Tunneling in perfect-fluid (minisuperspace) quantum cosmology

J. David Brown
Institute of Field Physics, Department of Physics and Astronomy, CB No. 3255, Phillips Hall, The University of North Carolina, Chapel Hill, North Carolina 27599
(Received 10 October 1989)

A minisuperspace model of general relativity with a positive cosmological constant coupled to a perfect isentropic fluid is presented. The classical equations of motion are solved for a tunneling solution that consists of a Euclidean instanton of the wormhole type, connected to Lorentzian Robertson-Walker universes. The path integral is then defined as a sum over Lorentzian geometries and it is shown that the tunneling solution dominates the semiclassical evaluation of the path integral.

I. INTRODUCTION

One of the fascinating consequences of the quantum nature of particles is that a particle can tunnel between regions of configuration space that are classically separated by a potential-energy barrier. Applying the principles of quantum mechanics to the Universe as a whole then suggests the possibility that three-space can tunnel between classically allowed regions of superspace.

In the context of quantum cosmology, there are numerous references to tunneling in the literature. Hartle and Hawking[1] discuss tunneling processes for gravity conformally coupled to a scalar field, where the scalar field is in a particular excited state. Vilenkin[2] proposes that the Universe has "tunneled from nothing," and that this condition determines the quantum state of the Universe. Most recently, interest in "wormhole physics" has led to the discovery of a number of Euclidean solutions to general relativity plus matter that can be interpreted as instantons.[3-9] That is, the geometry and matter field configurations for such a solution exhibit "turning points," boundary surfaces where Lorentzian geometries can be smoothly attached. Then by connecting Lorentzian solutions to the "ends" of the instanton, the Euclidean region provides a bridge between real classical motions of the system. It is generally believed that such solutions—mixtures of Lorentzian and Euclidean geometries—represent quantum tunneling in that they give a dominant contribution to the path integral for a transition between configurations of the system that are classically allowed, but separated by some effective potential barrier.

The purpose of this paper is first to present a class of such tunneling solutions for general relativity with a positive cosmological constant, coupled to a perfect isentropic fluid. (An isentropic fluid is one in which the entropy per baryon is a constant.) This includes, as a special choice of the equation of state for the fluid, the case of coupling to a massless scalar field,[5] which in turn is related to a two-indexed antisymmetric gauge field[3,4] through dualization. For the general isentropic fluid coupling, the Lorentzian solutions that attach to the Euclidean instanton are portions of distinct Robertson-Walker universes. The system is then described using a minisuperspace reduction to homogeneous, isotropic cosmologies, and the path integral for this model is defined as a sum over Lorentzian spacetime geometries. It is shown that the classical tunneling solution dominates the semiclassical evaluation of the Lorentzian path integral, and indeed represents tunneling in the quantum theory.

As discussed in Refs. 10 and 11, general relativity based on the usual action principle in which the cosmological constant is a fixed parameter, is formally analogous to nonrelativistic mechanics based on Jacobi's action. In Jacobi's action, the energy of the system is fixed, whereas the total time for the system to evolve between initial and final configurations is not fixed. The relationship between gravity and nonrelativistic mechanics is such that the cosmological constant Λ corresponds to energy E while the four-volume of the Universe corresponds to physical time. In minisuperspace models of cosmology, such as the one considered in this paper, the analogy between gravity and nonrelativistic mechanics becomes quite precise. This is because with the gravitational and matter degrees of freedom reduced to a finite number, the reduced action involves only a single constraint rather than the infinity of constraints—four for each space point—in the full theory. This one constraint serves to fix a function of the dynamical variables equal to the cosmological constant Λ; by analogy, the constraint in Jacobi's action for nonrelativistic mechanics sets a function of the dynamical variables, namely, the kinetic plus potential energies, equal to a constant E. Much of the analysis in this paper is inspired by the close formal relationship between general relativity and nonrelativistic mechanics, for which quantization and tunneling are of course relatively well understood.

The independent variables that describe the perfect isentropic fluid are the total particle number, which enters the theory as a momentum, and its conjugate coordinate. That coordinate is cyclic, so then classically the particle number is conserved. (The particles are usually identified with baryons.) It is therefore possible to eliminate the fluid variables by the standard procedure due to Routh.[12] Quantum mechanically, this amounts to assuming the Universe to be in an eigenstate of the total particle number. When this is done, what remains is a system with a single degree of freedom, namely, the volume of three-space, which will be denoted x. Viewed as a nonrelativistic mechanics problem, this system is equivalent to a particle of "mass" $2/3\kappa$ and "energy" Λ/κ moving in the one-dimensional potential $V(x)$ shown in Fig. 1. (Here, κ is 8π times Newton's constant.) If Λ/κ is less than the maximum of $V(x)$, then three-space must tunnel in order to evolve between initial and final configurations x' and x'' that are on opposite sides of the barrier. The under-barrier motion of three-space corresponds to a Euclidean four-geometry, the instanton, and represents tunneling between Robertson-Walker universes.

It is possible to give an alternative description of general relativity,[10,13] that is the analogue of nonrelativistic mechanics based on Hamilton's action principle. In Hamilton's action, the total time interval is fixed by boundary conditions, but the energy of the system is not fixed; in the analogous formulation of gravity, the total spacetime volume of the Universe between initial and final configurations is specified by boundary conditions, but the cosmological constant is not fixed.[14] If the spacetime volume is given a real value, then the classical solution to the variational problem yields a geometry that is Lorentzian everywhere, with no instanton tunneling. This happens because the cosmological constant is free to adjust to a value for which tunneling is not necessary in order for the three-geometry to evolve from the initial to the final configuration.

Then as far as tunneling is concerned, there is a crucial difference between the usual Jacobi-type formulation and the Hamilton-type formulation of general relativity. In terms of the particle analogy, this distinction is easily understood: write the energy equation $\frac{1}{2}m(dx/dt)^2 + V(x)=E$ as

$$dt = dx \left[\frac{m}{2[E-V(x)]} \right]^{1/2} , \tag{1.1}$$

and consider a potential such as in Fig. 1. If the energy of the particle is fixed to a value less than the maximum

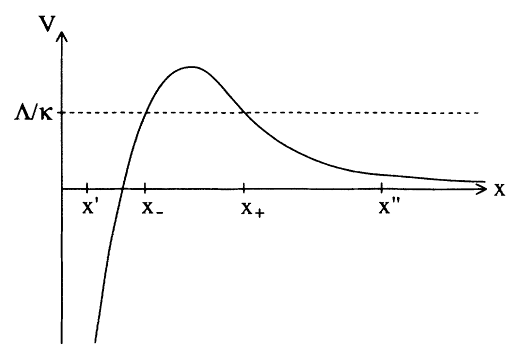

FIG. 1. The potential V that governs the motion of three-space for the minisuperspace model of perfect-fluid cosmology, as a function of spatial volume x. When the cosmological constant Λ/κ is less than the maximum of the potential, there are two classical turning points labeled x_- and x_+.

of $V(x)$, then the particle must tunnel in order to evolve from the initial configuration x' to the final configuration x''. From Eq. (1.1), dt is imaginary in the under-barrier region, and the total time interval, obtained by integrating the right-hand side of (1.1) from x' to x'', is complex. On the other hand, if the energy is not fixed but the total time interval is fixed to some real value, Eq. (1.1) can be integrated and then solved for the energy. In that case, the resulting value of E must necessarily be greater than the maximum of the potential $V(x)$, and tunneling does not occur.

So in order to investigate tunneling, I will make use of the usual formulation of general relativity with the cosmological constant fixed. This means that the wave function for the Universe is chosen to be in a Λ eigenstate, using the terminology of Refs. 10, 13, and 17. More generally, tunneling will occur if the wave function for the Universe is described by a wave packet that is a superposition of Λ eigenstates with values of Λ less than the barrier height.

Recently, one of the issues that has received considerable attention in quantum cosmology involves the definition of the path integral—should it be defined as a sum over Lorentzian or Euclidean four-geometries, or over some other set of complex geometries?[18,19] I will adopt a Lorentzian point of view throughout this paper and define the path integral as a sum over Lorentzian geometries. Then Euclidean solutions to the classical equations of motion are considered to be stationary configurations of the action that lie off the "axis" of integration. The question of choosing Lorentzian or Euclidean definitions is discussed more fully in Sec. II.

In order to deal with the quantization of perfect fluids, an action principle is needed. This is obvious for path-integral quantization, in which the classical action determines the phase for each path. But also for canonical quantization, it is necessary to have an action for the system so that the canonical variables and their Poisson-brackets structure can be identified. Only then can the quantum operators and their algebra be constructed. Thus, in Sec. III, I will present an action principle for perfect fluids. The action for a general perfect fluid[20,21] is rather involved, so for the purpose of this paper, I will only deal with isentropic fluids, and furthermore write the action only for the minisuperspace model to be quantized.

In Sec. IV the minisuperspace model of gravity coupled to an isentropic perfect fluid is presented. The fluid variables are effectively eliminated as a cyclic coordinate and its conserved conjugate momentum, leaving a one-dimensional problem for the spatial three-volume x. For physically reasonable fluid equations of state, the effective potential $V(x)$ has the general shape drawn in Fig. 1. Real Lorentzian solutions to the classical equations of motion are discussed, and tunneling solutions are explicitly obtained [see Eq. (4.11)]. In the latter case, the four-geometry consists of portions of Lorentzian Robertson-Walker universes connected by a Euclidean instanton.

The path integral for the minisuperspace model of perfect-fluid cosmology is considered in Secs. V and VI, based on the Jacobi-type action for the system. In Sec. V the path integral is constructed by adopting Teitelboim's notion of causality,[22] and it is therefore a Green function for the Wheeler-DeWitt equation. Causality here means that the path integral involves a sum over histories with positive values of the spacetime volume between initial and final surfaces. Furthermore, the path integral is constructed for states of fixed particle number and is then equal to a delta function of the final minus initial particle numbers, times the Green function for the effective one-dimensional system characterized by the potential $V(x)$. The one-dimensional Green function is evaluated semiclassically in Sec. VI. Some aspects of the calculation are patterned after results first developed by McLaughlin[23] in his path-integral description of barrier penetration in nonrelativistic mechanics. In particular, the contour of integration for spacetime volume is distorted in the complex plane to pass through an appropriate stationary point, so that the path integral is approximated by the behavior of the system near the classical tunneling configuration. The resulting Green function contains the expected barrier penetration factor [see Eq. (6.24)].

