Anomalous commutators and functional integration

C. Fosco and R. C. Trinchero
Centro Atómico Bariloche and Instituto Balseiro, 8400 Bariloche, Argentina
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We derive a method for the calculation of commutators within the functional-integral formulation of quantum mechanics. Anomalous terms in commutators are expressed in terms of the transformation properties of the functional-integral measure. As an application we compute some commutators for the case of the chiral Schwinger model and for chiral QED in 3+1 dimensions.

I. INTRODUCTION

Thirty years ago it was recognized that in the realm of quantum field theory quantum commutators need not coincide with canonical commutators. In addition to the canonical contribution there could be extra terms, the so-called Schwinger terms (ST's) or anomalous terms (AT's), that in general depend on the theory under consideration. Indeed, in Ref. 1 it was shown that for certain commutators the ST's cannot be zero if the spectrum of the theory is bounded from below. With the advent of current algebra the importance and utility of computing commutators became clear. Furthermore, the relevance of ST's in obtaining results in agreement with experiment was clearly recognized.[2] The Bjorken-Johnson-Low (BJL) method[3] for the calculation of commutators arose in this context. This method leads to an expression for the commutator of two operators in terms of their time-ordered product (TOP). This TOP can be computed by means of perturbation theory. Therefore this method has the virtue of ensuring that the handling of infinities in the calculation of commutators is consistent with the procedure employed in getting physical results from perturbation theory. Of course, it is not possible in general to compute and sum up all the diagrams appearing in the perturbative expansion of the TOP associated with the commutator in question. However, in practical calculations it turns out that only some diagrams contribute to the anomalous terms of the commutator in question and that these diagrams are in general anomalous. The denomination anomalous diagram refers to diagrams such that it is not possible to make them satisfy some Ward identity of the theory. This last statement requires some explanation, since in general for a given theory we will have a certain set of Ward identities, and we have anomalies when we cannot satisfy all of them simultaneously.

Once we have found possible to satisfy a certain subset of Ward identities by taking some specific regularization or what is the same by allowing for only certain counterterms then we are left with some diagrams that violate the rest of the Ward identities. These are what we call the anomalous diagrams. All these arguments lead us to expect that anomalies give rise to AT's in equal-time commutators (ETC's).

Let us now consider this situation from the point of view of the functional-integral framework. We know that in this formulation anomalies appear as the noninvariance of the functional-integral measure (FIM) under certain transformations.[4] Of course this is true if one adequately defines the FIM. If one wants to compare results with the ones of perturbation theory then an adequate definition of the FIM is one that reproduces the results of perturbation theory for the anomalies. This reproduction involves not only the definition of the FIM but also a regularization in correspondence with the one employed in perturbation theory. In any case the existence of anomalies is directly related to the FIM. Therefore one would expect AT's in commutators to be expressed in terms of some transformation properties of the FIM. The purpose of this work is to look for this expression.

This paper is organized as follows. Section II presents the method for computing commutators. In Sec. III we apply the results of Sec. II for the case of the chiral Schwinger model (CSM) and for chiral QED in 3+1 dimensions. Appendix A gives support to some formal manipulations used in Sec. II. Appendix B deals with the properties of antisymmetry and distributivity under addition of commutators.

II. COMMUTATORS AND AUXILIARY EVOLUTION

The basic idea behind this method for the calculation of commutators can be understood considering the following expression for the equal-time commutator (ETC) of two operators A and B:

$$[A(t),B(t)]=\lim_{\alpha\to 0}\frac{\hbar}{i\alpha}[e^{(i/\hbar)\alpha A(t)}B(t)e^{-(i/\hbar)\alpha A(t)}-B(t)]\ . \tag{2.1}$$

If we define

$$B(t,\alpha)=e^{(i/\hbar)\alpha A(t)}B(t)e^{-(i/\hbar)\alpha A(t)} \tag{2.2}$$

with the convention

$$B(t,0)=B(t)\ , \tag{2.3}$$

then (2.1) can be written as

$$[A(t),B(t)]=\frac{\hbar}{i}\frac{\partial}{\partial\alpha}B(t,\alpha)\bigg|_{\alpha=0}\ . \tag{2.4}$$

We interpret (2.2) as giving the α dependence of the operator B with respect to the evolution dictated by operator A. That is, in the same way as the Hamiltonian generates the evolution of the system in time, the operator A in (2.2) generates the evolution in the parameter α. In this respect Eq. (2.4) gives the equation of motion for the operator B under α evolution. Therefore in computing commutators we are only interested in infinitesimal α evolution. In order to compute the effects of this evolution we would like to consider an appropriate functional integral. However, it is not obvious to write down such a functional integral, essentially because we do not know what FIM should be associated with this evolution. Now, if we assume to have a well-defined FIM for time evolution, then a possible way to obtain the FIM for α evolution would be to relate the integration measures by means of some transformation. If this can be achieved then the knowledge of the t-integration measure would lead to the α-integration measure. Essentially the idea that allows us to connect these two FIM's is the following. We know that time evolution consists just in the application of a time-dependent unitary transformation to the states. The same can be asserted about α evolution. Therefore, in order to relate these two evolutions we have to undo the time-dependent unitary transformation and perform the α-dependent one. In terms of the functional integral this amounts to performing canonical transformations. For reasons that will become clear below we refer to this transformations as Poisson-bracket-preserving transformations (PBPT). The application of these PBPT's will allow us to transform the functional integral for time evolution into another for α evolution. In Appendix A we show how the connection between these two functional integrals works. This is done by considering the usual partition of time connecting the matrix elements of infinitesimal evolution operators with the functional integral and performing unitary transformations in the corresponding expressions.

Now we proceed to develop these ideas more precisely. Let us consider a regular system with an arbitrary number of degrees of freedom described by the phase-space variables $\{q_i,p_i\}$ where i denotes a set of indices that can be either continuous or discrete. For the sake of simplicity we restrict consideration to systems where the Hamiltonian is not explicitly time dependent. The transition amplitude for this system is given by

$$\begin{aligned}\langle \{q_i'\},\Delta|\{q_i\},0\rangle &= \langle \{q_i'\}|e^{-(i/\hbar)H\Delta}|\{q_i\}\rangle \\ &= \int [d\mu]_H \exp\left[\frac{i}{\hbar}\int_0^\Delta dt \left[\sum_i p_i\dot{q}_i - H(\{p_i,q_i\})\right]\right] . \end{aligned} \tag{2.5}$$

As we stated before we assume that we have a well-defined FIM for time evolution denoted by $[d\mu]_H$ in (2.5). The boundary conditions for the right-hand side (RHS) of (2.5) are

$$\begin{aligned} q_i(t=0)&=q_i, \quad \forall i , \\ q_i(t=\Delta)&=q_i', \quad \forall i . \end{aligned} \tag{2.6}$$

