Truncation of Schwinger-Dyson equations and the $1/N$ expansion in the O(N) model

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We use the O(N) model as a nonperturbative example to study the effect of truncating the Schwinger-Dyson sequence of equations. We show that truncation of this hierarchy leads in the O(N) model to solutions for the propagators and vertex functions correct to a definite power of $1/N$. The procedure can be used to generate a systematic expansion in $1/N$ for an arbitrary Green's function of the model.

The truncation of Schwinger-Dyson (SD) equations[1] provides a convenient method to study nonperturbative effects (see, for example, Refs. 2–5). While the SD hierarchy is often thought of as a method of summing loop expansions, in fact the SD equations are derived without use of perturbation theory. The approximation used to close the hierarchy of equations is thus in principle different from the most common nonperturbative method of expanding around the finite classical fields. Unfortunately, the error introduced by neglecting or using approximations for higher-point functions is, in general, not known. It would be very useful to have a nonperturbative example displaying the effect of the truncation.

In this paper we examine the solutions of truncated SD equations in the O(N)-symmetric model with N real scalar fields Φ^a. We show that there is a close connection between the truncation of SD equations and the $1/N$ expansion, in the sense that the Green's functions obtained by solving the truncated SD equations are correct to a definite power of $1/N$. For example, the propagator may be obtained to order $(1/N)^n$ for $n>1$ by solving a closed set of SD equations involving up to $(n+1)$-point one-particle-irreducible (1PI) vertices, neglecting $(n+2)$-point and higher 1PI vertices.

The O(N) model provides a highly nontrivial case for testing nonperturbative methods,[6–10] as has been seen in studies analyzing the effective potential in the $1/N$ expansion. It has been shown[9] that O(N) is unbroken to leading order in $1/N$. This result is readily obtained by truncating the SD hierarchy at low order: substituting the 2-point propagator into the 1-point mean-field equation leads to the same equations as the effective-potential method of Ref. 9. By contrast, the classical or tree approximation, often employed for nonperturbative studies, yields a broken-symmetry vacuum if $m^2<0$. If a renormalized perturbation theory is defined with respect to this incorrect vacuum in four dimensions, tachyon poles appear as a symptom of this vacuum's instability, destroying the validity of perturbation theory and of the $1/N$ expansion. Thus the truncated SD series is more reliable than perturbations based on the classical or tree vacuum.

While O(N) remains unbroken for large N, it might be broken for small values of N. It is conceivable that a suitably high order of the SD hierarchy would also possess broken-symmetry solutions; however, it is clear that this question cannot be answered within the $1/N$ expansion which could no longer be justified if the symmetry is broken. We will thus assume unbroken symmetry, returning to comment on the symmetry-breaking case at the end of this paper.

The Lagrangian density we consider is

$$L=\tfrac{1}{2}\partial_\mu\Phi^a\partial^\mu\Phi^a-\tfrac{1}{2}m_0^2\Phi^a\Phi^a-\frac{\lambda_0}{8N}(\Phi^a\Phi^a)^2 \tag{1}$$

with a running from 1 to N, m_0 and λ_0 being the bare mass and coupling. Using the well-known trick[7] of adding to the above Lagrangian the term

$$\frac{N}{2\lambda_0}\left[\chi-\frac{\lambda_0}{2N}(\Phi^a\Phi^a)\right]^2 ,$$

where χ is a new field, one obtains the equivalent Lagrangian

$$L=\tfrac{1}{2}\partial_\mu\Phi^a\partial^\mu\Phi^a-\tfrac{1}{2}m_0^2\Phi^a\Phi^a-\tfrac{1}{2}\chi\Phi^a\Phi^a+\frac{N}{2\lambda_0}\chi^2 . \tag{2}$$

The generating functional G for connected Green's functions is

$$\exp(G)\equiv\int D(\Phi^a,\chi)\exp(i\sigma) , \tag{3}$$

where

$$\sigma\equiv S+\int[J^a(x)\Phi^a(x)+\eta(x)\chi(x)]d^4x , \tag{4}$$

with S being the action corresponding to Lagrangian (2). Using the standard procedure we obtain the mean-field and propagator equations of the Schwinger-Dyson sequence:

$$\frac{N}{\lambda_0}\frac{\delta G}{i\delta\eta(x)}-\frac{1}{2}\frac{\delta G}{i\delta J^a(x)}\frac{\delta G}{i\delta J^a(x)}-\frac{1}{2}\frac{\delta^2 G}{i\delta J^a(x)i\delta J^a(x)}+\eta(x)=0 , \tag{5}$$

$$(-\partial^2-m_0^2)\frac{\delta G}{i\delta J^a(x)}-\frac{\delta G}{i\delta\eta(x)}\frac{\delta G}{i\delta J^a(x)}-\frac{\delta^2 G}{i\delta\eta(x)i\delta J^a(x)}+J^a(x)=0 . \tag{6}$$