Section VII contains comments and miscellaneous results. Included is a discussion of the relationship between the classical tunneling solutions found in Sec. IV and the complex stationary configuration that arises in the semiclassical evaluation of the path integral. Section VII also contains a brief discussion of the WKB transmission coefficient for tunneling between Robertson-Walker universes.

For the minisuperspace calculations in this paper, the ansatz for the four-metric is given by

$$dS^2 = -\frac{\alpha^2}{x^2}d\sigma^2 + \left[\frac{x}{2\pi^2}\right]^{2/3} d\Omega^2 \;, \tag{1.2}$$

where the metric variables α and x depend only on the coordinate σ, and $d\Omega^2$ denotes the metric on a unit three-sphere. As defined in (1.2), x is the volume of three-space,

$$x = \int d^3\Omega \sqrt{^{(3)}g} \;, \tag{1.3}$$

where $^{(3)}g$ denotes the determinant of the spatial metric and $d^3\Omega$ denotes the volume measure for a three-sphere. Likewise, α is given by

$$\alpha = \int d^3\Omega \sqrt{-^{(4)}g} \;, \tag{1.4}$$

with $^{(4)}g$ the determinant of the four-metric. Thus, α is proportional to the scalar lapse function multiplied by $\sqrt{^{(3)}g}$, and will be referred to as the lapse density.[10,13] The total spacetime volume is given by the integral $\int d\sigma\, \alpha$ of the lapse density. There will also be occasion to make use of the scalar lapse function, which will be denoted $\tilde{\alpha}$ and is related to the lapse density by $\alpha = \tilde{\alpha} x$.

It should be noted that the specification of variables in (1.2) may not be an inconsequential matter of choice. This is because different choices of the density weight for the lapse lead to Wheeler-DeWitt equations that are inequivalent due to differences in factor ordering, assuming the derivative terms are ordered to give the minisuperspace Laplacian. In more than one dimension, this inequivalence can be removed by including in the Wheeler-DeWitt equation an appropriate term proportional to the minisuperspace scalar curvature.[24,25] However, for problems with just one minisuperspace dimension, which is effectively the case here, the discrepancy between different choices of variables remains. Note however that the calculations of Sec. VI are limited to the WKB approximation, and the rapidly varying exponential factors in the Green functions are insensitive to such factor ordering ambiguities. On the other hand, the slowly varying preexponential factors would be affected by a change of variables.

II. LORENTZIAN OR EUCLIDEAN?

The complete tunneling configuration consists of Lorentzian Robertson-Walker universes connected by a Euclidean instanton. It is obtained directly in Sec. IV as a solution to the Einstein and matter equations of motion, as opposed to obtaining the Lorentzian and Euclidean parts of the solution independently. Here it is important to recognize that, as a set of differential equations for the metric tensor components $g_{\mu\nu}$, there is no distinction between "Lorentzian" and "Euclidean" Einstein equations. However, *solutions* to the Einstein equations can be Lorentzian or Euclidean; in cases such as tunneling, solutions can be a mixture of Lorentzian in some regions and Euclidean in others. Of course, in solving the Einstein equations, we are often prejudiced towards finding Lorentzian or Euclidean solutions by writing the metric coefficient g^{00} as either $-1/\tilde{\alpha}^2$ or $+1/\tilde{\alpha}^2$, where $\tilde{\alpha}$ is the scalar lapse function. This is still completely general though, if it is allowed for $\tilde{\alpha}$ to be complex. Tunneling solutions are of this type: defining $g^{00} = -1/\tilde{\alpha}^2$, they consist of two (Lorentzian) regions with real lapse, separated by a (Euclidean) region with imaginary lapse. Accordingly, instantons can be referred to as "complex classical solutions," since they are associated with evolution in imaginary time (that is, imaginary lapse).

In the quantum theory, the question of whether the path integral is Lorentzian or Euclidean is just the question of how the contours of integration should be chosen for the lapse $\tilde{\alpha}$. If $\tilde{\alpha}$ is integrated over real values, then the path integral is a sum over Lorentzian geometries; if $\tilde{\alpha}$ is integrated over imaginary values, then the path integral is a sum over Euclidean geometries.

For minisuperspace models of quantum cosmology, the path integral for the Green function can be written as a single ordinary integral over the spacetime volume of an appropriate kernel.[26] (The kernel is itself proportional to the path integral that follows from the Hamilton-type action for general relativity plus matter. That path integral is the transition amplitude for the system to evolve between initial and final configurations, with the total spacetime volume fixed but the cosmological constant not fixed.) In particular, the path integral does not include the full functional integration over $\alpha(\sigma)$, due to the removal of redundant histories by gauge fixing. Then the functional integral is a sum over Lorentzian or Euclidean geometries depending on whether the integration of the four-volume is taken over real or imaginary values. In Sec. VI it is argued that for the minisuperspace model of gravity coupled to a perfect isentropic fluid, the Lorentzian path integral is well defined in the sense that it consists of an infinite product of convergent integrals.[30] Here, there appears to be no reason to choose the Euclidean definition of the path integral over the Lorentzian one—in fact, as is well known, the Euclidean path integral is problematic because of the negative conformal mode.

The demonstration that the Lorentzian path integral is well defined is given in two steps. First, it is argued that within a time-slicing formulation, the kernel is a product of integrals each of which converges at least as quickly as the standard Fresnel integral. Having established its convergence, the kernel is then evaluated semiclassically using WKB methods. The second step is to analyze the convergence of the single integral over spacetime volume of the WKB kernel. By comparison again with a Fresnel integral, it is shown that the integral over positive real values of the spacetime volume converges. The path integral therefore appears to be well defined as a sum over Lorentzian four-geometries.

The path integral analyzed here is particularly interesting, because with Λ/κ chosen to be less than the effective potential barrier height, there is no Lorentzian four-geometry that extremizes the action. This, of course, just reflects the fact that the transition considered involves tunneling, which is a purely quantum effect and cannot be approximated by any (real) classical history. So in the final integration of the kernel over spacetime volume, the steepest-descent values of the four-volume lie off the real axis of integration, and are in fact complex. In order to evaluate this integral semiclassically, the contour of integration must be distorted from the real axis to pass through the appropriate saddle points.[23] The complex four-geometry that dominates this approximate evaluation of the path integral is just the classical tunneling solution obtained in Sec. IV. It is in this sense that the instantons of Sec. IV represent tunneling in the path integral.

III. ACTION FOR A PERFECT ISENTROPIC FLUID

In this section I will present an action principle for a perfect isentropic fluid that is suitable for the minisuperspace model to be analyzed in the following sections. In general, perfect fluids are composed of particles, such as baryons, that are conserved, and the fluid must furthermore satisfy the first law of thermodynamics: energy conservation.[34] Applied to a fluid element consisting of a fixed number of baryons, the first law reads

$$d(\rho/n) = -P d(1/n) + T\, ds \ , \tag{3.1}$$

where ρ is the energy density and n is the baryon-number density of the fluid. Thus, ρ/n is the energy per baryon and $1/n$ is the volume per baryon. In addition, P is the pressure, T the temperature, and s the entropy per baryon. All the thermodynamic potentials appearing in Eq. (3.1) are to be measured in the rest frame of the fluid element.

An isentropic fluid is one in which the entropy per baryon is a constant, so that $ds=0$. In that case, the first law can be written as

$$P = n\frac{d\rho}{dn} - \rho \ . \tag{3.2}$$

This relationship can be enforced in the following way. First, *choose* $\rho(n)$, the energy density as a function of the number density, as an equation of state. Second, use (3.2) to *identify* $P(n) \equiv n\rho' - \rho$ as the pressure, with the prime denoting differentiation with respect to n. Then given an equation of state $\rho(n)$, the local thermodynamic properties of an isentropic fluid are described in terms of only one independent variable, the number density n.

In addition to the thermodynamic properties discussed above, a perfect fluid is also characterized by its four-velocity U^μ, and energy-momentum tensor

$$T_{\mu\nu} = \rho(n) U_\mu U_\nu + P(n)(g_{\mu\nu} + U_\mu U_\nu) \ . \tag{3.3}$$

The equations of motion for the fluid are the equations $T^{\mu\nu}{}_{;\nu}=0$, expressing the conservation of energy momentum in the fluid flow. Using (3.2), the component $T^{\mu\nu}{}_{;\nu}U_\mu=0$ in the direction of flow reduces to $(nU^\mu)_{;\mu}=0$, implying the conservation of baryons; accordingly, baryon-number conservation and energy conservation were used to deduce the expression for pressure $P(n)$ that occurs in the energy momentum tensor (3.3). The components of $T^{\mu\nu}{}_{;\nu}=0$ perpendicular to the fluid flow yield the Euler equations $(\rho+P)a_\mu + (\delta_\mu^\nu + U_\mu U^\nu)P_{;\nu}=0$, which relate the acceleration a_μ of the fluid flow lines to the gradient of the pressure.

The action principle for an isentropic perfect fluid must reproduce the energy-momentum tensor (3.3), as well as baryon-number conservation and the Euler equations as equations of motion. In general, such an action[20,21] is rather complicated, but it will be sufficient for present purposes to consider the action for homogeneous, isotropic fluid flow. Thus, assume that the number density is a function of the coordinate σ only, $n(\sigma)$, and that the fluid four-velocity is orthogonal to the spatial three-spheres, $\|U_\mu\| = (-\sqrt{-g_{\sigma\sigma}},0,0,0)$. Here, the notation for the minisuperspace metric is

$$dS^2 = g_{\sigma\sigma}(\sigma)d\sigma^2 + g_{\ll}(\sigma)d\Omega^2 \tag{3.4}$$

and will later be changed to the notation of Eq. (1.2). (The "angle-angle component" of the metric is denoted $g_{\ll}$.) It is also convenient to change fluid variables from the baryon-number density $n(\sigma)$ to the total particle number $N(\sigma)$, defined by

$$N = \int d^3\Omega (g_{\ll})^{3/2} n = 2\pi^2 (g_{\ll})^{3/2} n \ . \tag{3.5}$$

Then the equation of state is a specified function $\rho(N/2\pi^2(g_{\ll})^{3/2})$.