For our present purposes we can take $\Delta \ll 1$ in (2.5). Following the general idea outlined above we first perform a time-dependent PBPT on the integration variables in (2.5) so as to undo time evolution. We take the generating functional of this transformation to be

$$\Phi^{(1)}(\{q_i,\bar{p}_i\},t)=\sum_i q_i\bar{p}_i - tH(\{q_i,\bar{p}_i\}) ; \tag{2.7}$$

hence, the relation between the q_i,p_i and the new variables $\bar{q}_i,\bar{p}_i$ is

$$\begin{aligned} p_i &= \frac{\partial\Phi^{(1)}}{\partial q_i} = \bar{p}_i - t\frac{\partial H}{\partial q_i}(\{q_i,\bar{p}_i\}) , \\ \bar{q}_i &= \frac{\partial\Phi^{(1)}}{\partial \bar{p}_i} = q_i - t\frac{\partial H}{\partial \bar{p}_i}(\{q_i,\bar{p}_i\}) . \end{aligned} \tag{2.8}$$

This PBPT can be considered as an infinitesimal one because in the expression under consideration the values of t are bounded to be less than Δ and we have $\Delta \ll 1$. The relation between the new Hamiltonian and the old one is given by

$$\begin{aligned} \bar{H}(\{\bar{q}_i,\bar{p}_i\}) &= H(\{q_i,p_i\}) + \frac{\partial}{\partial t}\Phi^{(1)}(\{q_i,\bar{p}_i\},t) \\ &= H(\{q_i,p_i\}) - H(\{q_i,\bar{p}_i\}) = O(t) . \end{aligned} \tag{2.9}$$

Neglecting the $O(t)$ terms we have

$$\bar{H}=0 ; \tag{2.10}$$

therefore, (2.5) transforms to

$$\begin{aligned}\langle \{\bar{q}_i'\},\Delta|\{q_i\},0\rangle = \int [d\mu]_H J(\{\bar{q},\bar{p}\},g_{-H}) \exp\left[\frac{i}{\hbar}\int_0^\Delta dt \sum_i \bar{p}_i\dot{\bar{q}}_i\right] , \end{aligned} \tag{2.11}$$

where we have neglected the border terms associated with the PBPT (2.8) in accordance with the argument appearing in Appendix A. $J(\{\bar{q},\bar{p}\},g_{-H})$ denotes the Jacobian associated with the transformation (2.8). For a system with a finite number of degrees of freedom $J(\{\bar{q},\bar{p}\},g_{-H})$ should be one due to Liouville theorem. For infinite dimensional phase spaces this needs not be the case due to the eventual presence of infinities that should be regularized in order that this Jacobian make sense. However, if for a given system the definition of $[d\mu]_H$ and the regularization of J_{-H} would be such that J_{-H} turns out to be dependent on the $\{\bar{q}_i,\bar{p}_i\}$ then some curious property would arise. We would have that the expectation value of the Hamiltonian is time dependent or, what is the same, that the ETC of the Hamiltonian with itself would not vanish. We think that this property is not desirable; therefore from now on we will assume that $[d\mu]_H$ and the regularization are defined in such a way that $J_{-H}=1$. We remark that this assumption is not a too restrictive one, since FIM's are usually defined in terms of eigenfunctions of a given operator, and these eigenfunctions are related to the eigenfunctions of the first quantization Hamiltonian.

The next step is to make a time-dependent PBPT corsponding to the α evolution. The generating functional for this transformation is

$$\Phi^{(2)}(\{\bar{q},P'\})=\sum_i \bar{q}_i P'_i+\alpha(t)A(\{\bar{q},P'\}) , \tag{2.12}$$

where for the moment we take α to be a smooth function of t such that $\alpha(0)=0$ and $\alpha(t)\ll 1$ if $t \ll 1$. The corresponding transformation of variables is

$$\begin{aligned}\bar{p}_i&=\frac{\partial\Phi^{(2)}}{\partial\bar{q}_i}=P'_i+\alpha(t)\frac{\partial}{\partial\bar{q}_i}A(\{\bar{q},P'\}) ,\\ Q'_i&=\frac{\partial\Phi^{(2)}}{\partial P'_i}=\bar{q}_i+\alpha(t)\frac{\partial}{\partial P'_i}A(\{\bar{q},P'\}) ,\end{aligned} \tag{2.13}$$

and the new Hamiltonian is determined by

$$\begin{aligned}H'(\{Q',P'\})&=\frac{\partial\Phi^{(2)}}{\partial t}(\{\bar{q},P'\})=\dot{\alpha}(t)A(\{\bar{q},P'\})\\ &=\dot{\alpha}(t)A(\{Q',P'\})+O(\alpha^2) .\end{aligned} \tag{2.14}$$

Neglecting the $O(\alpha^2)$ terms we get

$$H'=\dot{\alpha}(t)A(\{Q',P'\}) \tag{2.15}$$

and (2.11) transforms to

$$\begin{aligned}&\langle\{Q'_i\}',\Delta|\{Q'_i\},0\rangle\\ &\quad=\int[d\mu']_H J(\{Q',P'\},g_A)\\ &\qquad\times\exp\left[\frac{i}{\hbar}\int_0^\Delta dt\left[\sum_i P'_i\dot{Q}'_i-\dot{\alpha}(t)A(\{P',Q'\})\right]\right] .\end{aligned} \tag{2.16}$$

$J(\{Q',P'\},g_A)$ in (2.16) stands for the Jacobian corresponding to the transformation (2.13). This expression is the one that corresponds to evolution in α; therefore, we define the FIM for the evolution dictated by the operator A as

$$[d\mu]_A=[d\mu]_H J(\{q,p\},A) . \tag{2.17}$$

Having stated this definition we now proceed to see whether or not it satisfies the following consistency requirement. By means of (2.17) we can write a FIM associated to any well-defined function of the phase-space variables. Let A, B, and C be three of these functions such that

$$C=A+B , \tag{2.18}$$

so considering the infinitesimal transformations of the type (2.13) generated by these functions we have

$$g_C=g_A\circ g_B , \tag{2.19}$$

so that the FIM associated with C is

$$[d\mu]_C=[d\mu]_H J(\{q,p\},g_A\circ g_B) . \tag{2.20}$$

Because of (2.19) it should be also possible to obtain $[d\mu]_C$ by performing an infinitesimal transformation generated by A on $[d\mu]_B$, obtaining

$$\begin{aligned}[d\mu]_C&=([d\mu]_B)^{g_A}\\ &=[d\mu]_H J(\{q^{g_A},p^{g_A}\},g_B)J(\{q,p\},g_A) ,\end{aligned} \tag{2.21}$$

where the superscript g_A denotes that the corresponding quantity is the one obtained by applying the transformation generated by A. If these two expressions for $[d\mu]_C$ are to be the same, we obtain

$$J(\{q,p\},g_A\circ g_B)=J(\{q^{g_A},p^{g_A}\},g_B)J(\{q,p\},g_A) , \tag{2.22}$$

this expression is the Wess-Zumino consistency condition[5] for the group of PBPT's of the phase space under consideration. We remark that the consistency requirement for the FIM leading to (2.22) only involves infinitesimal transformations. As will be seen below, condition (2.22) turns out to be essential in the proof of the desirable properties of antisymmetry and distributivity under addition of the ETC's. Therefore, we make the further assumption that $[d\mu]_H$ and the regularizations involved in the calculation of Jacobians should be such that (2.22) holds.