Setting external sources to zero we get

$$\hat{\chi}\equiv\langle\chi\rangle+m_0^2=m_0^2+\frac{\lambda_0}{2N}\langle T\Phi^a(0)\Phi^a(0)\rangle\ . \tag{7}$$

Although we assume the O(N) symmetry is not broken spontaneously, the χ field, since it is an O(N) scalar, will have a nonzero vacuum expectation value. From Eq. (7) we get

$$\hat{\chi}=m_0^2+\frac{\lambda_0}{2}\int\frac{d^4k}{(2\pi)^4}D_{\phi\phi}(k)\ , \tag{8}$$

where $D_{\phi\phi}(k)$ is the full ϕ propagator. Functionally differentiating Eqs. (5) and (6) we obtain the equations for the propagators:

$$\begin{aligned}D_{\chi\chi}(p)=D^{(0)}_{\chi\chi}-\frac{N}{2}D^{(0)}_{\chi\chi}\Gamma^{(0)}_{\chi\phi\phi}&\\ \times\int\frac{d^4k}{(2\pi)^4}\Gamma_{\chi\phi\phi}(-p,p+k,-k)&\\ \times D_{\chi\chi}(p)D_{\phi\phi}(p+k)D_{\phi\phi}(k)\ ,&\end{aligned} \tag{9}$$

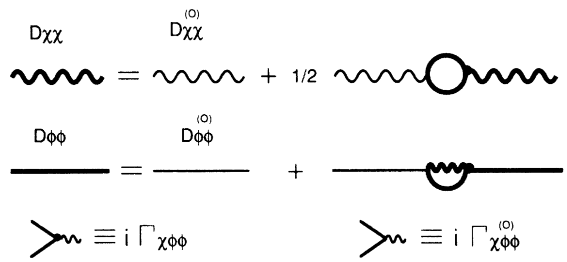

FIG. 1. Schwinger-Dyson equations for the propagators. The wavy line is always proportional to $1/N$, while a solid line loop gives a factor of N because of all the fields Φ^a going around the loop.

$$\begin{aligned}D_{\phi\phi}(p)=D^{(0)}_{\phi\phi}(p)-D^{0}_{\phi\phi}(p)\Gamma^{(0)}_{\chi\phi\phi}&\\ \times\int\frac{d^4k}{(2\pi)^4}\Gamma_{\chi}\phi\phi(p+k,-k,-p)&\\ \times D_{\chi\chi}(p+k)D_{\phi\phi}(p)D_{\phi\phi}(k)\ .&\end{aligned} \tag{10}$$

In the above equations $D^{(0)}_{\chi\chi}=i\lambda_0/N$, $D^{(0)}_{\phi\phi}(p)=i/p^2-\hat{\chi}$, $\Gamma^{(0)}_{\chi\phi\phi}$ is the bare 3-point vertex (equal to -1) and $\Gamma_{\chi\phi\phi}$ is the full vertex function corresponding to the connected 3-point function $G^{(3)}_{\chi\phi\phi}(x,y,z)\equiv\langle T\chi(x)\phi(y)\phi(z)\rangle_c$. Diagrammatically, Eqs. (9) and (10) are shown in Fig. 1.

A short digression about the conventions used is in order now. The momentum-space Green's functions $G^{(n)}(p_1,\ldots,p_n)$ are defined only for $p_1+\cdots+p_n=0$ (all momenta are oriented toward the vertex) by

$$G^{(n)}(x_1,\ldots,x_n)=\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^4(p_1+\cdots+p_n)\exp[i(p_1x_1+\cdots+p_nx_n)]G^{(n)}(p_1,\ldots,p_n)\ . \tag{11}$$

Also, we use the conventional relationship between the connected Green's functions $G^{(n)}_{a\cdots b}(p_1,..,p_n)$ and one-particle-irreducible vertices $\Gamma^{(n)}_{c\cdots d}(p_1,\ldots,p_n)$:

$$G^{(n)}_{a\cdots b}(p_1,\ldots,p_n)=i\Gamma^{(n)}_{c\cdots d}(p_1,\ldots,p_n)D_{ac}(p_1)\cdots D_{bd}(p_n)+\text{one-particle-reducible terms}\ , \tag{12}$$

where $D_{ac}(p)$ is the full propagator.