The action for an isentropic fluid in this minisuperspace model is given by

$$S_f[N,\varphi,g_{\sigma\sigma},g_{<<}]=\frac{1}{2\pi^2}\int d\sigma\, d^3\Omega\{N\dot{\varphi}-\sqrt{-g_{\sigma\sigma}}[2\pi^2(g_{<<})^{3/2}\rho(N/2\pi^2(g_{<<})^{3/2})]\}\ , \tag{3.6}$$

where the overdot denotes differentiation with respect to σ. (The integral over angles is retained for the purpose of computing functional derivatives below.) Notice that the action (3.6) is in canonical form, with the particle number $N(\sigma)$ entering as a momentum conjugate to a coordinate $\varphi(\sigma)$. Also, φ is cyclic, so that the momentum N is conserved; this is how baryon-number conservation is maintained. The equation of motion obtained by varying S_f with respect to N is $\dot{\varphi}/\sqrt{-g_{\sigma\sigma}}=\rho'$. The left-hand side of this equation is just the derivative $U^\mu\varphi_{,\mu}$ of φ along the fluid flow, while the right-hand side is the chemical potential $(\rho+P)/n$ for the fluid. Thus, the variable φ may be described as a "velocity potential"[20] for the chemical potential. Also notice that the Euler equations are identically satisfied in this model; that is, the acceleration a_μ and the orthogonal projection of the pressure gradient $(\delta^\nu_\mu+U^\nu U_\mu)P_{;\nu}$ are both zero by virtue of the minisuperspace ansatz.

The reduced energy-momentum tensor for the fluid is obtained from the action S_f by the definitions

$$T^{\sigma\sigma}\equiv\frac{2}{\sqrt{-{}^{(4)}g}}\frac{\delta S_f}{\delta g_{\sigma\sigma}}\ , \tag{3.7a}$$

$$T^{<<}\equiv\frac{1}{3}\frac{2}{\sqrt{-{}^{(4)}g}}\frac{\delta S_f}{\delta g_{<<}}\ . \tag{3.7b}$$

The extra factor of $\frac{1}{3}$ in (3.7b) is included because $g_{<<}$ is a metric coefficient for all three spatial directions. Likewise, the Einstein tensor component $G^{<<}$ is obtained from the reduced gravitational action S_g of Sec. IV by defining

$$G^{<<}+\Lambda g^{<<}\equiv-\frac{1}{3}\frac{2\kappa}{\sqrt{-{}^{(4)}g}}\frac{\delta S_g}{\delta g_{<<}}\ ,$$

with an extra factor of $\frac{1}{3}$. From the fluid action S_f, (3.7a) gives $T^\sigma_\sigma=-\rho$, as expected from the general form (3.3) for the energy-momentum tensor. Correspondingly, the action (3.6) shows that the matter contribution to the scalar weight Hamiltonian constraint is just the energy density ρ. Furthermore, (3.7b) gives $T^<_<=[N/2\pi^2(g_{<<})^{3/2}]\rho'-\rho$, which is identified as the pressure $P(n)$ and is again the expected result.

To summarize, S_f as given in (3.6) is the correct action for an isentropic perfect fluid, restricted to homogeneous isotropic flow in the spacetimes specified by the metric (3.4). The Euler equations are automatically satisfied, while baryon conservation follows as an equation of motion by varying the potential $\varphi(\sigma)$. Furthermore, S_f produces the correct energy-momentum tensor when varied with respect to the gravitational variables. In preparation for its use in the subsequent sections, the action S_f can be written with the metric in the notation of Eq. (1.2), giving

$$S_f[\varphi,N,\alpha,x]=\int d\sigma\{N\dot{\varphi}-\alpha[\rho(N/x)]\}\ . \tag{3.8}$$

Finally, it is worthwhile noting that for the particular equation of state $\rho(n)=(1/4\pi)k^2n^2$, with k an appropriate dimensionful constant, the fluid action S_f is that of an ordinary scalar field ϕ. This can be seen by starting with the usual scalar field action, specializing to the minisuperspace metric (1.2), and to homogeneous fields $\phi(\sigma)$. The resulting reduced action is

$$S_\phi=\pi\int d\sigma\frac{x^2}{\alpha}(\dot{\phi})^2\ . \tag{3.9}$$

By defining $\phi=\varphi/k$ and passing to the canonical form of the action (3.9) with respect to the matter field, the resulting scalar field action is seen to coincide exactly with the fluid action (3.8) with equation of state $\rho(n)=(1/4\pi)k^2n^2$.

IV. CLASSICAL SOLUTIONS

With the spacetime metric written in terms of spatial volume x and lapse density α as in Eq. (1.2), the action for general relativity reduces to

$$S_g[\alpha,x]=\frac{3}{\kappa}\int d\sigma\left[-\frac{\dot{x}^2}{9\alpha}+\alpha\left[\frac{2\pi^2}{x}\right]^{2/3}-\alpha\left[\frac{\Lambda}{3}\right]\right]\ . \tag{4.1}$$

The momentum conjugate to x will be denoted p. It is defined by $p=-(2/3\kappa)(\dot{x}/\alpha)$, and equals $(2/3\kappa)$ times the trace of the extrinsic curvature for the spatial surfaces $\sigma=$const. Passing to the canonical form of the gravitational action (4.1) and adding the fluid action (3.8), the full action for the minisuperspace model of general relativity coupled to an isentropic fluid is given by

$$S[\alpha,x,p,\varphi,N]=\int_{\sigma'}^{\sigma''}d\sigma(p\dot{x}+N\dot{\varphi}-\alpha\mathcal{H})\ , \tag{4.2}$$

where $\mathcal{H}$ is the Hamiltonian constraint for the system:

$$\mathcal{H}\equiv-\left[\frac{3\kappa}{4}p^2+\frac{3}{\kappa}\left[\frac{2\pi^2}{x}\right]^{2/3}-\rho(N/x)-\frac{\Lambda}{\kappa}\right]\ . \tag{4.3}$$

In (4.2), σ' and σ'' denote the values of the coordinate σ labeling the initial and final spatial surfaces. The action S is to be varied with $x(\sigma'),\varphi(\sigma')$ and $x(\sigma'')$, $\varphi(\sigma'')$ held fixed.

Because the variable φ is cyclic, it can be eliminated from the action (4.2) using the standard procedure due to Routh.[12] In the canonical formalism, this amounts first to passing to a new action S_N in which N is fixed initially and finally, rather than φ. S_N is obtained from S by subtracting the boundary term $N\varphi|^{\sigma''}_{\sigma'}$, which effectively replaces the term $N\dot{\varphi}$ in (4.2) by $-\dot{N}\varphi$. Then invoking the equation of motion $\dot{N}=0$, baryon-number conservation, N can be taken to be constant and the term $-\dot{N}\varphi$ discarded. The resulting action is

$$S_N[\alpha,x,p]=\int_{\sigma'}^{\sigma''}d\sigma(p\dot{x}-\alpha\mathcal{H})\ , \tag{4.4}$$

where the Hamiltonian constraint is again given by (4.3), but with N understood to be a fixed constant.

From the considerations of Ref. 10, it is clear that the action (4.4) is essentially the Jacobi action for a single point particle of "mass" $2/3\kappa$ moving in the one-dimensional "potential"

$$V(x) \equiv \frac{3}{\kappa}\left[\frac{2\pi^2}{x}\right]^{2/3} - \rho(N/x) \; . \tag{4.5}$$

[Observe that $V(x)$depends on the fixed value of N.] Thus, the classical evolution of the spatial volume x in this minisuperspace cosmological model can be conveniently visualized as particle motion in the potential $V(x)$. The Hamiltonian constraint (4.3) fixes the total "energy" $(3\kappa/4)p^2+V(x)$ of the particle to the value Λ/κ. It is worth observing that the Hamiltonian constraint could be written without the overall minus sign that appears in (4.3), which would be the natural convention for a true nonrelativistic mechanics problem. However, that minus sign is a reflection of the fact that x is a conformal factor for the three-geometry, and the corresponding (one-dimensional) superspace metric is negative. The minus sign will be left in place.

As a result of the Hamiltonian constraint, the one-dimensional system described by the action (4.4) formally has no degrees of freedom. The equations of motion for x and p imply only that the "energy" $(3\kappa/4)p^3+V(x)$ is a constant; from the Hamiltonian constraint, that constant is Λ/κ. Solving the classical system amounts to solving nothing more than the Hamiltonian constraint for the motion of the spatial volume x in "time." Here, the appropriate notion of "time" is an increment in the lapse density $\alpha\, d\sigma$, which according to Eq. (1.4) is an increment in the spacetime volume. Defining the spacetime-volume time t through $dt=\alpha\, d\sigma$, the evolution $x(t)$ of the spatial volume is found by substituting the definition $p=-(2/3\kappa)(\dot{x}/\alpha)$ for momentum into the Hamiltonian constraint, giving

$$\mathcal{H} = -\left[\frac{1}{3\kappa}\left[\frac{dx}{dt}\right]^2 + V(x) - \frac{\Lambda}{\kappa}\right] = 0 \; . \tag{4.6}$$

For a given equation of state $\rho(N/x)$, this equation can be intergrated to yield $x(t)$. Then from Eq. (1.2), the resulting spacetime metric is

$$dS^2 = -\frac{dt^2}{x(t)^2} + \left[\frac{x(t)}{2\pi^2}\right]^{2/3} d\Omega^2 \; , \tag{4.7}$$

which describes a Robertson-Walker universe.

When the general shape of the function $V(x)$ in (4.5) is known, explicit solutions to the Hamiltonian constraint (4.6) are not needed in order to understand qualitatively the real classical motions $x(t)$ of three-space. This is made possible by invoking the analogy discussed above: namely, that $x(t)$ represents the position of a particle with fixed energy moving in the potential $V(x)$. Furthermore, for equations of state $\rho(n)$ that satisfy certain natural restrictions, the potential has a generic shape, which is displayed in Fig. 1. Indeed, the behavior of $V(x)$ as $x\to 0$ and ∞ can be deduced by simply assuming an equation of state for which the energy density ρ is positive and the pressure P is non-negative. By Eq. (3.2), this restriction on the pressure gives $(\rho/n)'\geq 0$. Then since $\rho>0$, as $n\to\infty$ the energy density ρ must increase to positive infinity at least as fast as n. This implies that for small values of the spatial volume x, the matter term $-\rho(N/x)$ dominates the potential $V(x)$. Therefore, $V(x)$ tends toward minus infinity as $x\to 0$, as shown in Fig. 1. The behavior of $V(x)$ as $x\to\infty$ can likewise be deduced by expressing the same inequality used above in the form $\partial(x\rho(N/x))/\partial x\leq 0$. Again recalling that ρ is positive, this shows that ρ vanishes at least as fast as $1/x$ in the limit of infinitely large x. Thus, in the limit $x\to\infty$, the term $(3/\kappa)(2\pi^2/x)^{2/3}$ dominates the potential, and $V(x)$ tends toward zero through positive values as shown in Fig. 1.