Having obtained the FIM's associated with any phase-space function $A(\{p,q\})$ we proceed to derive the equations of motion that lead through (2.4) to the ETC's. We obtain these equations of motion using a procedure similar to the one employed in deriving Ward identities from the functional integral. That is, we perform a transformation on the integration variables and then we use the invariance of the resulting functional integral under variations with respect to certain parameters of the transformation. Let us consider the following infinitesimal PBPT:

$$\begin{aligned}P'&\to P=P'+\epsilon(t)\frac{\partial B}{\partial Q}(P,Q) ,\\ Q'&\to Q=Q'-\epsilon(t)\frac{\partial B}{\partial P}(P,Q), \quad \epsilon(t)\ll 1,\forall t .\end{aligned} \tag{2.23}$$

When applying this change of variables to (2.16), the exponent in that expression transforms to

$$\begin{aligned}E=\int_0^\Delta dt&\left[\sum_i\left[P_i-\epsilon\frac{\partial B}{\partial Q_i}\right]\frac{d}{dt}\left[Q_i+\epsilon\frac{\partial B}{\partial P_i}\right]\right.\\ &\left.-\dot{\alpha}A\left[P_i-\epsilon\frac{\partial B}{\partial Q_i},Q_i+\epsilon\frac{\partial B}{\partial P_i}\right]\right] .\end{aligned} \tag{2.24}$$

Neglecting terms of $O(\epsilon^2)$ and $O(\alpha\epsilon)$, we obtain

$$E=\int_0^\Delta dt\left[\sum_i P_i\dot{Q}_i-\dot{\alpha}A(P,Q)-\dot{\epsilon}B(P,Q)+\dot{\alpha}\epsilon\{A,B\}\right] , \tag{2.25}$$

so that (2.16) becomes

$$\langle\{Q_i\}',\Delta|\{Q_i\},0\rangle=\int[d\mu]_H J(\{Q,P\},g_A\circ g_B)\exp\left[\frac{i}{\hbar}E\right] , \tag{2.26}$$

where we have used (2.22) in rewriting the composition of Jacobians. The functional variation of (2.26) with respect to $\epsilon(t)$ for fixed extremes, so as not to disturb boundary conditions, should vanish. Therefore we obtain the equation

$$0=\frac{\delta}{\delta\epsilon(t)}\langle\{Q_i\}',\Delta|\{Q_i\},0\rangle\Big|_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}}$$

$$=\left\langle\{Q_i\}',\Delta\left|\frac{i}{\hbar}\left[\frac{dB}{dt}+\{A,B\}\right]+\frac{\delta}{\delta\epsilon(t)}J(\{Q,P\},g_A\circ g_B)\Big|_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}}\right|\{Q_i\},0\right\rangle , \quad (2.27)$$

where the border terms appearing in E give no contribution due to the limit $\Delta\to 0$. Hence taking into account the expression (2.4) for the ETC we obtain, as an identity between matrix elements, the equation

$$[A,B]=i\hbar\{A,B\}-\hbar^2\frac{\delta}{\delta\epsilon}J(\{Q,P\},g_B\circ g_A)\Big|_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} . \quad (2.28)$$

This expression does not take into account the case of ETC between composite operators. In that case we should specify the ordering of factors in the composite operators in question. This case is considered in Appendix A, where the following general expression is justified:

$$[:A:,:B:]=[:A:,:B:]_{\rm can}-\hbar^2\frac{\delta}{\delta\epsilon}J(\{Q,P\},g_B\circ g_A)\Big|_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} . \quad (2.29)$$

Regarding this point we remark that in field theory the definition of composite operators not only involves a certain ordering of factors but also a procedure for the handling of infinities. As will become clear in the next section this procedure will be determined by the regularization one employs in the calculation of the RHS of (2.29).

Equation (2.29) is the main result of this work. We remark that the noncanonical part of the ETC is given by the second term in (2.29) (if we regard as canonical the additional terms appearing through operator ordering effects). These terms are only present when we consider transformations [the ones denoted by g_A and g_B in (2.29)] that in spite of leaving Poisson brackets invariant do not preserve areas in phase space; i.e., they lead to nontrivial Jacobians. For systems with a finite number of degrees of freedom this cannot be the case due to the Liouville theorem.

It is not obvious from (2.29) that the ETC's are antisymmetric and satisfy the distributive law under addition. These properties are derived in Appendix B, the crucial ingredients in this derivation being the assumption $J_{-H}=1$ and the Wess-Zumino consistency condition (2.22).

III. APPLICATIONS TO FIELD THEORY

We first consider the application of the method developed in Sec. II to the CSM (Ref. 6). This method has been devised for regular systems (without constraints), and the CSM is a system with constraints. We will consider the CSM as if it had no constraints; such a model is by itself interesting and its quantization is a first step in a recently proposed quantization method for the CSM (Ref. 7). Furthermore, such a model is also obtained by naively fixing the temporal gauge ($A_0=0$) in the CSM.

The Lagrangian we consider is

$$\mathcal{L}=\bar\psi(i\partial\!\!\!/-eA\!\!\!/P_L)\psi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} , \quad (3.1)$$

where we have adopted the following conventions:

$$\begin{aligned}&F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\quad \mu,\nu=0,1,\quad \hbar=1 ,\\ &\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu},\quad g^{\mu\nu}=\delta^{\mu0}\delta^{\nu0}-\delta^{\mu1}\delta^{\nu1} ,\\ &\gamma^{0\dagger}=\gamma^0,\quad \gamma^{1\dagger}=-\gamma^1,\quad \gamma^5=i\gamma\cdot\gamma^1 ,\\ &P_L=\tfrac{1}{2}(1+i\gamma^5),\quad P_R=\tfrac{1}{2}(1-i\gamma^5) .\end{aligned} \quad (3.2)$$

The generating functional is therefore given by

$$G(J,\bar\eta,\eta)=\int\mathcal{D}\pi_0\mathcal{D}E\,\mathcal{D}A_0\mathcal{D}A_1\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left[i\int d^2x(\pi_0\partial_0A_0+E\partial_0A_1+i\bar\psi\gamma^0\partial_0\psi-\mathcal{H}+J\cdot A+\bar\eta\psi+\bar\psi n)\right] , \quad (3.3)$$

where

$$\mathcal{H}=\tfrac{1}{2}E^2+E\partial_1A_0-i\bar\psi\gamma^1\partial_1\psi+e\bar\psi A\!\!\!/P_L\psi . \quad (3.4)$$

We will first evaluate the commutators between gauge field operators. In calculating the commutator $[A_0(t,x^1),\pi_0(t,y^1)]$, we choose first to make the PBPT generated by $\pi_0(t,y^1)$, and then the one generated by $A_0(t,x^1)$.

The only changes produced in the fields are, respectively,

$$\begin{aligned}&A_0(t,z^1)\to A_0(t,z^1)+\alpha(t)\delta(z^1-y^1),\quad \forall z^1 ,\\ &\pi_0(t,z^1)\to\pi_0(t,z^1)-\epsilon(t)\delta(z^1-x^1),\quad \forall z^1 .\end{aligned} \quad (3.5)$$

We can see that the FIM of the gauge fields does not change under (3.5) (they are mere translations of the fields). It can also be seen that for this model the fermionic integration measure (including regularization), is invariant under (3.5). Consequently, we obtain for (3.5) the trivial Jacobian $J=1$.

