Nonlinear constraint in scale-invariant theories

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We show the existence of a nonlinear constraint satisfied by fields in scale-invariant theories. Our model consists of a dilaton field and additional scalar fields in curved space-time. Using the constraint one of the fields becomes auxiliary, decreasing the number of physical degrees of freedom by 1. Implementation of the constraint in a Weyl-transformed scale-invariant model generates exponential interactions from polynomial potentials.

Scale invariance in field-theoretical models has a long-standing history.[1] Scale- (dilatational-) invariant theories were originally envisaged as a device to provide for a field-dependent evolution of dimensional coupling constants, thereby reducing them to "no-scale" theories. Dilational invariance is achieved by means of the addition of an extra scalar field, the dilaton. In more modern approaches dilatons emerge as an inherent part of higher-dimensional unified theories, in which they play the role of the radius of the compactified manifold and/or a remainder of a spontaneously broken conformal invariance.[1] The particular feature that leads to the common denomination stems from their peculiar couplings to gravity and matter fields. There are nevertheless higher-dimensional theories, such as "no-scale supergravity models,[3]" in which the dilaton originates solely from the spontaneous breaking of scale invariance.

Four-dimensional, scale-invariant effective theories in the standard cosmological context were extensively studied by Wetterich.[4] He concluded that the existence of the dilaton may indeed be compatible with standard cosmology. In a previous work we have investigated a scale-invariant theory of inflation,[5] finding special characteristics, such as a two-phase evolution of the cosmological scale factor. Scale invariance was also called upon in trying to solve the cosmological-constant problem,[6] but it proved unsuccessful when loop corrections were included in the calculations.[7]

Scale invariance holds only at the classical level as it is broken at the quantum level by anomalies.[1,6] This explicit symmetry breaking renders the dilaton massive; the vacuum expectation value of the dilaton fixes the masses and coupling constants, providing us with some means to obtain testable effects.

In what follows we will show that in a broad class of cosmological applications, scale invariance implies a nonlinear condition in terms of the fields, yielding a compactified field manifold. By use of this constraint we can remove one degree of freedom, for example, the dilaton field itself.

The field theory we are considering is given by the action

$$I=\int d^4x\sqrt{-g}\left[-\frac{1}{16\pi GM^2}\chi^2R+\tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi+\tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i -V(\chi,\phi_i)\right], \tag{1}$$

where $\chi=Me^{S/M}$, S being the dilaton field. M is a mass scale, usually taken to be of the order of the Planck mass, ϕ_i are N scalar ("matter") fields and R is the Ricci scalar. For simplicity we will assume in the following $N=1$ and drop the index on ϕ. The most general scale-invariant potential is given by[4]

$$V=v(\phi/\chi)\phi^4\ , \tag{2}$$

where v is an arbitrary function of ϕ/χ.

The action of Eq. (1) is invariant under the global scale transformations

$$g_{\mu\nu}\to e^{-2\alpha}g_{\mu\nu},\quad \phi\to e^{\alpha}\phi,\quad \chi\to e^{\alpha}\chi\ , \tag{3}$$

for α space-time independent. After having exploited the equations of motion, the corresponding conserved Noether current reduces to

$$j^\mu=h\chi\partial^\mu\chi+\phi\partial^\mu\phi \tag{4}$$

with $h=1+3/4\pi GM^2$. The conservation equation for j^μ may be recast in the form of a wave equation for $\psi\equiv h\chi^2+\phi^2$: namely,

$$\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\psi)=0\ . \tag{5}$$

For time-independent fields Eq. (5) becomes the Laplace equation in curved space. This equation can be easily solved for fields tending asymptotically at spatial infinity to their vacuum expectation value, $\psi\to$const, and for asymptotically flat space. With these boundary conditions, the unique solution of the Laplace equation is $\psi=$const everywhere: i.e.,

$$h\chi^2+\phi^2=K\ , \tag{6}$$

where K is the constant value of ψ. This nonlinear constraint for the fields applies also to the exterior solution of a static black hole. (Bekenstein[8] has proven that the unique solution of the Laplace equation in the black-hole background, assuming regularity of the scalar field at the horizon in addition to our above-mentioned boundary conditions at spatial infinity, is indeed a constant.) Moreover, the same constraint is also satisfied for time-dependent, but space-*independent* fields, as long as these settle down at their vacuum expectation values at future infinity. However, for general space-time-dependent fields, this relation cannot be established.