Now we turn to the problem of determining $\Gamma_{\chi\phi\phi}$ from its SD equation. We start with the equation for the 3-point function $G^{(3)}_{\chi\phi\phi}$, obtained by functionally differentiating the equation for $D_{\phi\phi}$:

$$(k^2-\hat{\chi})G^{(3)}_{\chi\phi\phi}(p,k,-p-k)=-\Gamma^{(0)}_{\chi\phi\phi}D_{\chi\chi}(p)D_{\phi\phi}(-p-k)-\Gamma^{(0)}_{\chi\phi\phi}\int\frac{d^4r}{(2\pi)^4}G^{(4)}_{\chi\chi\phi\phi}(p,r,-p-k,k-r)\ . \tag{13}$$

Equation (13) is graphically shown in Fig. 2. Decomposing the connected 4-point function $G^{(4)}_{\chi\chi\phi\phi}$ into one-particle-reducible and 1PI pieces (see Fig. 3) and truncating the propagators on the legs (using the SD equation for $D_{\phi\phi}$) we obtain the equation for the 3-point vertex $\Gamma_{\chi\phi\phi}$ (see Fig. 4):

$$\begin{aligned}\Gamma_{\chi\phi\phi}(p,k,-p-k)=\Gamma^{(0)}_{\chi\phi\phi}-\Gamma^{(0)}_{\chi\phi\phi}&\int\frac{d^4r}{(2\pi)^4}D_{\chi\chi}(r)D_{\chi\chi}(p-r)D_{\phi\phi}(k+r)\\ &\times\Gamma_{\chi\phi\phi}(p-r,k+r,-p-k)\Gamma_{\chi\chi\chi}(p,-r,r-p)\\ &-\Gamma^{(0)}_{\chi\phi\phi}\int\frac{d^4r}{(2\pi)^4}D_{\chi\chi}(r)D_{\phi\phi}(k-r)D_{\phi\phi}(r-p-k)\Gamma_{\chi\phi\phi}(p,k-r,r-p-k)\Gamma_{\chi\phi\phi}(-r,k,r-k)\\ &+i\Gamma^{(0)}_{\chi\phi\phi}\int\frac{d^4r}{(2\pi)^4}D_{\chi\chi}(r)D_{\phi\phi}(k-r)\Gamma_{\chi\chi\phi\phi}(r,p,k-r,-p-k)\ .\end{aligned} \tag{14}$$

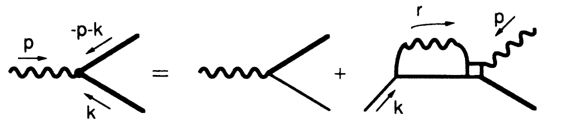

FIG. 2. SD equation for the 3-point connected Green's function $G^{(3)}_{\chi\phi\phi}$.

The above equation also involves the vertex $\Gamma_{\chi\chi\chi}$, which is determined by its SD equation (shown in Fig. 5), in terms of the propagator $D_{\phi\phi}$ and vertices $\Gamma_{\chi\phi\phi}$ and $\Gamma_{\chi\chi\phi\phi}$. Hence, the only unknown is the 4-point vertex $\Gamma_{\chi\chi\phi\phi}$. The equation for it can be obtained from the one for the 4-point connected Green's function $G^{(4)}_{\chi\chi\phi\phi}$. A much simpler way is to start from the equation for $\Gamma_{\chi\phi\phi}$ (with sources kept nonzero) and observe that functional differentiation pulls out a leg from existing vertices or propagators. The vertices created in this way are also 1PI.

In fact this procedure generalizes to an n-point vertex. We start from the SD equation for a 1PI vertex and want to derive the equation for the vertex with one more, say, χ leg. We represent the vertices as functional derivatives of the effective action, which is a generator of 1PI vertices. To pull out an additional χ leg we have to apply a functional derivative $\delta/\delta\chi(x)$. On the left-hand side this gives the desired vertex (note that we keep all the sources nonzero). On the right-hand side we have a sum of products of vertices and propagators (see, e.g., Fig. 4). When the derivative hits a vertex, it transforms it into one with one more χ leg. When it hits a propagator (for example, D_{ab}) we obtain (the sum over repeated indices includes also space-time integration)

$$\frac{\delta D_{ab}}{\delta\chi}=\frac{\delta D_{ab}}{\delta\eta_c}\frac{\delta\eta_c}{\delta\chi}=-G^{(3)}_{abc}\Gamma^{(2)}_{c\chi}=\Gamma_{df\chi}D_{ad}D_{bf}\ , \qquad (15)$$

where we used the decomposition (12) of the connected Green's function $G^{(3)}_{abc}$ into a product of a 3-point vertex and propagators, and also the orthogonality of the 2-point vertex $\Gamma^{(2)}_{c\chi}$ and the propagator. The net effect is pulling out a (truncated) leg of each internal propagator. This procedure can be continued and, although the number of terms in the equations increases quickly, it is clear what types of terms will be present.