As $J=1$, we can say that the commutator will be "canonical" (including in the term "canonical" the cases where the Schwinger's terms are due to operator-ordering problems). The resulting commutator is then

$$[A_0,(t,x^1),\pi_0(t,y^1)]=i\delta(x^1-y^1) \ . \tag{3.6}$$

Taking into account that the transformation generated by E also leads to a trivial Jacobian, we conclude that all the commutators between gauge fields do not contain anomalous terms for this model.

Now we consider the more interesting case of commutators between currents and gauge field operators.

Since the fermionic currents are composite operators, we should regularize them, in order to have a well-defined commutator. Different regularizations give place to different composite operators, all of them with the same classical limit when $\hbar\to 0$. In order to be definite about which current we are speaking about, we give their definitions in terms of the point-splitting method.[6,8] The first definition for the left-handed current we consider is

$$j_L^\mu(x,a)=\lim_{\epsilon\to 0}\bar\psi(x-\epsilon/2)\gamma^\mu P_L\psi(x+\epsilon/2) \times\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\mu A^\mu(y)\right] , \tag{3.7}$$

where a symmetrical limit is understood.

Corresponding to this definition of the current there should exist a regularization of the Jacobian associated with the transformation generated by the classical current. This transformation is

$$\begin{aligned}\psi(x)&\to\psi(x)+i\rho(x)\gamma^0\gamma^\mu P_L\psi(x) \ ,\\ \bar\psi(x)&\to\bar\psi(x)-i\rho(x)\bar\psi(x)\gamma^\mu\gamma^0 P_R \ .\end{aligned} \tag{3.8}$$

One way to obtain the above-mentioned regularization of the Jacobian in correspondence with the definition (3.7) is the following. From (3.7) we can calculate the divergence of the current, then we look for a regularization of the Jacobian associated with (3.8) for $\mu=0$ such that the anomaly so obtained is the same as the divergence of the current (3.7). In this way one obtains the same Ward identity for the divergence of the current in the path-integral framework. The regularization operator one has to choose in accordance with (3.7) is

$$\not{D}=\not\partial+ie\not{A}(P_L+aP_R) \ . \tag{3.9}$$

In order to evaluate the Jacobian, we choose the basis of eigenfunctions,

$$\begin{aligned}\not{D}\phi_n(x)&=\lambda_n\phi_n(x) \ ,\\ \int d^2x\,\phi_n^\dagger(x)\phi_m(x)&=\delta_{n,m}\end{aligned} \tag{3.10}$$

[the operator $\not{D}$ can be made Hermitian by a "smooth continuation" of the axial component of the field A^μ (Ref. 9)], and following the standard method,[10] we find the Jacobian for (3.8),

$$J=\exp\left[\frac{ie}{8\pi}\int d^2x\,\rho(x)[(1+a)\epsilon^{\mu\nu}-(1-a)g^{\mu\nu}]\partial_\mu A_\nu\right] . \tag{3.11}$$

Having obtained the desired regularization, we consider first the commutator between the fields $\pi_0(t,x^1)$ and $j^0(t,y^1)$. We perform first the transformation generated by $\pi_0(t,x^1)$ and, as we know, we obtain $J=1$.

According to (2.29), the evaluation of this commutator involves the calculation of

$$\left\langle\frac{\delta J}{\delta\epsilon}\Bigg|_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}}\right\rangle ,$$

where the expectation value is evaluated with the operator $\pi_0(t,x^1)$ as Hamiltonian:

$$\left\langle\frac{\delta J}{\delta\epsilon}\Bigg|_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}}\right\rangle=-\frac{ie}{8\pi}\langle[(a-1)g^{\mu\nu}+(a+1)\epsilon^{\mu\nu}]\partial_\mu A_\nu(t,y^1)\rangle \ . \tag{3.12}$$

Then, as the "canonical" commutator between $j^0(t,y^1)$ and $\pi_0(t,x^1)$ is zero, we find

$$\begin{aligned}\langle[j^0(t,y^1),\pi_0(t,x^1)]\rangle=\frac{ie}{8\pi}[&(a-1)\langle\partial_0A_0(t,y^1)\rangle+(a-1)\langle\partial_1A^1(t,y^1)\rangle\\ &+(a+1)\langle\partial_0A_1(t,y^1)\rangle-(a+1)\langle\partial_1A_0(t,y^1)\rangle] \ .\end{aligned} \tag{3.13}$$

So, we finally have to evaluate the expectation values of the derivatives of the field A^μ, with $\pi_0(t,x^1)$ as Hamiltonian. To this end we consider the functional integral associated with the evolution generated by $\pi_0(t,x^1)$,

$$Z=\int\mathcal{D}\pi_0\mathcal{D}E\,\mathcal{D}A_0\mathcal{D}A_1\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left[i\int d^2z[\pi_0\partial_0A_0+E\partial_0A_1+i\bar\psi\gamma^0\partial_0\psi-\pi_0\delta(x^1-z^1)]\right] . \tag{3.14}$$

Performing the transformation

$$\pi_0(t,z^1)\to\pi_0(t,z^1)+\delta\pi_0(t,z^1) \tag{3.15}$$

and noting that Z cannot depend on $\delta\pi_0$, we find

$$\frac{\delta Z}{\delta\pi_0(z)}=0\to\langle\partial_0A_0(t,y^1)\rangle=\delta(x^1-y^1) \ . \tag{3.16}$$

Analogously, performing the change of variables

$$E(t,z^1)\to E(t,z^1)+\delta E(t,z^1)\ , \tag{3.17}$$

we obtain

$$\langle \partial_0 A_1(t,y^1)\rangle = 0 \tag{3.18}$$

and employing similar calculations we also find

$$\begin{aligned}&\langle \partial_1 A_0(t,y^1)\rangle = 0\ ,\\ &\langle \partial_1 A^1(t,y^1)\rangle = 0\ .\end{aligned} \tag{3.19}$$

Inserting the results (3.16), (3.18), and (3.19) in (3.12), we obtain

$$[j^0(t,y^1),\pi_0(t,x^1)]=\frac{ie}{8\pi}(a-1)\delta(x^1-y^1)\ . \tag{3.20}$$

As in our definition (3.7) of the current we have

$$j^0(x,a)=-j^1(x,a),\quad \forall x,\ \forall a\ , \tag{3.21}$$

the exponent of the Jacobian for j^1 will differ only in a sign with that of j^0. Then we can obtain the commutator between j^1 and π_0,

$$[j^1(t,y^1),\pi_0(t,x^1)]=-\frac{ie}{8\pi}(a-1)\delta(x^1-y^1)\ . \tag{3.22}$$

In an analogous fashion we calculate the commutators between the currents and the electric field, obtaining

$$[j^0(t,y^1),E(t,x^1)]=\frac{ie}{8\pi}(a+1)\delta(x^1-y^1)\ , \tag{3.23}$$

$$[j^1(t,y^1),E(t,x^1)]=-\frac{ie}{8\pi}(a+1)\delta(x^1-y^1)\ . \tag{3.24}$$

It can also be easily seen that the commutators between the A_μ's and the currents all vanish.