Now consider the extrema of $V(x)$. They must satisfy $\partial V/\partial x=0$, which reduces in essence to $\rho'\sim x^{1/3}$. The left-hand side of this relationship is just the chemical potential $(\rho+P)/n$ for the fluid, and notice that the right-hand side is a positive, monotonically increasing function of x. Now assume the chemical potential is a nonincreasing function of x, so the relationship $\rho'\sim x^{1/3}$ will have just a single solution. The additional assumption being made here is that the chemical potential is a nondecreasing function of n, or in mathematical terms, $\rho''=[(\rho+P)/n]'\geq 0$. But from Eq. (3.2), this is equivalent to $P'\geq 0$. So in physical terms, the assumption is that the pressure does not decrease with increasing number density. In light of the asymptotic behavior discussed previously, the potential $V(x)$ then has one extremum, which must be a maximum as shown in Fig. 1.

Summarizing the arguments just presented, the potential $V(x)$ will have the generic shape depicted in Fig. 1 for any equation of state $\rho(n)$ that satisfies $\rho>0$, $P\geq 0$, and $P'\geq 0$. It would seem that these restrictions on the equation of state are physically reasonable, at least for classical matter. A useful example of an equation of state with such properties is the γ-law equation of state $\rho(n)=Cn^{\gamma}$, where $1\leq\gamma\leq 2$ and C is a positive constant.[35] According to Eq. (3.2), the pressure and energy density in this case are related by $P=(\gamma-1)\rho$. Notice that the isentropic fluid described by this equation of state with $\gamma=2$ can be identified with a massless scalar field, as discussed at the end of Sec. III.

The real classical solutions $x(t)$ of the Hamiltonian constraint (4.6) can now be easily identified. They naturally fall into different categories, depending on the value of the cosmological constant. For Λ/κ greater than the maximum $V_{\max}$ of $V(x)$, there is a solution in which the Universe expands from an initial singularity $x=0$. As three-space becomes large, the fluid is diluted so that the matter term in the potential becomes negligible. Then for large t, the resulting spacetime asymptotically approaches de Sitter spacetime. There is also the time-reversed version of the solution just described, in which three-space contracts from an approximately de Sitter-like phase to a final singularity. When $\Lambda/\kappa=V_{\max}$, there is a solution for which the spatial volume x equals, for all t, the value that maximizes the potential. This is the Einstein static universe. Likewise, there are solutions in which three-space expands or contracts, asymptotically approaching the Einstein static universe as $t \to \infty$. For $\Lambda/\kappa < V_{max}$, including negative Λ, there is a classical solution to (4.6) in which three-space expands from an initial singularity to some maximum size x_-, "reflects" off the potential barrier, and recontracts to a final singularity. When $\Lambda/\kappa < V_{max}$ and Λ is positive as well, there is a second classical solution in which three-space contracts from a de Sitter-like phase to a minimum size x_+, "reflects" off the potential barrier, then reexpands and again approaches a de Sitter-like phase.

The case in which the cosmological constant takes intermediate values, $0 < \Lambda/\kappa < V_{max}$, is the one that is of primary interest for the remainder of this paper. In this situation there are two distinct classical motions $x(t)$, as just described. In one, three-space remains small; its evolution is dominated by the fluid matter, which halts the initial expansion and causes collapse to a final singularity. In the other, three-space is relatively large and the fluid is dilute; the evolution $x(t)$ is dominated by the cosmological constant, so the resulting spacetime is approximately de Sitter spacetime.

These solutions to the Hamiltonian constraint (4.6) are obtained by integrating the equation

$$dt = \pm \frac{dx}{\sqrt{3\kappa[\Lambda/\kappa - V(x)]}} , \tag{4.8}$$

where the positive or negative square-root must be chosen appropriately, depending on whether three-space is expanding or contracting. For example, consider the matter-dominated solution that begins and ends with a singularity and has a maximum spatial volume x_-. During expansion when $dx/dt > 0$, Eq. (4.8) gives

$$t(x) = \int_0^x \frac{dy}{\sqrt{3\kappa[\Lambda/\kappa - V(y)]}} , \tag{4.9a}$$

while during contraction when $dx/dt < 0$, Eq. (4.8) gives

$$t(x) = -\int_{x_-}^{x} \frac{dy}{\sqrt{3\kappa[\Lambda/\kappa - V(y)]}} + \int_0^{x_-} \frac{dy}{\sqrt{3\kappa[\Lambda/\kappa - V(y)]}} . \tag{4.9b}$$

Similarly for the cosmological-constant-dominated solution, $t(x)$ is found by integrating (4.8) using simple physical reasoning to fix the signs.

Both the fluid-dominated and the cosmological-constant-dominated solutions are Robertson-Walker universes with four-metrics given by (4.7); see Fig. 2. Each spacetime consists of two symmetric regions, one expanding and one contracting, that differ from one another only in the sign of dx/dt. For each expanding and contracting half, the spatial volume x is a monotonic function of t and can therefore serve as a coordinate covering that half of the spacetime. Accordingly, the metric (4.7) is written using the Hamiltonian constraint as

$$dS^2 = -\frac{dx^2}{3\kappa x^2[\Lambda/\kappa - V(x)]} + \left[\frac{x}{2\pi^2}\right]^{2/3} d\Omega^2 . \tag{4.10}$$

The four-geometry for the fluid-dominated solution consists of two copies of the metric (4.10) with $0 < x \le x_-$, one copy for the expanding phase and the other for the contracting phase. Likewise, the four-geometry for the cosmological-constant-dominated solution consists of two copies of the metric (4.10) with $x \ge x_+$.

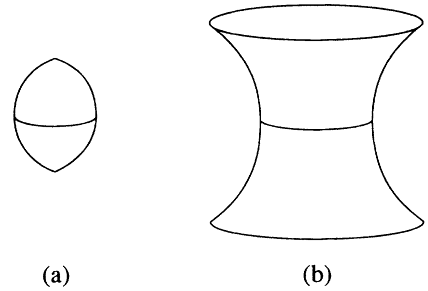

FIG. 2. (a) The fluid-dominated and (b) the cosmological-constant-dominated Robertson-Walker universes.

The classical solutions discussed above are Lorentzian geometries. The tunneling solution, on the other hand, consists of a Euclidean four-geometry connected to two Lorentzian spacetime regions. The tunneling solution is naturally described by the metric (4.10) where x ranges over the entire positive real line:

$$dS^2 = -\frac{dx^2}{3\kappa x^2[\Lambda/\kappa - V(x)]} + \left[\frac{x}{2\pi^2}\right]^{2/3} d\Omega^2, \quad x > 0 . \tag{4.11}$$

For $x < x_-$, (4.11) describes the expanding (or contracting) half of the fluid-dominated Lorentzian solution, while for $x > x_+$ (4.11) describes the expanding (or contracting) half of the cosmological-constant-dominated Lorentzian solution. For $x_- < x < x_+$, the geometry (4.11) is Euclidean—it is this region that is referred to as the instanton. Note that although the metric coefficients g_{xx} in the solutions (4.10) and (4.11) are infinite at the turning points $x_\pm$, the geometry is completely regular there. In particular, the curvature invariants are all finite and well behaved at $x = x_\pm$. The complete four-geometry for the tunneling solution is drawn schematically in Fig. 3.

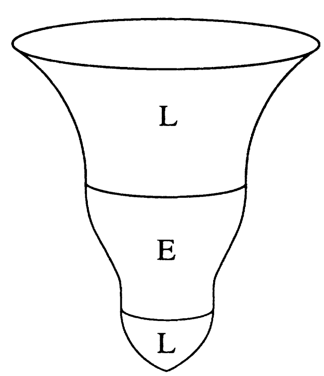

FIG. 3. The tunneling solution, consisting of a Euclidean (E) instanton connecting portions of Lorentzian (L) Robertson-Walker universes.

The tunneling solution is a complex classical solution in the sense that increments dt in the spacetime volume are imaginary in the Euclidean region $x_- < x < x_+$. This is seen in Eq. (4.8), where the argument of the square root is negative because $\Lambda/\kappa < V(x)$. In this "under barrier" region, the sign of the square root can be chosen so that dt is either positive or negative imaginary. Choosing the signs in (4.8) so that dt/dx is positive real or positive imaginary (as shown later, this choice yields the solution that dominates the path integral), the spacetime volume t as a function of spatial volume x for the tunneling solution is

$$t(x)=\begin{cases} \int_0^x dy\{3\kappa[\Lambda/\kappa - V(x)]\}^{-1/2}\ , & x \le x_-\ , \\ t(x_-)+i\int_{x_-}^x dy\{3\kappa[V(x)-\Lambda/\kappa]\}^{-1/2}\ , & x_- < x \le x_+\ , \\ t(x_+)+\int_{x_+}^x dy\{3\kappa[\Lambda/\kappa - V(x)]\}^{-1/2}\ , & x_+ < x\ . \end{cases} \tag{4.12}$$

First, as x increases from 0 to x_-, the spacetime volume t increases through real values. Next, as x increases from x_- to x_+, the imaginary part of t increases while its real part remains unchanged. Finally, as x increases beyond x_+, the real part of t again increases while its imaginary part remains unchanged.

The metric (4.11) completely characterizes the geometry of the tunneling solution, displaying explicitly the instanton and the Lorentzian Robertson-Walker universes that it connects. It was possible essentially to ignore the matter variables in constructing this solution, because the potential φ was eliminated from the beginning as a cyclic coordinate while the baryon number N became a fixed constant. Thus, φ does not enter the equations of motion arising from the action (4.4). But an equation for φ *can* be obtained from the action (4.2) by varying the baryon number N. This gives $d\varphi/dt = \rho'/x$, so from (4.8) φ is obtained by integrating

$$d\varphi = \pm\frac{\rho' dx}{x\sqrt{3\kappa[\Lambda/\kappa - V(x)]}}\ . \tag{4.13}$$

This equation shows that for the tunneling solution, in which x increases monotonically, the potential φ is complex: φ aquires an imaginary contribution as three-space passes through the Euclidean region $x_- < x < x_+$.

