9] B. Buschbeck, R. Lipa, and R. Pechanski, Phys. Lett. B **215**, 788 (1988).

[10] NA22 Collaboration, I. V. Ajinenko *et al.*, Phys. Lett. B **222**, 306 (1989).

[11] R. Holynski *et al.* Phys. Rev. Lett. **62**, 733 (1989).

[12] A. Bialas and R. Peschanski, Phys. Lett. B **207**, 59 (1988).

[13] A. Bialas and R. Peschanski, Nucl. Phys. **B308**, 857 (1988).

[14] W. Ochs and J. Wosiek, Phys. Lett. B **214**, 617 (1988).

[15] P. Carruthers, E. M. Friedlander, C. C. Shih, and R. M. Weiner, Phys. Lett B **222**, 487 (1989).

[16] L. Van Hove, Ann. Phys. (N.Y.) **192**, 66 (1989).

[17] R. C. Hwa, Nucl. Phys. **B328**, 59 (1989).

[18] P. Berge, Y. Pomeau, and C. Vidal, *Order within Chaos* (Wiley, New York, 1984).

[19] P. Cvitanovic, *Universality in Chaos* (Adam Hilger, Bristol, 1984).

[20] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1977).

[21] G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).

[22] H. G. E. Hentschel and I. Procaccia, Physica **8D**, 435 (1983).

[23] P. Grassberger and I. Procaccia, Physica **13D**, 34 (1984).

[24] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[25] T. C. Halsey and M. H. Jensen, Physica **23D**, 112 (1986).

[26] P. Carruthers and Minh Duong-Van, Los Alamos Report No. LA-UR-83-2419 (unpublished).

[27] I. M. Dremin, Mod. Phys. Lett. A **3**, 1333 (1988).

[28] R. Lipa and B. Buschbeck, Phys. Lett B **223**, 465 (1989).

[29] *Multiparticle Dynamics,* proceedings of the Workshop, Perugia, Italy, 1988, edited by R. C. Hwa, G. Pancheri, and Y. Srivastava (World Scientific, Singapore, 1989).

[30] UA5 Collaboration, R. E. Ansorge *et al.*, Z. Phys. C **43**, 357 (1989).

[31] NA22 Collaboration, M. Adamus *et al.*, Z. Phys. C **37**, 215 (1988).

[32] K. Sengupta, G. Singh, and P. L. Jain, Phys. Lett. B **213**, 548 (1988). The data points for C_5 were provided by private communication.

[33] I am grateful to K. Fialkowski for pointing this out to me.

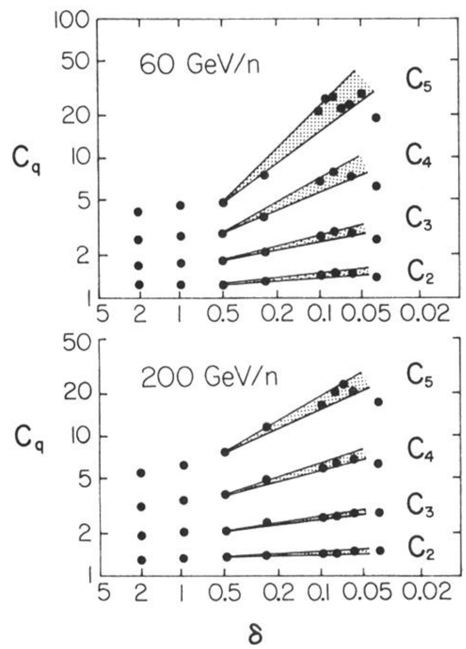

FIG. 1. Normalized moments C_q^{Buff} of Ref. 32 are denoted by C_q here for brevity. The data are for ^{16}O-emulsion collisions at 60 GeV/nucleon and 200 GeV/nucleon.