Next, we consider the evaluation of the same commutators (3.20), and (3.22)–(3.24) but with another definition of the current. In terms of the point-splitting method this current is given by

$$\tilde{j}^\mu(x,a)=\lim_{\epsilon\to 0}\bar{\psi}(x-\epsilon/2)(\gamma^\mu P_L-iea\epsilon\cdot A\gamma^\mu\gamma^5)\psi(x+\epsilon/2)\times\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\mu A^\mu(y)\right]\ . \tag{3.25}$$

This definition corresponds to the consistent gauge current given by the derivative of the effective action respect to the field A_μ. One way of calculating these commutators is by means of the relation between the two different definitions of the current we have given (11). This relation is

$$\tilde{j}^\mu(x)=j^\mu(x)-\frac{ea}{8\pi}\epsilon^{\mu\nu}A_\nu(x)\ . \tag{3.26}$$

Now, using (3.26), we can obtain the commutators involving $\tilde{j}^\mu$. For example,

$$\begin{aligned}[\tilde{j}^1(t,y^1),\pi_0(t,x^1)]&=[j^1(t,y^1),\pi_0(t,x_1)]\\&\quad+\frac{ea}{8\pi}[A_0(t,y^1),\pi_0(t,x^1)]\\&=\frac{ie}{8\pi}\delta(x^1-y^1)\ .\end{aligned} \tag{3.27}$$

Analogously, we find

$$[\tilde{j}^0(t,y^1),\pi_0(t,x^1)]=\frac{ie}{8\pi}(a-1)\delta(x^1-y^1)\ , \tag{3.28}$$

$$[\tilde{j}^0(t,y^1),E(t,x^1)]=\frac{ie}{8\pi}\delta(x^1-y^1)\ , \tag{3.29}$$

$$[\tilde{j}^1(t,y^1),E(t,x^1)]=-\frac{ie}{8\pi}(a-1)\delta(x^1-y^1)\ . \tag{3.30}$$

Another way of calculating these commutators consists in directly looking for a regularization of the Jacobian associated with (3.8), in accordance with the definition (3.25). It is clear from (3.25) that we should regularize in a different fashion the transformations generated by $\tilde{j}^0$ and $\tilde{j}^1$.

It is easy to check by explicit calculation that the regularization operators we have to use are

$$\not{D}=\not{\partial}+ie\not{A}(P_L+aP_R)\ , \tag{3.31}$$

for $\tilde{j}^0$, and

$$\not{D}=(1+a)^{1/2}[\not{\partial}+ie\not{A}(P_L+aP_R)]\ , \tag{3.32}$$

for $\tilde{j}^1$.

Now we consider the calculation of some commutators in (3+1)-dimensional chiral QED taken as a theory without constraints.

The Lagrangian for this theory is given by

$$\mathcal{L}=\bar{\psi}i\not{D}\psi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ , \tag{3.33}$$

where now

$$\begin{aligned}&\not{D}=\not{\partial}+ie\not{A}P_L,\quad P_L=\tfrac{1}{2}(1+\gamma^5)\ ,\\&\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3\ ,\end{aligned} \tag{3.34}$$

and the gauge field A^μ is Abelian.

In consequence, the generating functional we consider is

$$G(J,\bar{\eta},\eta)=\int\mathcal{D}\pi_0\mathcal{D}E\,\mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\times\exp\left[i\int d^4x(\pi_0\partial_0A_0+\mathbf{E}\partial_0\mathbf{A}+i\bar{\psi}\gamma^0\partial_0\psi-\mathcal{H}+J\cdot A+\bar{\psi}\eta+\bar{\eta}\psi)\right]\ . \tag{3.35}$$

We will calculate the commutators between the charge density and the canonical momenta of the gauge fields. The charge density operator is chosen as the temporal component of the gauge-covariant current. Therefore under the transformation

$$\begin{aligned}&\psi(x)\to e^{i\alpha(x)P_L}\psi(x)\ ,\\&\bar{\psi}(x)\to\bar{\psi}(x)e^{-i\alpha(x)P_R}\ ,\end{aligned} \tag{3.36}$$

the Jacobian for the fermionic measure is[10]

$$J=\exp\left[\frac{ie^2}{16\pi^2}\int d^4x\ \alpha(x)\tilde{F}_{\mu\nu}F^{\mu\nu}\right]\ . \tag{3.37}$$

For the commutator between π_0 and j^0, we have the expression

$$[j^0(t,\bar{y}),\pi_0(t,\bar{x})]=-\left\langle\frac{\delta J}{\delta\epsilon(t)}\right\rangle_{H=\pi_0(t,\bar{x})}, \quad (3.38)$$

where the Jacobian appearing in (3.38) is the one corresponding to the transformation

$$\psi(t,\bar{z})\to\psi(t,\bar{z})-i\epsilon(t)\delta(\bar{x}-\bar{z})P_L\psi(t,\bar{z}),$$
$$\bar{\psi}(t,\bar{z})\to\bar{\psi}(t,\bar{z})+i\epsilon(t)\delta(\bar{x}-\bar{z})\bar{\psi}(t,\bar{z})P_R. \quad (3.39)$$

Then we find

$$[j^0(t,\bar{y}),\pi_0(t,\bar{x})]=\frac{ie^2}{16\pi^2}\langle\tilde{F}_{\mu\nu}(t,\bar{y})F^{\mu\nu}(t,\bar{y})\rangle|_{H=\pi_0(t,\bar{x})}=-\frac{ie^2}{4\pi^2}\langle E^j(t,\bar{y})B^j(t,\bar{y})\rangle|_{H=\pi_0(t,\bar{x})}=0, \quad (3.40)$$

where the expectation value in presence of the Hamiltonian $\pi_0(t,\bar{x})$ is evaluated by a procedure analogous to the one employed in the CSM.

Finally, we can calculate the commutator between the same current and the field E. In this case we have

$$[j^0(t,\bar{y}),E_k(t,\bar{x})]=-\frac{ie^2}{4\pi^2}\langle E^j(t,\bar{y})B^j(t,\bar{y})\rangle|_{H=E_k(t,\bar{x})}=-\frac{ie^2}{4\pi^2}B_k(t,\bar{x})\delta(\bar{x}-\bar{y}). \quad (3.41)$$

Note that the different Hamiltonians give place to different expectation values of the same operator.