Equation (6) is the main result of the present work. Scale invariance forces the fields to exist in a compact manifold: namely, the sphere in $N+1$ dimensions (N being the number of matter-scalar fields) with radius $K^{1/2}$. K constrains the vacuum expectation values of the various scalar fields; spontaneous symmetry breaking of scale invariance occurs whenever $K\neq 0$. For complex scalar fields the squares in the nonlinear constraint are replaced by the moduli of the fields squared. Furthermore, the addition of minimally coupled gauge fields does not change the constraint.

In the cosmological model of a homogeneous universe, the primordial quantum phase will determine K, constraining uniquely the classical evolution of the fields in the subsequent epoch. In the special case of scale-invariant (homogeneous) inflation of Ref. 5 we have indeed verified the constraint on the actual numerical solutions.

We now implement the constraint in the action and reduce the number of degrees of freedom by 1. Note that this will yield a theory which is *a priori* only correct for a specific subgroup of all solutions: namely, those with the above-mentioned boundary conditions. First we cast the theory in a form with fixed Newton's constant. As it is commonly known, this is accomplished by the following Weyl transformation on the metric:[4,5]

$$g_{\mu\nu}\rightarrow\tilde{g}_{\mu\nu}=e^{2S/M}g_{\mu\nu}\ ,$$

obtaining ($\tilde{\phi}=\phi e^{-S/M}$)

$$\tilde{I}=\int d^4x\sqrt{-\tilde{g}}\left[-\frac{1}{16\pi G}R(\tilde{g})+\frac{1}{2}\left[h+\frac{\tilde{\phi}^2}{M^2}\right]\tilde{g}^{\mu\nu}\partial_\mu S\partial_\nu S+\tfrac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}+\frac{\tilde{\phi}}{M}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu S-\tilde{V}\right]\ . \qquad (7)$$

The rescaled potential becomes $\tilde{V}=v(\tilde{\phi}/M)\tilde{\phi}^4$.

We have called the action of Eq. (1) "manifestly scale invariant"[9] (MSI), whereas the transformed one, Eq. (7), is denominated "hidden scale invariant" (HSI). Both MSI and HSI theories are equivalent at the classical level, provided the Weyl transformation does not become singular. At the quantum level the theories differ: For example, we have seen in the treatment of dilaton stars[9] that massive stellarlike objects made of dilaton particles can exist in the MSI theory (in the quantum mean-field approach), but are absent in the HSI version. However, we here focus only on the classical aspects of the theories.

By defining $u=\sqrt{h}\,M\,\mathrm{arcsinh}(\tilde{\phi}/\sqrt{h}\,M)$ and substituting S as a function of $\tilde{\phi}$ as derived from the constraint, namely, $S=-\frac{1}{2}M\ln(h+\tilde{\phi}^2/M^2)+\mathrm{const}$, the HSI action of Eq. (7) transforms to

$$\tilde{I}_u=\int d^4x\sqrt{-\tilde{g}}\left[-\frac{1}{16\pi G}R(\tilde{g})+\tfrac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu u\partial_\nu u-W(u)\right]. \qquad (8)$$

The required boundary conditions on ϕ and S translate directly to conditions on u. Consider, for example, the special scale-invariant potential $\tilde{V}=\lambda\tilde{\phi}^4$ in the HSI theory. The transformed potential $W(u)$ will read now $W(u)=\bar{\lambda}\sinh^4(u/\sqrt{h}\,M)$. This theory describes the evolution of a scalar field (with regular canonical kinetic term), subjected to an exponential potential. Any scale-invariant potential in the MSI theory will naturally appear as a combination of exponential potentials in the transformed HSI theory. Such models have become recently quite fashionable in the inflationary scenario.[10]

In summary, if scale invariance is a true symmetry of nature (broken only spontaneously) and the conditions are given for the above-cited nonlinear constraint, the fields essentially exist in a compact manifold. This, in turn, enables us to estimate the effects of those fields in complicated cases without resorting to explicit solutions.

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