The fact that the potential φ is complex for the tunneling solution should not be considered a problem. In fact, the behavior of φ is qualitatively no different from the behavior of the spacetime volume t; indeed, the analogy between φ and t is quite close. Like φ, t can also be viewed as a cyclic variable, with Λ/κ its conjugate momentum. As discussed in the Introduction, there is a Hamilton-type action for the system in which t and Λ/κ appear as dynamical variables. The action (4.2) [or (4.4)] is then obtained when t is eliminated by Routh's procedure, with the consequence that Λ is a fixed constant.[10] The close relationship between spacetime volume t and the potential φ shows that since t is imaginary for Euclidean geometries, φ can be imaginary as well. This feature of wormhole-type instantons is discussed in Ref. 5 for gravity coupled to a massless scalar field, which here is the special case $\rho(n)=(1/4\pi)k^2n^2$.

Finally, notice that other complex solutions to the classical equations of motion can be obtained by patching together in various ways the Lorentzian and Euclidean portions of the geometry shown in Fig. 3. In the language of the particle analogy, these solutions correspond to the particle undergoing multiple reflections under the barrier between the turning points x_- and x_+. Also note that when $\Lambda=0$, the tunneling solution (4.11) consists of a Lorentzian spacetime connected to an instanton that becomes asymptotically flat as $x\to\infty$. There is no turning point x_+, or loosely speaking, the second turning point is at infinity. Then the instanton may be imagined to connect portions of flat Lorentzian spacetimes at infinity.[3]

V. THE PATH INTEGRAL

The action (4.2) that describes the minisuperspace model of perfect-fluid cosmology is invariant under the transformation

$$\delta x^a = \epsilon[x^a, \mathcal{H}]\ , \tag{5.1a}$$

$$\delta p_a = \epsilon[p_a, \mathcal{H}]\ , \tag{5.1b}$$

$$\delta\alpha = \dot\epsilon\ , \tag{5.1c}$$

where x^a and p_a denote collectively x,φ and p,N. Also, the square brackets are the Poisson brackets and ϵ is an infinitesimal function of σ that parametrizes the transformation. The action is invariant under (5.1) only for parameters that vanish at the end points, $\epsilon(\sigma')=\epsilon(\sigma'')=0$. The transformation (5.1) defines a "gauge" equivalence class of histories—it is simply the canonical statement of reparametrization invariance, that is, invariance under a relabeling of the σ-coordinate values for the spatial three-spheres. The path integral for such a reparametrization-invariant theory is most elegantly obtained from the Becchi-Rouet-Stora-Tyutin (BRST) formalism, which is used in constructing a sum over histories that includes just a single history from each equivalence class.[29] The resulting path integral is furthermore independent of the choice of a representative history from each equivalence class.[36,37] Since the result of the BRST construction of the path integral for minisuperspace quantum cosmology is well known,[24] it will not be repeated here. Instead, I will motivate the result from a direct study of the invariance (5.1).

Consider two histories, one specified by a set of functions $x^a(\sigma),p_a(\sigma),\alpha(\sigma)$ and the other by a "barred" set $\bar x^a(\sigma),\bar p_a(\sigma),\bar\alpha(\sigma)$. Notice that the total spacetime volume for any history, given by the integral of the lapse density, is invariant under the transformation (5.1). Thus, if $\int_{\sigma'}^{\sigma''}d\sigma\,\alpha(\sigma)$ does not equal $\int_{\sigma'}^{\sigma''}d\sigma\,\bar\alpha(\sigma)$, then the two histories are gauge inequivalent. So consider the situation in which the two histories describe the same spacetime volume, $\int_{\sigma'}^{\sigma''}d\sigma\,\alpha(\sigma)=\int_{\sigma'}^{\sigma''}d\sigma\,\bar\alpha(\sigma)$. In this case, define a gauge transformation parameter.

$$\epsilon(\sigma)=\int_{\sigma'}^{\sigma}d\sigma\,\alpha(\sigma)-\int_{\sigma'}^{\sigma}d\sigma\,\bar\alpha(\sigma) \tag{5.2}$$

and notice that $\epsilon(\sigma)$ vanishes at the end points $\sigma=\sigma'$ and $\sigma=\sigma''$, as it must. [In general the parameter (5.2) will not be infinitesimal, so the transformation equations (5.1) should be integrated to allow for finite $\epsilon(\sigma)$. However, (5.1c) is valid as it stands for the change in the lapse density due to a finite gauge transformation.] Next, apply the transformation [(5.1) and (5.2)] to the "barred" history, and thereby obtain a gauge-equivalent history denoted by carets:

$$\bar{x}^{a}(\sigma),\bar{p}_a(\sigma),\bar{\alpha}(\sigma)\rightarrow\hat{x}^{a}(\sigma),\hat{p}_a(\sigma),\hat{\alpha}(\sigma)=\alpha(\sigma) \ . \tag{5.3}$$

By construction, the new lapse density $\hat{\alpha}(\sigma)$ equals that for the original history, $\alpha(\sigma)$. Now observe that in the path integral, both histories $x^a(\sigma),p_a(\sigma),\alpha(\sigma)$ and $\hat{x}^a(\sigma),\hat{p}_a(\sigma),\alpha(\sigma)$ will be included in the sum over histories if, for the lapse density equal to $\alpha(\sigma)$, the canonical variables x^a and p_a are integrated over all functional values. The "barred" history $\bar{x}^a(\sigma),\bar{p}_a(\sigma),\bar{\alpha}(\sigma)$ will also be included as the gauge-related history $\hat{x}^a(\sigma)$, $\hat{p}_a(\sigma),\alpha(\sigma)$. This leads to the following prescription: for each possible value of spacetime volume, choose a representative function $\alpha(\sigma)$ with $\int_{\sigma'}^{\sigma''}d\sigma\,\alpha(\sigma)$ equaling that volume; then in the sum over histories, include only those histories whose lapse density equals a representative $\alpha(\sigma)$. In this way, the path integral is defined as a functional integral over all $x^a(\sigma),p_a(\sigma)$, along with an ordinary integral over spacetime volume $\int_{\sigma'}^{\sigma''}d\sigma\,\alpha(\sigma)$. The path integral then includes precisely one history from each gauge equivalence class.

The arguments above do not fix the measure in the sum over histories, but for the functional integration over the canonical variables $x^a(\sigma),p_a(\sigma)$ it is natural to choose the Liouville phase-space measure, which will be denoted $\mathcal{D}x\,\mathcal{D}p$. For the remaining integral over spacetime volume, the naive choice $d(\int_{\sigma'}^{\sigma''}d\sigma\,\alpha(\sigma))$ is correct, that is, the same as that obtained from a BRST analysis. Then the resulting path integral is

$$G(x^{a''},x^{a'}) =\int d\left[\int_{\sigma'}^{\sigma''}d\sigma\,\alpha(\sigma)\right] \times\int\mathcal{D}x\,\mathcal{D}p\,\exp\left[\frac{i}{h}\int_{\sigma'}^{\sigma''}d\sigma(p_a\dot{x}^{a}-\alpha\mathcal{H})\right] . \tag{5.4}$$

Here, the end-point values of the coordinates $x^a(\sigma'')=x^{a''}$ and $x^a(\sigma')=x^{a'}$ are fixed, and not integrated; this is inferred from the classical variational principle in which $x^a(\sigma)$ is fixed at the end points. Notice that in (5.4), the choice of a representative function $\alpha(\sigma)$ for each value of spacetime volume, the "gauge choice," is irrelevant. To see this explicitly, define as in previous sections a spacetime-volume time t through $t(\sigma)\equiv(\int_{\sigma'}^{\sigma}d\sigma\,\alpha(\sigma))+t'$ and denote $t(\sigma'')\equiv t''$. Then using $dt/d\sigma=\alpha(\sigma)$, the path integral becomes

$$G(x^{a''},x^{a'}) =\int d(t''-t') \times\int\mathcal{D}x\,\mathcal{D}p\,\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}dt[p_a(dx^a/dt)-\mathcal{H}]\right] , \tag{5.5}$$

independent of any particular parametrization $\alpha(\sigma)$.

The result (5.5) is the correct expression of the path integral for any system with a reparametrization invariance given by the transformation (5.1). In particular, (5.5) is valid for nonrelativistic mechanics based on Jacobi's action.[10] For the case at hand, minisuperspace perfect-fluid cosmology described by the action (4.2), the path integral is explicitly

$$G_\Lambda(x'',\varphi'';x',\varphi')=\int d(t''-t')e^{-i\Lambda(t''-t')/\kappa\hbar} \times K(x'',\varphi'',t''|x',\varphi',t') , \tag{5.6a}$$

where

$$K(x'',\varphi'',t''|x',\varphi',t') =\int\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}\varphi\,\mathcal{D}N \times\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}dt[p\dot{x}+N\dot{\varphi}-h(x,p,N)]\right] . \tag{5.6b}$$

The overdot now refers to differentiation with respect to t. In (5.6b), the "Hamiltonian" h is the part of the Hamiltonian constraint [(4.3) and (4.5)] excluding the cosmological constant term:

$$h(x,p,N)\equiv-\left[\frac{3\kappa}{4}p^2+V(x)\right] . \tag{5.7}$$

The path integral as written above is the integral transform of the quantum-mechanical kernel or amplitude $K(x'',\varphi'',t''|x',\varphi',t')$. In turn, this amplitude can be considered to be the matrix elements of an evolution operator:

$$K(x'',\varphi'',t''|x',\varphi',t')=\langle x'',\varphi''|e^{-i\hat{h}(t''-t')/\hbar}|x',\varphi'\rangle , \tag{5.8}$$

where $\hat{h}$ is the quantum operator corresponding to the Hamiltonian (5.7). Thus, the kernel satisfies a time-dependent Wheeler-DeWitt equation (see Refs. 10, 13, and 15–17)

$$\left[\frac{\hbar}{i}\frac{\partial}{\partial t''}+\hat{h}''\right]K(x'',\varphi'',t''|x',\varphi',t')=0 , \tag{5.9}$$

with

$$\hat{h}''\equiv h\left[x'',\frac{\hbar}{i}\frac{\partial}{\partial x''},\frac{\hbar}{i}\frac{\partial}{\partial\varphi''}\right]$$

denoting the Hamiltonian operator in the coordinate representation, acting on the arguments x'',φ''.