Although the antisymmetry of the commutators is not explicit in the above calculations, this property has been proved in general (Appendix B) and can be checked in each case. As an example we calculate the commutator (3.29) between $E(t,x^1)$ and $\tilde{j}^0(t,y^1)$ in the reverse order to the one chosen in (3.29). Following (2.29), we first consider the transformation generated by $\tilde{j}^0(t,y^1)$ with parameter $\alpha(t)$:

$$\psi(t,z^1)\to\psi(t,z^1)-i\alpha(t)\delta(z^1-y^1)P_L\psi(t,z^1),$$
$$\bar{\psi}(t,z^1)\to\bar{\psi}(t,z^1)+i\alpha(t)\delta(z^1-y^1)\bar{\psi}(t,z^1)P_R, \quad (3.42)$$

under which the measure acquires the Jacobian

$$J(\alpha)=\exp\left[-\frac{ie}{8\pi}\int dt\,\alpha(t)[\epsilon^{\mu\nu}-(1-a)g^{\mu\nu}]\partial_\mu A_\nu(t,y^1)\right]. \quad (3.43)$$

Now we perform the transformation generated by $E(t,x^1)$,

$$A_1(t,z^1)\to A_1(t,z^1)+\epsilon(t)\delta(z^1-x^1), \quad (3.44)$$

under which the only modification is that the Jacobian (3.43) will change to

$$J(\alpha,\epsilon)=\exp\left[-\frac{ie}{8\pi}\int dt\{\alpha(t)[\epsilon^{\mu\nu}-(1-a)g^{\mu\nu}]\partial_\mu A_\nu(t,y^1)+\alpha(t)[\dot{\epsilon}(t)\delta(x^1-y^1)(1+a)+\epsilon(t)\delta'(y^1-x^1)(1-a)]\}\right]. \quad (3.45)$$

Integrating by parts and recalling (2.29), we have to calculate

$$\frac{\delta J}{\delta\epsilon(t)}(\alpha,\epsilon)=\frac{ie}{8\pi}[\dot{\alpha}(t)\delta(x^1-y^1)+(1-a)\alpha(t)\delta'(y^1-x^1)], \quad (3.46)$$

and then taking the limits $\dot{\alpha}\to 1$, $\epsilon\to 0$, $\Delta\to 0$ we obtain

$$[E(t,x^1),\tilde{j}^0(t,y^1)]=-\frac{ie}{8\pi}\delta(x^1-y^1), \quad (3.47)$$

which must be compared with (3.29).

Finally we make some comments about the calculation of current-current commutators. It is readily verified that the second term in the RHS of (2.29) does not contribute to these commutators. In Refs. 12 and 13 it is shown that using a point-splitting definition of the currents and canonical anticommutators for the fermionic fields, the commutators between currents so obtained are in agreement with the results obtained with the BJL limit. Therefore the result for these commutators is correctly given by the first term in (2.29). However, we remark that the point-splitting definitions employed in Refs. 12 and 13 do not agree with our definitions, but using them the same results can be obtained.[14]

IV. CONCLUSIONS

We have derived a method for the calculation of quantum commutators in the functional integral formulation of quantum mechanics. This method leads to a general expression for ETC's where anomalous contributions arise when the FIM has anomalous transformation properties. In order to get results in agreement with the BJL method based on perturbation theory one should make the correspondence between the regularization involved in the perturbative calculation of anomalous Ward identities and the associated regularization of ill-defined transformations of the FIM.

Having obtained a general expression for ETC's a natural step is to analyze general properties of them. In particular it would be interesting to know under what conditions one could expect a violation of Jacobi identity. In this respect the application of this method to the calculation of the ETC between electric fields in the (3+1)-dimensional non-Abelian chiral gauge theory is certainly interesting. This method can be very useful in the search of a consistent quantization of chiral gauge theories, particularly if one employs the method proposed in Ref. 7.

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In this appendix we reexamine the results obtained in Sec. II in terms, of the usually employed relation between the operator formulation and functional integral formulation of quantum mechanics. We think that this appendix will help to clarify certain points of Sec. II. In particular we are interested in questions such as operator ordering, border terms, and the operator version of the first term in the RHS of (2.28).

Let us consider the transition amplitude

$$T^H_{FI} = \langle q^F, t | q^I, 0 \rangle = \langle q^F | e^{-(i/\hbar)tH} | q^I \rangle, \quad t \ll 1 , \tag{A1}$$

where q^F and q^I denote certain values for the coordinates of all the degrees of freedom of the system. For further use and in order to fix notation we derive, following the standard procedure, the functional-integral version of (A1). We rewrite (A1) as

$$\begin{aligned} T^H_{FI} &= \langle q^F | (e^{-(i/\hbar)\Delta t H})^N | q^I \rangle \\ &= \int dp_N \prod_{i=1}^{N-1} dp_i dq_i \langle q^F | p_N \rangle \langle p_N | e^{-(i/\hbar)\Delta t H} | q_{N-1} \rangle \langle q_{N-1} | p_{N-1} \rangle \\ &\qquad \times \langle p_{N-1} | e^{-(i/\hbar)\Delta t H} | q_{N-2} \rangle \cdots \langle q_2 | p_1 \rangle \langle p_1 | e^{-(i\Delta t/\hbar)H} | q^I \rangle, \quad N\Delta t = t . \end{aligned} \tag{A2}$$

In order to evaluate the matrix elements appearing in (A2) we choose a definite operator ordering in the Hamiltonian. Following Ref. 15 we adopt the convention that in H all the p's are to the left of the q's. We denote this particular operator ordering by semicolons on each side of the operator in question. With this convention we have

$$\langle p | e^{-(i/\hbar)\Delta t :H:} | q \rangle = \frac{1}{(2\pi)^{1/2}} \exp\left[\frac{-i}{\hbar} [pq + H(p,q)] \Delta t \right], \quad \Delta t \ll 1 ; \tag{A3}$$

inserting (A3) in (A2) we obtain the result

$$T^H_{FI} = \int \frac{dp_N}{2\pi} \prod_{i=1}^{N-1} \frac{dp_i dq_i}{2\pi} \exp\left[\frac{i}{\hbar} \sum_{n=0}^{N-1} \left[p_{n+1} \frac{q_{n+1} - q_n}{\Delta t} - H(p_{n+1}, q_n) \right] \Delta t \right] . \tag{A4}$$

Now we proceed to undo time evolution. Let us consider the following family of unitary operators:

$$G_n = \exp\left[\frac{i}{\hbar} n \Delta t :H: \right], \quad n = 0, 1, \ldots, N . \tag{A5}$$

Inserting these operators in (A2) we can rewrite T^H_{FI} as

$$\begin{aligned} T^H_{FI} &= \int dp_N \prod_{i=1}^{N-1} dp_i dq_i \langle q^F | G_N^\dagger | p_N \rangle \langle p_N | G_N e^{-(i/\hbar)\Delta t :H:} G_{N-1}^\dagger | q_{N-1} \rangle \\ &\qquad \times \langle q_{N-1} | G_{N-1} G_{N-1}^\dagger | p_{N-1} \rangle \cdots \langle q_1 | G_1 G_1^\dagger | p_1 \rangle \langle p_1 | G_1 e^{-(i/\hbar)\Delta t :H:} G_0^\dagger | q^I \rangle , \end{aligned} \tag{A6}$$

noting that

$$\begin{aligned} & G_n e^{-(i/\hbar)\Delta t :H:} G_{n-1}^\dagger = 1 , \\ & \langle q^F | G_N^\dagger | p_N \rangle = \frac{1}{(2\pi)^{1/2}} \exp\left[-\frac{i}{\hbar} t H(p_N, q^F) \right] \langle q^F | p_N \rangle, \quad T \ll 1 , \end{aligned} \tag{A7}$$

we get

$$T^H_{FI} = \int \frac{dp_N}{2\pi} \prod_{i=1}^{N-1} \frac{dp_i dq_i}{2\pi} \exp\left[\frac{i}{\hbar} \sum_{n=0}^{N-1} \left[p_{n+1} \frac{q_{n+1} - q_n}{\Delta t} - tH(p_N, q_F) \right] \right] . \tag{A8}$$