Now we want to establish the leading-order (in $1/N$) behavior of the propagators and vertices. Using the $1/N$ perturbation theory and the properties of free propagators and vertices (especially that $D^{(0)}_{\chi\chi}$ is of order $1/N$) it is not difficult to see (Fig. 6) that $D_{\phi\phi}$ is of order 1, $D_{\chi\chi}$ is

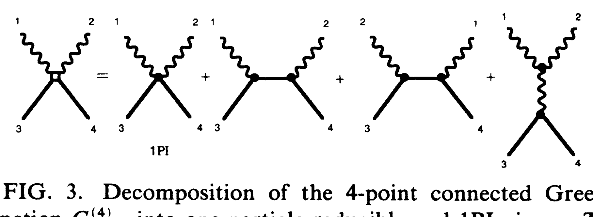

FIG. 3. Decomposition of the 4-point connected Green's function $G^{(4)}_{\chi\chi\phi\phi}$ into one-particle reducible and 1PI pieces. The filled out blob always denotes a 1PI vertex.

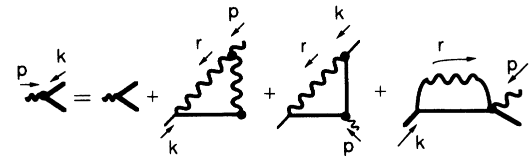

FIG. 4. SD equation for the 3-point vertex $\Gamma_{\chi\phi\phi}$. The external legs have been truncated, the short legs drawn serve only to identify the type of the vertex.

$O(1/N)$, $\Gamma_{\chi\cdots\chi}$ (only χ legs) is $O(N)$ and $\Gamma_{\chi\cdots\chi\phi\phi}$ (two ϕ legs and two or more χ legs) is $O(1/N)$.

We now make the assumption of the inductive proof. Suppose we know the propagators to order $(1/N)^p$, the vertex $\Gamma_{\chi\phi\phi}$ to order $(1/N)^{p-1}$, $\Gamma_{\chi\chi\phi\phi}$ to $(1/N)^{p-2},\ldots,$ and, finally, the vertex with p χ legs and two ϕ legs to order 1. This also implies that we know $\Gamma_{\chi\chi\chi}$ to order $(1/N)^{p-3}$ $(p\geq 2)$ and, in general, the vertex with k χ legs to order $(1/N)^{p-k}$.

To make the next step we start with the equation for the vertex with p χ legs and two ϕ legs, to calculate it to order $1/N$. We can then neglect the contribution coming from the (unknown) vertex with $p+1$ χ legs and two ϕ legs, since that term is of order $1/N^2$ because of the presence of a $D_{\chi\chi}$ in the loop. But we need the vertex with $p+2$ χ legs to order N; fortunately that being the leading order we can get it simply. We also need the vertices with $p+1, p, \ldots, 3$ χ legs to order N, but by the above assumption we know them. For the vertices involving two ϕ legs and any number of χ legs up to p, we need them to order 1 only. Then we use the SD equation for the vertex with $p-1$ χ legs and two ϕ legs to calculate it to order $1/N^2$, using the above result for the vertex with one more χ legs. We also use that result to calculate the vertex with $p+1$ χ legs to order 1, which we also need. This procedure can be continued without encountering any difficulty. Indeed, consider the case when we want to calculate the vertex with two ϕ legs and $p-k$ χ legs to order $(1/N)^{k+1}$. We need the vertices with no ϕ legs and any number of χ legs up to $p-k+2$ to order $(1/N)^{k-1}$. By the assumption of the previous paragraph we know the vertices with up to $p-k+1$ χ legs to that accuracy. The vertex with $p-k+2$ χ legs we can get to $(1/N)^{k-1}$ using the updated value of the vertex with two ϕ legs and $p-k+1$ χ legs, known to $(1/N)^k$. We also need the vertices with two ϕ legs and up to $p-k$ χ legs to order $(1/N)^k$, and we know them by the inductive assumption. In the above considerations we have not mentioned the propagators since it is clear that we never need them to accuracy better than $(1/N)^p$, since the most accurate calculation (that of $\Gamma_{\chi\phi\phi}$) is to that order. Once we have

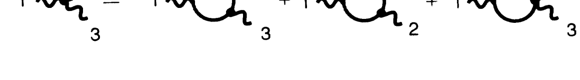

FIG. 5. SD equation for the vertex $\Gamma_{\chi\chi\chi}$.