At this point, it is necessary to specify a range of integration for the spacetime volume $(t''-t')$ in the path integral (5.6). I will restrict the $(t''-t')$ integration to positive real values. This choice incorporates the view of causality advocated by Teitelboim:[22] namely, that the histories included in the path integral should be those for which the final three-surface is located to the future of the initial three-surface. The path integral (5.6) is then written as

$$G_\Lambda(x'',\varphi'';x',\varphi') = \int_0^\infty dT\, e^{-i\Lambda T/\kappa\hbar} K(x'',\varphi'',T|x',\varphi',0)\ , \qquad (5.10)$$

where for notational simplicity $(t''-t')$ is denoted by T. $G_\Lambda(x'',\varphi'';x',\varphi')$ can now be recognized as the Fourier transform in T of the retarded Green function $K\theta(T)$ for the time-dependent Wheeler-DeWitt operator $-i\hbar\partial/\partial t''+\hat{h}''$, where $\theta(T)$ is the usual step function. The path integral is therefore seen to be a Green function for the time-independent Wheeler-DeWitt operator

$$(\Lambda/\kappa+\hat{h}'')G_\Lambda(x'',\varphi'';x',\varphi') = -i\hbar\delta(x'',x')\delta(\varphi'',\varphi')\ . \qquad (5.11)$$

The relationship between the kernel K and the Green function G_Λ written in (5.10) is formally the same as the standard relationship from nonrelativistic mechanics:[38] let T correspond to physical time and $-\Lambda/\kappa$ correspond to energy, then G_Λ is a Green function for the time-independent Schrödinger equation. Notice that in order for the Green-function equation (5.11) to follow from (5.10) and the equation satisfied by $K\theta(T)$, the kernel must vanish as $T\to\infty$. For the minisuperspace model of perfect-fluid cosmology, the calculations of the next section show that the WKB kernel indeed has this property. It is possible that, for other models, the kernel does not vanish for infinitely large T. But (5.11) may nevertheless hold if a small negative imaginary part is added to the cosmological constant.

The Green function (5.10) is most easily obtained by computing first its Fourier transform in φ' and φ'',

$$G_\Lambda(x'',N'';x',N') = \frac{1}{2\pi\hbar}\int d\varphi'' d\varphi' e^{-iN''\varphi''/\hbar} e^{+iN'\varphi'/\hbar} \times G_\Lambda(x'',\varphi'';x',\varphi')\ . \qquad (5.12)$$

This is a Green function for the time-independent Wheeler-DeWitt operator with the Hamiltonian

$$\hat{h}'' = h\left[x'', \frac{\hbar}{i}\frac{\partial}{\partial x''}, N''\right]$$

in a mixed coordinate-momentum representation. $G_\Lambda(x'',N'';x',N')$ can be expressed as a path integral similar to (5.6), but with the $N\dot{\varphi}$ term in the action replaced by $-\dot{N}\varphi$. Because φ does not occur in $h(x,p,N)$, the functional integral over $\varphi(t)$ can be executed trivially, giving an infinite product over t of delta functions of $\dot{N}$. These delta functions restrict $N(t)$ to be a constant, so the $N(t)$ functional integration leaves a single factor $\delta(N''-N')$. As a result, the Green function (5.12) reduces to

$$G_\Lambda(x'',N'';x',N') = \delta(N''-N'')G_{\Lambda,N'}(x'';x')\ . \qquad (5.13)$$

Here, $G_{\Lambda,N'}(x'';x')$ is the path integral

$$G_{\Lambda,N}(x'';x') = \int_0^\infty dT\, e^{-i\Lambda T/\kappa\hbar} K_N(x'',T|x',0)\ , \qquad (5.14a)$$

with

$$K_N(x'',T|x',0) = \int \mathcal{D}x\,\mathcal{D}p\, \exp\left[\frac{i}{\hbar}\int_0^T dt[p\dot{x}-h(x,p,N)]\right]. \qquad (5.14b)$$

The kernel (5.14b) can also be expressed as a matrix element,

$$K_N(x'',T|x',0) = \langle x''|e^{-i\hat{h}T/\hbar}|x'\rangle\ , \qquad (5.15)$$

and $G_{\Lambda,N}(x'';x')$ is itself a Green function for the Wheeler-DeWitt operator,

$$(\Lambda/\kappa+\hat{h}'')G_{\Lambda,N}(x'';x') = -i\hbar\delta(x'',x')\ . \qquad (5.16)$$

In (5.15) and (5.16), the momentum operator $\hat{N}$ appearing in $\hat{h}$ is replaced by the constant value N. Likewise, in Eqs. (5.14) the total particle number appears as a fixed constant.

The functional integration over the variables φ,N that led to (5.13) can be made precise through a time-slicing procedure.[38] Alternatively, the same result can be obtained from the interpretation of the kernel as a matrix element. By eliminating the variables φ,N, the original path integral for the minisuperspace model is reduced to the one-dimensional path integral (5.14). Observe that this Green's function $G_{\Lambda,N}(x'';x')$ can also be obtained directly as the path integral for the effective one-dimensional system described by the action (4.4), in which φ has been eliminated. The reduction of the path integral presented here is the quantum counterpart of Routh's classical procedure for the elimination of a cyclic variable.

VI. EVALUATION OF THE PATH INTEGRAL

In this section, the Green function $G_{\Lambda,N}(x'';x')$ in Eq. (5.14) is evaluated as a path integral. Apart from some minus signs, $G_{\Lambda,N}$ is just the Green function for a particle moving in the one-dimensional potential $V(x)$. Stated more carefully, $G_{\Lambda,N}$ can be viewed as the path integral for a particle of "mass" $-2/3\kappa$ and "energy" $-\Lambda/\kappa$, moving in a potential $-V(x)$. Although the minus signs affect the details, the evaluation of $G_{\Lambda,N}$ here is in spirit no different from the evaluation of the path integral for nonrelativistic mechanics.[23,39] For the following analysis, it is assumed that Λ/κ is less than the maximum $V_{\max}$ of the potential, and that the initial and final three-volumes x' and x'' are separated by the potential barrier as in Fig. 1.

A. The kernel

As a first step in evaluating the Green function $G_{\Lambda,N}(x'',x')$, the kernel $K_N(x'',T|x',0)$ will be calculated in the WKB approximation. The kernel (5.14b) is the transition amplitude for the spatial three-volume to evolve from x' to x'' within a total four-volume time T, but where the cosmological constant is not fixed.

Before computing the kernel, consider the issue of its convergence. The path integral (5.14b) for K_N can be made precise by a time-slicing prescription.[38] Then each $p(t)$ integration is Gaussian, and converges as a Fresnel integral.[40] For the remaining integration over $x(t)$, the action—that is, the argument of the exponential—consists of quadratic terms from the "kinetic energy," and terms from the potential $V(x)$. For large values of x, the potential term $V(x)$ tends toward zero as in Fig. 1, and can be ignored in comparison with the quadratic terms. Therefore at large x the integrals behave as Gaussians and converge.

For small x, the arguments of Sec. IV show that the potential tends toward minus infinity at least as fast as $-1/x$, so that $V(x)$ dominates over the quadratic terms in the action. Then near $x=0$, the integrals behave like $dx\,\exp(-i/x^p)$ where $p\geq 1$. These integrals converge—to see this, make the change of variables $y=x^{-p/2}$, resulting in

$$dx\,\exp(-i/x^p)\Big|_{x\sim 0}=-\frac{2}{p}dy\,y^{-1-2/p}\exp(-iy^2)\Big|_{y\sim\infty}\ . \tag{6.1}$$

For all $p\geq 1$, the integral of the right-hand side above converges at large y, because the integrand goes to zero more quickly than the integrand for a Fresnel integral (which is the case $p=-2$). In conclusion, it appears that the functional integral (5.14b) for the kernel can be well-defined using a time-slicing procedure in the sense that it will consist of a product of convergent integrals.

At this point, it is appropriate to pause and consider the range of integration for the spatial three-volume x. (See Ref. 24 for relevant comments.) Physically, it seems that x should be restricted to positive values, but mathematically it is much simpler to define a quantum theory for which the variables range over the entire real line. It is commonly felt that this situation can be dealt with by modifying the effective potential $V(x)$ to include an infinite wall at $x=0$. For the most part, I will ignore these issues. However, note that the convergence arguments given above appear to be equally valid whether x has a half infinite range or a fully infinite range. Also, the kernel is evaluated below using the WKB approximation, which takes into account only the behavior of the system near the classical stationary paths. Since the classical solution that dominates K_N has x everywhere positive, it would make the same semiclassical contribution regardless of the range of integration for x. On the other hand, if the effect of restricting x to a half infinite range is to create an infinite potential barrier at $x=0$, then there is a second classical path that contributes to the semiclassical kernel, but is ignored below. That is the path in which three-space reflects off the barrier at $x=0$. In Sec. VII I will comment on the effect of including such a reflection term in the WKB evaluation of the path integral.

Now turn to the WKB approximation of the kernel (5.14b). It is constructed by evaluating the action for a stationary path. Such a path satisfies the classical equations of motion that are generated by the Hamiltonian h through the Poisson brackets. Then the Hamiltonian itself is a constant of the motion, which will be denoted by $-\Lambda_{\rm cl}/\kappa$ where $\Lambda_{\rm cl}$ is the "classical" cosmological constant. Thus, a first integral of the equations of motion is

$$\frac{3\kappa}{4}p^2+V(x)=\Lambda_{\rm cl}/\kappa\ . \tag{6.2}$$

From the form of the potential in Fig. 1 and the particle analogy, it is clear that a classical solution with boundary values $x(0)=x'$ and $x(T)=x''$ always exists, for any value of T. For very small T, the "energy" $\Lambda_{\rm cl}/\kappa$ will be large so the particle moves quickly from x' to x''; for very large T, the "energy" $\Lambda_{\rm cl}/\kappa$ will be just slightly greater than the maximum of the potential $V(x)$, so the particle passes slowly over the peak of the potential as it moves from x' to x''. In every case, the particle moves directly from the initial position x' to the final position x'', with an "energy" $\Lambda_{\rm cl}/\kappa$ that is positive and decreases with increasing T.

The classical solution is found by combining (6.2) with the equation of motion $p=-(2/3\kappa)\dot{x}$ to give

$$\dot{x}=\sqrt{3\kappa[\Lambda_{\rm cl}/\kappa-V(x)]}\ . \tag{6.3}$$

Integrating (6.3) yields $x(t)$. Here, the positive square root is chosen because, as described above, the spatial volume (the "particle position") evolves directly from x' to x'', with $\dot{x}>0$. The constant of motion $\Lambda_{\rm cl}$ is obtained by integrating (6.3) from the initial to the final end point, resulting in

$$T=\int_{x'}^{x''}\frac{dx}{\sqrt{3\kappa[\Lambda_{\rm cl}/\kappa-V(x)]}}\ . \tag{6.4}$$

This equation is to be thought of as determining $\Lambda_{\rm cl}$ as a function of the end-point data x',x'',T.