The last term in the exponent of (A8) is exactly the contribution of the border terms,

$$\int_0^t \frac{d}{d\tau} \left[\Phi^{(1)} + \sum_i p_i q_i \right] d\tau , \tag{A9}$$

that we have neglected in (2.11). Now neglecting this border term corresponds to setting $G_N^\dagger = 1$ in (A6). And this is the same as considering the trivial transition amplitude

$$T_{FI}^{0}=\langle q^{F}|G_{N}e^{-(i/\hbar)tH}|q^{I}\rangle=\langle q^{F}|q^{I}\rangle\ , \tag{A10}$$

that is what we wanted to consider in (2.11). The procedure employed in obtaining α evolution from the trivial transition amplitude (A10) is the same. We consider the following family of unitary operators:

$$U_{n}=\exp\left[-\frac{i}{\hbar}\alpha_{m}:A:\right],\quad \alpha_{0}=0\ . \tag{A11}$$

Inserting the U_n's in the trivial transition amplitude (A10) in the same way as the G_n's were inserted in (A6) and evaluating matrix elements one gets

$$T_{FI}^{0}=\int\frac{dp_{N}}{2\pi}\prod_{i=1}^{N-1}\frac{dp_{i}dq_{i}}{2\pi}\exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1}\left[p_{n+1}\frac{q_{n+1}-q_{n}}{\Delta t}-\frac{\alpha_{n+1}-\alpha_{n}}{\Delta t}A(p_{n+1},q_{n})\right]-\frac{i}{\hbar}\alpha_{N}A(p_{N},q_{F})\right]\ . \tag{A12}$$

Neglecting the border terms in (A12) is equivalent to considering the transition amplitude

$$T_{FI}^{A}=\langle q^{F}|U_{N}|q^{I}\rangle=\left\langle q^{F}\left|\exp\left[-\frac{i}{\hbar}\alpha_{n}:A:\right]\right|q^{I}\right\rangle\ , \tag{A13}$$

that corresponds to (2.16).

In order to obtain the ETC between the operators A and B we consider the following unitary operators:

$$V_{n}=\exp\left[-\frac{i}{\hbar}\epsilon_{n}:B:\right]\ . \tag{A14}$$

Inserting these operators in (A13) we have

$$T_{FI}^{A}=\int dp_{N}\prod_{i=1}^{N-1}dp_{i}dq_{i}\langle q^{F}|V_{N}^{\dagger}|p_{N}\rangle\langle p_{N}|V_{N}e^{-(i/\hbar)\alpha_{m}:A:}V_{N-1}^{\dagger}|g_{N-1}\rangle$$
$$\times\langle q_{N-1}|V_{N-1}V_{N-1}^{\dagger}|p_{N-1}\rangle\cdots\langle q_{1}|V_{1}V_{1}^{\dagger}|p_{1}\rangle\langle p_{1}|V_{1}e^{-i/\hbar:A:}V_{0}^{\dagger}|q^{I}\rangle\ , \tag{A15}$$

noting that

$$V_{n}e^{-(i/\hbar)\alpha_{n}:A:}V_{n-1}^{\dagger}=V_{n}e^{-(i/\hbar)\alpha_{n}:A:}V_{n}^{\dagger}V_{n}V_{n-1}^{\dagger}$$
$$=\exp\left\{-\frac{i}{\hbar}(\dot{\alpha}_{n}:A:-\dot{\alpha}_{n}\epsilon_{n}[:A:,:B:])\Delta t\right\}\exp\left[-\frac{i}{\hbar}\frac{\epsilon_{n}-\epsilon_{n-1}}{\Delta t}:B:\Delta t\right]\ , \tag{A16}$$

where

$$\dot{\alpha}_{n}=\frac{\alpha_{n}-\alpha_{n-1}}{\Delta t}\ . \tag{A17}$$

Replacing (A16) in (A15) we get

$$T_{FI}^{A}=\int dp_{N}\prod_{i=1}^{N-1}dp_{i}dq_{i}\langle q^{F}|V_{N}^{\dagger}|p_{N}\rangle\langle p_{N}|e^{-(i/\hbar)\dot{\epsilon}_{n}\Delta t:B:}e^{-(i/\hbar)\dot{\alpha}_{N}(:A:-\epsilon_{n}[:A:,:B:])\Delta t}|q_{N-1}\rangle$$
$$\times\langle q_{N-1}|p_{N-1}\rangle\cdots\langle q_{1}|p_{1}\rangle\langle p_{1}|e^{-(i/\hbar)\dot{\epsilon}_{1}\Delta t:B:}e^{-(i/\hbar)\dot{\alpha}_{1}\Delta t(:A:-\epsilon_{1}[:A:,:B:])}|q^{I}\rangle\ . \tag{A18}$$

As T_{FI}^{A} is independent of the ϵ's, then taking the derivative of (A18) with respect to ϵ_n at $\epsilon_j=0$, $\forall_j$ and equating to zero one gets an expression that shows us that the first term in the RHS of (2.28) is the canonical commutator of the ordered operators A and B.

All this derivation does not touch upon the appearance of AT's in ETC's. In order to get an idea of the origin of these terms in the present discussion we briefly reconsider some points of the preceding derivation. Let us consider the identity component of the group of PBPT's of the phase space of the system. Each element of this group can be represented by its generating functional,

$$\Phi_{A}(q,P)=\sum_{i}q_{i}P_{i}+\alpha A(q,P)\ . \tag{A19}$$

To the group product of two elements there corresponds a generating functional that is obtained as a Baker-Haussdorf type of expansion (16). The algebra of this group is the Poisson-bracket algebra; even for only one degree of freedom this is an infinite-dimensional Lie algebra. In formulas such as (A5), (A11), and (A14) we are dealing with an infinite-dimensional representation of this group. To a group element of the generic form (A19) we associate the following unitary operator acting on the space of states of the system:

$$U_A = \exp[-i\alpha :\hat{A}(\hat{p},\hat{q}):] , \tag{A20}$$

it is important to note the ordering symbol in (A20). This representation is in general a projective one (17). If this were not the case, then the following equation would be valid,

$$[:\hat{A}:,:\hat{B}:] = :\{A,\hat{B}\}: , \tag{A21}$$

which is not in general true. The representation (A20) will in general produce phase factors in the group composition law and the corresponding extensions of the Lie algebra of this group. Anomalous terms arising from the second term in the RHS of (2.29) arise in a similar way. If for certain subgroups we have, in addition to the operator-ordering phase factor appearing in the composition law, extra projective phases, then the Lie algebra will have additional extensions that are the anomalous terms. From this point of view the present work just gives a method for calculating these extensions in terms of the transformation properties of the FIM.