$\Gamma_{\chi\phi\phi}$ to order $(1/N)^p$ we can calculate the propagators to $(1/N)^{p+1}$ from the SD equations of Fig. 1.

It is simple to check that, starting from the free propagators and vertices, using the SD equations (8)–(10) we can obtain the propagators to order $1/N$, and then, calculating the first correction to the vertex $\Gamma_{\chi\phi\phi}$, solve for the propagators to order $1/N^2$. This completes the inductive proof that we can use the truncated SD equations to obtain a systematic expansion in $1/N$ of the Green's functions.

Suppose we just truncate the SD equations and solve them self-consistently. The error we make is still determined by the neglected term and we can give it as a power of $1/N$, but the solution will not be in the form of a polynomial in $1/N$. Thus, the solution will be correct to a definite power of $1/N$, but it will, in general, contain terms of all the higher powers also.

All the equations we considered so far were formal ones, containing the bare parameters and fields. The renormalization can be implemented, at least in the $1/N$ expansion, rendering the Green's functions finite,[7–9] and preserving the $1/N$ structure of the equations.

Using the above scheme we can also discuss the case of the spontaneous breakdown of $O(N)$ symmetry. Assuming that only one of the fields Φ^a has a nonzero vacuum expectation value and choosing that field to be Φ^N, we denote $\langle \Phi^N \rangle \equiv \hat{\phi}$, $\langle \chi \rangle \equiv \hat{\chi} - m_0^2$. Equations (5) and (6) then give (external sources are put to zero)

$$\hat{\chi} = m_0^2 + \frac{\lambda_0}{2N}\hat{\phi}^2 + \frac{\lambda_0}{2N}\langle T\Phi^a(0)\Phi^a(0)\rangle , \tag{16}$$

$$\hat{\chi}\hat{\phi} + \langle T\chi(0)\Phi^N(0)\rangle = 0 . \tag{17}$$

The SD equations for the propagators and higher-point functions can be written down in the same way as before and a systematic expansion in $1/N$ seems possible. A major complication is caused by the presence of the nondiagonal propagator $D_{\chi\sigma}$ ($\sigma \equiv \Phi^N - \hat{\phi}$) and new types of vertices not forbidden anymore by the symmetry of the original Lagrangian. The equations for the propagators can still be solved easily to leading order in $1/N$, but then a tachyon pole appears (a signature of choosing the incorrect vacuum)[7–9] and that makes the whole procedure of $1/N$ expansion meaningless. We can imagine finding a symmetry-breaking solution by solving the truncated SD equations self-consistently, thus circumventing the explicit expansion in powers of $1/N$. However, analytically that seems impossible and it would require significant numerical effort.

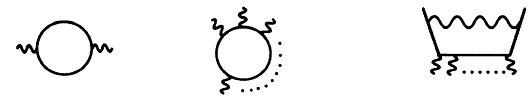

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FIG. 6. (a) Leading-order contribution to χ self-energy, proportional to N. (b) Leading-order contribution to the vertex with any number of χ legs, proportional to N. (c) Leading-order diagram for the 1PI vertex with two Φ legs and any number of χ legs, proportional to $1/N$. Note that the wavy line here represents the corrected propagator $D_{\chi\chi}$ calculated to order $1/N$.

The Gross-Neveu model[10] offers another possibility for studying spontaneous symmetry breaking, the only disadvantage being the necessity to work in three dimensions.[11] The connection between the truncation of SD equations and the $1/N$ expansion, which we established for the scalar $O(N)$ model, also holds for the Gross-Neveu Lagrangian; however, complications similar to those of $O(N)$ arise in the Gross-Neveu model in two dimensions.[12]

To conclude, we examined the effect of truncating the Schwinger-Dyson equations for the $O(N)$-symmetric model and showed that the errors in Green's functions are bounded by powers of $1/N$. We examined in detail the case when the $O(N)$ symmetry is not broken spontaneously, but the same conclusion seems to hold for the case of the vacuum breaking the symmetry to $O(N-1)$.

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