The action in the path integral for the kernel K_N, evaluated at the classical solution, is easily found to be

$$S(x'',T|x',0)=-\frac{2}{3\kappa}\int_{x'}^{x''}dx\sqrt{3\kappa[\Lambda_{\rm cl}/\kappa-V(x)]}+\Lambda_{\rm cl}T/\kappa\ . \tag{6.5}$$

Then the WKB approximation to the kernel is[39]

$$K_N(x'',T|x',0)\approx\left[\frac{1}{2\pi\hbar}\frac{\partial^2 S(x'',T|x',0)}{\partial x'\partial x''}\right]^{1/2}\times\exp\left[\frac{i}{\hbar}S(x'',T|x',0)+\frac{i\pi}{4}\right]\ . \tag{6.6}$$

The preexponential factor is just proportional to the square root of the Van Vleck–Morette determinant $\partial^2 S/\partial x'\partial x''$, and is positive as shown below. The square root is defined as the positive root. The overall factors in (6.6) are determined by considering the limit of small spacetime volume T, as follows. For very small T, the cosmological constant $\Lambda_{\rm cl}$ must be very large and the potential $V(x)$ in Eqs. (6.4) and (6.5) can be ignored. The spatial geometry evolves as what may be termed a "free universe." From (6.4) the cosmological constant in this limit takes the value $\Lambda_{\rm cl}=\frac{1}{3}[(x''-x')/T]^2$ and from (6.5) the action is $S=-(x''-x')^2/(3\kappa T)$. The WKB kernel in (6.6) then becomes

$$K_N(x'',T|x',0)\Big|_{T\sim 0}\approx\left[\frac{1}{3\pi\kappa\hbar T}\right]^{1/2}\times\exp\left[-\frac{i(x''-x')^2}{3\kappa\hbar T}+\frac{i\pi}{4}\right], \tag{6.7}$$

for small T. On the other hand, the kernel K_N can be computed straightforwardly in the small-T limit as a matrix element (5.15) by inserting a complete set of momentum eigenstates. Comparing the results of these two calculations shows that the kernel in (6.7) is indeed correct, which establishes the overall numerical factors in (6.6).

The Van Vleck–Morette determinant appearing in the WKB approximation to the kernel can be written less formally by using the explicit expression (6.5) for the action S. For convenience, define the notation

$$k(x)\equiv\sqrt{3\kappa[\Lambda_{\rm cl}/\kappa-V(x)]} \tag{6.8}$$

and notice that $k(x)$ depends on $\Lambda_{\rm cl}$ and therefore on the end-point data x',x'',T. First compute the derivative of S with respect to x'',

$$\frac{\partial S(x'',T|x',0)}{\partial x''}=-\frac{2}{3\kappa}k(x''), \tag{6.9}$$

which is the momentum for the classical motion evaluated at x''. Then the Van Vleck–Morette determinant is

$$\frac{\partial^2 S(x'',T|x',0)}{\partial x'\partial x''}=-\frac{1}{\kappa k(x'')}\frac{\partial\Lambda_{\rm cl}}{\partial x'}. \tag{6.10}$$

The factor $\partial\Lambda_{\rm cl}/\partial x'$ is obtained by taking the derivative of Eq. (6.4) with respect to x'. This can be compared to the result of differentiating (6.4) with respect to T, which yields the relationship

$$\frac{\partial\Lambda_{\rm cl}}{\partial x'}=\frac{1}{k(x')}\frac{\partial\Lambda_{\rm cl}}{\partial T}. \tag{6.11}$$

Therefore the Van Vleck–Morette determinant can be written in the convenient form

$$\frac{\partial^2 S(x'',T|x',0)}{\partial x'\partial x''}=-\frac{1}{\kappa k(x')k(x'')}\frac{\partial\Lambda_{\rm cl}}{\partial T}. \tag{6.12}$$

Recall that the constant of motion $\Lambda_{\rm cl}$ decreases with increasing T. Thus, $\partial\Lambda_{\rm cl}/\partial T$ is negative and the Van Vleck–Morette determinant is positive. The WKB kernel can finally be written as

$$K_N(x'',T|x',0)\approx\left[-\frac{1}{2\pi\kappa\hbar k(x')k(x'')}\frac{\partial\Lambda_{\rm cl}}{\partial T}\right]^{1/2}\times\exp\left[-\frac{2i}{3\kappa\hbar}\int_{x'}^{x''}dx\,k(x)+\frac{i}{\kappa\hbar}\Lambda_{\rm cl}T+\frac{i\pi}{4}\right], \tag{6.13}$$

where $k(x)$is defined in (6.8) and $\Lambda_{\rm cl}$ is determined through Eq. (6.4).

B. The Green function

Now turn to the evaluation of the Green function $G_{\Lambda,N}(x'',x')$, which is defined in Eq. (5.14a) as an integral transform in T of the kernel K_N. As discussed in Sec. V, the integration in spacetime volume T is taken over positive, real values. This corresponds to summing over real Lorentzian four-geometries in the path integral. Using the WKB approximate kernel, the Green function is given by

$$G_{\Lambda,N}(x'',x')\approx\int_0^\infty dT\left[-\frac{1}{2\pi\kappa\hbar k(x')k(x'')}\frac{\partial\Lambda_{\rm cl}}{\partial T}\right]^{1/2}\times\exp\left[-\frac{2i}{3\kappa\hbar}\int_{x'}^{x''}dx\,k(x)+\frac{i}{\kappa\hbar}(\Lambda_{\rm cl}-\Lambda)T+\frac{i\pi}{4}\right]. \tag{6.14}$$

I will first establish the convergence of this integral, and then approximate the integral by the method of steepest descents.

Consider the behavior of the integral near $T=0$. In this case, the kernel can be approximated as in Eq. (6.7). Now make the change of variables $T=z^{-2}$; apart from an overall constant, the integral behaves like

$$\frac{dT}{T^{1/2}}\exp[-i(x''-x')^2/(3\kappa\hbar T)]\Big|_{T\sim 0}=-\frac{2dz}{z^2}\exp[-i(x''-x')^2z^2/(3\kappa\hbar)]\Big|_{z\sim\infty}. \tag{6.15}$$

Comparing the right-hand side of (6.15) with a Fresnel integral[40] shows that it converges at $z\sim\infty$. Therefore the integral for $G_{\Lambda,N}$ converges at $T\sim 0$.

Next, consider the behavior of the T integral near infinity. Recall, from physical arguments, that $\Lambda_{\rm cl}$ must tend towards the limit $\kappa V_{\max}$ as $T\to\infty$. Then $k(x')$ and $k(x'')$ as well as $\int_{x'}^{x''}dx\,k(x)$ are all finite constants in this limit. So apart from an overall constant, the T integral behaves at infinity like

$$dT\left[-\frac{\partial\Lambda_{\rm cl}}{\partial T}\right]^{1/2}\exp\left[\frac{i}{\hbar}(V_{\max}-\Lambda/\kappa)T\right]\Big|_{T\sim\infty}. \tag{6.16}$$

In order to proceed, the asymptotic behavior of $\partial\Lambda_{cl}/\partial T$ is needed. For this purpose, observe that $\Lambda_{cl}-\kappa V_{max}$ is positive, and viewed as a function of $\ln T$, it has the limit $\Lambda_{cl}-\kappa V_{max}\to 0$ as $\ln T\to\infty$. Furthermore, $\partial\Lambda_{cl}/\partial(\ln T)$ is everywhere negative. Now assume that the limit of $\partial\Lambda_{cl}/\partial T$ as $T\to\infty$ exists; then the limit of $\partial\Lambda_{cl}/\partial(\ln T)$ as $\ln T\to\infty$ also exists. This later limit must in fact be zero, because otherwise $\Lambda_{cl}-\kappa V_{max}$ would become negative for sufficiently large $\ln T$. Therefore

$$\lim_{T\to\infty}\left[T\frac{\partial\Lambda_{cl}}{\partial T}\right]=0\ , \tag{6.17}$$

and it follows that $\partial\Lambda_{cl}/\partial T$ must tend towards zero faster than $1/T$. But even if $\partial\Lambda_{cl}/\partial T$ behaved like $1/T$ at infinity, the integral of expression (6.16) would still converge as a Fresnel integral. This shows that the integral (6.14) for the Green function converges at the upper limit of integration. The conclusion is that the integral over positive real values of spacetime volume T in (6.14) is well defined.

The T integration for $G_{\Lambda,N}$ cannot be done exactly, but can be approximated by the method of steepest descents. Unfortunately, the analysis is not straightforward because the explicit dependence of the integrand on T is not known—Λ_{cl} is defined implicitly through Eq. (6.4), and for general potentials $V(x)$ its T dependence is not explicitly known. An exception is the special case in which the potential is a quadratic function of x, and (6.4) can be solved for Λ_{cl} as a function of T. This is worth mentioning because a quadratic potential with $\partial^2V/\partial x^2<0$ provides a good example of the present mathematical problem, namely, the semiclassical evaluation of a Green function for fixed "energy" Λ/κ less than the maximum of a smooth, single-peaked potential. For the quadratic potential, such a calculation has been carried out by McLaughlin[23] in the context of true particle mechanics. Note, however, that according to the arguments of Sec. IV, the potential $V(x)$ for a classically reasonable fluid equation of state must have the general shape shown in Fig. 1, so $V(x)$ can never actually be quadratic. Nevertheless, the case of a quadratic potential contains the main features of the steepest-descent analysis presented below, and in this respect it is extremely valuable for the insights that it provides.

To begin the steepest-descent calculation, let the argument of the exponential in (6.14) be written as

$$\mathcal{S}(T)\equiv-\frac{2i}{3\kappa\hbar}\int_{x'}^{x''}dx\,k(x)+\frac{i}{\kappa\hbar}(\Lambda_{cl}-\Lambda)T+\frac{i\pi}{4}\ . \tag{6.18}$$

Observe that $\mathcal{S}(T)$ is extremized in T for $\Lambda_{cl}=\Lambda$. Denoting such a value of spacetime volume by T_0, Eq. (6.4) gives

$$T_0=\int_{x'}^{x''}\frac{dx}{\sqrt{3\kappa[\Lambda/\kappa-V(x)]}}\ . \tag{6.19}$$

The key observation here is that, since Λ/κ is less than V_{max}, T_0 is complex—the stationary points of the T integral lie off the real axis of integration.