APPENDIX B

In this appendix we consider the antisymmetry and distributivity under addition of the ETC's. We remark that *a priori* the expression (2.29) for the ETC does not seem to have any simple transformation property under interchange of its arguments A and B. Therefore for the moment we take the convention that the first operator in the commutator is the one that dictates evolution and the second argument is the one for which we calculate the equation of motion. The strategy we employ in proving the above-mentioned properties is the following. We first show that $[0,A]=0$; using this result we next prove that $[A,A]=0$; finally we derive distributivity in both arguments. From the last two properties antisymmetry follows.

$$\text{Property (1)} \quad [0,A]=0 . \tag{B1}$$

Let us consider the following matrix element for evolution with the null operator

$$A(\Delta) \equiv \langle q^F | e^{i\Delta 0} A e^{-i\Delta 0} | q^I \rangle = \int [d\mu]_H A(0) \exp\left[\frac{i}{\hbar}\int_{-\Delta}^{\Delta} dt \sum_i p_i \dot{q}_i\right] . \tag{B2}$$

We note that in writing (B2) we have used the assumption $J(\{q,p\},q_{-H})=1$. We can express $A(\Delta+\epsilon)$, $\epsilon \ll 1$, as follows:

$$A(\Delta+\epsilon) = A(\Delta) + \epsilon \frac{\partial}{\partial\Delta} A(\Delta)$$
$$= A(\Delta) + \epsilon \int [d\mu]_H A(0) \frac{\partial}{\partial\Delta}\left[\frac{i}{\hbar}\int_0^{\Delta} dt \sum_i p_i \dot{q}_i\right] \times \exp\left[\frac{i}{\hbar}\int_0^{\Delta} dt \sum_i p_i \dot{q}_i\right] . \tag{B3}$$

By means of the Hamilton-Jacobi equation,

$$\frac{\partial S}{\partial \Delta} = H , \tag{B4}$$

we obtain

$$A(\Delta+\epsilon) = A(\Delta) \tag{B5}$$

so that

$$-i\hbar \frac{dA}{d\Delta}(\Delta) = 0 = [0,A] . \tag{B6}$$

We note that using this result and the expression for the commutator as given by (2.29) we obtain the following identity between matrix elements:

$$[0,A] = \lim_{\substack{\epsilon\to 0\\ \Delta\to 0}} -\hbar^2 \frac{\delta}{\delta\epsilon} J(\{q,p\},g_A(\epsilon)) = 0 . \tag{B7}$$

$$\text{Property (2)} \quad [A,A]=0 . \tag{B8}$$

From (2.29) the expression for this commutator is

$$[A,A] = -\lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \hbar^2 \frac{\delta}{\delta\epsilon} J(\{q,p\},g_A(\epsilon)\circ g_A(\alpha))$$
$$= -\lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \hbar^2 \frac{\delta}{\delta\epsilon} J(\{q,p\},g_A(\epsilon+\alpha))$$
$$= -\lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \hbar^2 \frac{\delta}{\delta\epsilon} J(\{q,p\},g_A(\epsilon)) , \tag{B9}$$

where for the last equality we have used the fact that due to the initial condition $\alpha(0)=0$, if $\dot\alpha \to 1$ and $\Delta\to 0$ then $\alpha\to 0$. From (B9) and (B7), (B8) follows.

$$\text{Property (3a)} \quad [B+C,A] = [B,A]+[C,A] . \tag{B10}$$

The anomalous contribution to the commutator in the LHS of (B10) is given by

$$[B+C,A]_{\text{an}} = -\hbar^2 \lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \frac{\delta}{\delta\alpha} J(\{q,p\},g_A(\alpha)\circ g_B(\epsilon)\circ g_C(\epsilon)) . \tag{B11}$$

This contribution is the one for which distributivity is not obvious. By repeated application of (2.22) to the Jacobian appearing in (B11) we obtain the identity

$$J(\{q,p\},g_A(\alpha)\circ g_B(\epsilon)\circ g_C(\epsilon)) = J(\{q,p\}^{g_A(\alpha)\circ g_B(\epsilon)},g_C(\epsilon)) J(\{q,p\}^{g_A(\alpha)},g_B(\epsilon)) \times J(\{q,p\},g_A(\alpha)) , \tag{B12}$$

now inserting (B12) in (B11) we have

$$[B+C,A]_{\text{an}} = -\hbar^2 \lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \left[\frac{\delta J}{\delta\alpha}(\{q,p\}^{g_A(\alpha)\circ g_B(\epsilon)},g_C(\epsilon)) + \frac{\delta}{\delta\alpha} J(\{q,p\}^{g_A(\alpha)},g_B(\epsilon)) + \frac{\delta}{\delta\alpha} J(\{q,p\},g_A(\alpha))\right] \tag{B13}$$

the last term in the RHS of (B13) vanishes because of (B7). The second term gives just $[B, A]_{\rm an}$ and the first term gives $[C, A]_{\rm an}$, using the same argument that lead to the last equality in (B9). So we obtain

$$[B+C, A]_{\rm an}=[B, A]_{\rm an}+[C, A]_{\rm an}\ , \tag{B14}$$

which leads to (B10).

Property (3b) $[A,B+C]=[A,B]+[A,C]$. (B15)

We have

$$[A,B+C]_{\rm an} = -\hbar^2 \lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \frac{\delta}{\delta\epsilon} J(\{q,p\}, g_B(\epsilon)\circ g_C(\epsilon)\circ g_A(\alpha))\ ; \tag{B16}$$

by using (2.22) we can rewrite (B16) as

$$[A,B+C]_{\rm an} = -\hbar^2 \lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \left[\frac{\delta}{\delta\epsilon} J(\{q,p\}^{g_B(\epsilon)\circ g_C(\epsilon)}, g_A(\alpha)) + \frac{\delta}{\delta\epsilon} J(\{q,p\}, g_B(\epsilon)\circ g_C(\epsilon)) \right] . \tag{B17}$$

The last term vanishes because of the analog of (B7) for the operator $B+C$. Neglecting terms of $O(\epsilon^2)$, which vanish in the limit $\epsilon\to 0$, the first term can be written as

$$-\hbar^2 \lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \frac{\delta}{\delta\epsilon} J(\{q,p\}^{g_B(\epsilon)\circ g_C(\epsilon)}, g_A(\alpha))$$
$$= -\hbar^2 \lim_{\substack{\dot\alpha\to 1\\ \epsilon\to 0\\ \Delta\to 0}} \left[\frac{\delta}{\delta\epsilon} J(\{q,p\}^{g_B(\epsilon)}, g_A(\alpha)) + \frac{\delta}{\delta\epsilon} J(\{q,p\}^{g_C(\epsilon)}, g_A(\alpha)) \right] , \tag{B18}$$

so that we obtain

$$[A,B+C]_{\rm an}=[A,B]_{\rm an}+[A,C]_{\rm an}\ , \tag{B19}$$

which leads to (B15).

We note that (B8) together with (B10) and (B15) imply

$$[A,B]=-[B,A]\ . \tag{B20}$$

A direct proof of (B20) can also be obtained by using (B7), (B8), and (2.22).

It is important to remark that in proving properties (3a) and (3b), we have assumed that the Jacobians associated with the transformations generated by the two operators appearing in the commutator satisfy the condition (2.22). If one of them or both do not satisfy this condition then we have no proof of these properties.

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