Previously, the relationship (6.4) was defined by taking the positive square root. This was unambiguous because for all positive real T, $\Lambda_{cl}/\kappa>V_{max}$. But now the same integral must be defined in (6.19) for $\Lambda/\kappa<V_{max}$. It is best to proceed by allowing for all possibilities in defining the integral (6.19). That is, allow the sign of the square root to be chosen independently for each of the three regions $x'\le x\le x_-$, $x_-\le x\le x_+$, and $x_+\le x\le x''$. Furthermore, the x integration need not be monotonically increasing, because the integral ranging from $x_+\to x_-\to x_+$ is not zero if the sign of the square root is changed between the two halves of the "circuit." The result is an infinite array of stationary points in the complex T plane, as shown in Fig. 4. The eight points closest to the real axis correspond to a direct integration of (6.19) from x' to x'', and the eight ways of choosing the signs in the three integration regions. In general, the stationary points can be categorized according to the magnitude of their imaginary part, which depends on the number of circuits $x_+\to x_-\to x_+$ included in integrating (6.19). Each category contains eight stationary points which differ from one another in the signs chosen for the three integration regions. The physical origin and interpretation of the extrema of $\mathcal{S}(T)$ does not depend on the details of the potential $V(x)$. In particular, the same structure arises for a quadratic potential barrier, in which case the stationary points can be derived explicitly.

The goal now is to distort the original integration contour—which was along the real T axis—into a contour that passes through the appropriate stationary points. Again, the quadratic potential provides crucial insight into this problem. In that case, the important stationary points are those in the upper-right quadrant of the complex T plane that are closest to the real axis;[23] the steepest-descent contours in this region appear as in Fig. 5. The more general potentials of Fig. 1 should be qualitatively the same. The contours shown in Fig. 5 are curves of constant imaginary part of $\mathcal{S}(T)$, and the arrows show the direction of incresing real part of $\mathcal{S}(T)$. I will denote the stationary point on the left by T_L and the one on the right by T_R.

As discussed by McLaughlin[23] for the quadratic poten-

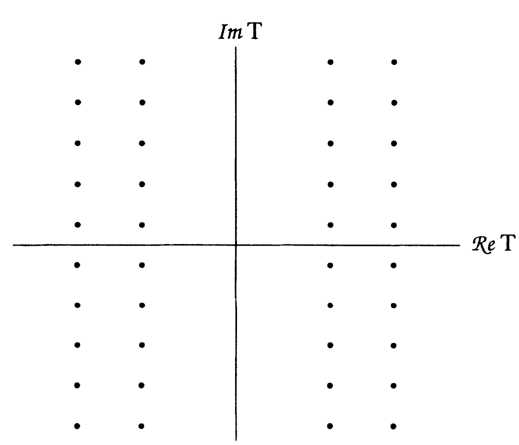

FIG. 4. The values of spacetime volume T that extremize the phase $\mathcal{S}(T)$ in the path integral.

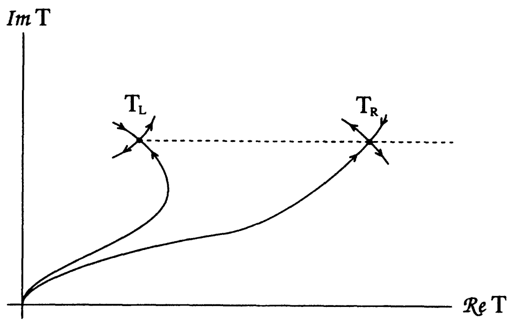

FIG. 5. The steepest-descent contours for the phase $\mathcal{S}(T)$, near the stationary points T_L and T_R that are closest to the real axis and in the first quadrant of the complex T plane.

tial in particle mechanics, the integration contour can naturally be chosen in two ways. One possibility is to take the contour from the origin to T_L along the steepest-descent path, then out parallel to the real T axis, passing through T_R along the dashed line of Fig. 5. The other possibility is to take the contour from the origin to T_R along the steepest-descent path, then out parallel to the real axis. The results obtained from these two contours are identical, for the following reasons. For the first contour described, the contribution from the steepest-descent path near T_L exactly cancels the contribution from the stationary phase path, the dashed line, near T_L. Thus, T_L does not contribute to the integral. This is not entirely surprising, because T_L is an "unphysical" stationary point for which the signs in (6.19) are opposite in the two regions $x' \le x \le x_-$ and $x_+ \le x \le x''$. So for both contours described above, the integral is dominated entirely by the stationary point T_R. Furthermore, the fact that the two contours pass through that stationary point in different ways does not matter—just as the Fresnel integral $\int dz\, e^{iz^2} = \sqrt{\pi} e^{i\pi/4}$ can be evaluated either directly by integrating along the real axis, or by distorting the contour through the steepest-descent path.

Before continuing with the evaluation of the Green function, observe that the original contour along the real T axis must be distorted into the upper-right quadrant rather than the lower-right quadrant. This is because the stationary points below the real axis are along contours of steepest *ascent*. This can be seen generally by considering the small-T limit, in which the argument of the exponential in (6.18) is $\mathcal{S}(T) = -i(x''-x')^2/(3\kappa\hbar T) + i\pi/4$. An analysis of the steepest-descent contours for small T then shows that the real part of $\mathcal{S}(T)$ decreases as T moves away from the origin in the fourth quadrant.

The contour of integration should be distorted to pass through the stationary point T_R in Fig. 5. This value of spacetime volume has a positive imaginary part, that comes from choosing the square root in (6.19) to be negative imaginary when $\Lambda/\kappa < V(x)$. Thus, $k(x)$ is negative imaginary for x between the turning points, and $\mathcal{S}(T)$ will include a negative real part when evaluated at the stationary point T_R:

$$\mathcal{S}(T_R) = -\frac{2i}{3\kappa\hbar}\int_{x'}^{x_-} dx\sqrt{3\kappa[\Lambda/\kappa - V(x)]} - \frac{2}{3\kappa\hbar}\int_{x_-}^{x_+} dx\sqrt{3\kappa[V(x) - \Lambda/\kappa]} - \frac{2i}{3\kappa\hbar}\int_{x_+}^{x''} dx\sqrt{3\kappa[\Lambda/\kappa - V(x)]} + \frac{i\pi}{4}\,. \tag{6.20}$$

It is already evident that the effect of the "under barrier" region $x_- \le x \le x_+$ is to contribute an exponential suppression in the Green function (6.14). This also shows why only the stationary points of Fig. 4 that are closest to the real axis are important: if the integration contour were somehow distorted to pass through stationary points further from the real axis, they would make contributions to the Green function that are suppressed by even larger exponential factors.

The semiclassical value of the Green function (6.14) is approximated by integrating T near the stationary point T_R, with the argument of the exponential $\mathcal{S}(T)$ expanded to second order in T. The constant term is just $\mathcal{S}(T_R)$ given in (6.20), and the linear term is of course zero; from (6.18), the quadratic term is seen to be proportional to

$$\left.\frac{\partial^2 \mathcal{S}(T)}{\partial T^2}\right|_{T_R} = \frac{i}{\hbar\kappa}\left.\frac{\partial \Lambda_{\rm cl}}{\partial T}\right|_{T_R}. \tag{6.21}$$

Thus, the Green function can be written as

$$G_{\Lambda,N}(x'',x') \approx \left[-\frac{1}{2\pi\kappa\hbar}\left.\left[\frac{1}{k(x')k(x'')}\frac{\partial\Lambda_{\rm cl}}{\partial T}\right]\right|_{T_R}\right]^{1/2} \times \int dT \exp\left[\mathcal{S}(T_R) + \frac{i}{2\hbar}\left.\left[\frac{1}{\kappa}\frac{\partial \Lambda_{\rm cl}}{\partial T}\right]\right|_{T_R} \times (T-T_R)^2\right]. \tag{6.22}$$

This integral is easily evaluated. There is, however, one subtle point to be addressed. The preexponential factor was previously defined in (6.14) as the positive square root of a positive argument. But now, because the integration contour has been distorted, the argument is no longer real and positive. So the square root must be defined with some degree of care. In particular, it should depend continuously on Λ, and not jump by a factor of -1 by crossing a branch cut. But the factors $k(x')$ and $k(x'')$ are positive real. Furthermore, the real part of $-\partial\Lambda_{\rm cl}/\partial T|_{T_R}$ must be positive; this is seen by computing $\partial\Lambda_{\rm cl}/\partial T$ from Eq. (6.4) and using the fact that at T_R, the square-root term is real and positive in the "over barrier" region. Thus, the square root in (6.22) can be safely defined by choosing a branch cut along the negative real axis.

Performing the integration over T in (6.22) gives a factor

$$\int dT \exp\left[\frac{i}{2\kappa\hbar}\left.\frac{\partial\Lambda_{\rm cl}}{\partial T}\right|_{T_R}(T-T_R)^2\right] = \left[\frac{2\pi i\kappa\hbar}{(\partial\Lambda_{\rm cl}/\partial T)|_{T_R}}\right]^{1/2}, \tag{6.23}$$

where the branch cut for this square root is also along the negative real axis. Combining this term with the corresponding preexponential factors in (6.22) leaves an overall factor of $e^{-i\pi/4}$ that cancels the term $e^{+i\pi/4}$ from $\mathcal{S}(T_R)$. Using the definition (6.8) for $k(x)$ and (6.20) for $\mathcal{S}(T_R)$, the final result for the Green function is

$$G_{\Lambda,N}(x'',x')\approx\{3\kappa[\Lambda/\kappa-V(x')]\}^{-1/4}\{3\kappa[\Lambda/\kappa-V(x'')]\}^{-1/4}$$
$$\times\exp\left[-\frac{2i}{3\kappa\hbar}\int_{x'}^{x_-}dx\sqrt{3\kappa[\Lambda/\kappa-V(x)]}-\frac{2}{3\kappa\hbar}\int_{x_-}^{x_+}dx\sqrt{3\kappa[V(x)-\Lambda/\kappa]}\right.$$
$$\left.-\frac{2i}{3\kappa\hbar}\int_{x_+}^{x''}dx\sqrt{3\kappa[\Lambda/\kappa-V(x)]}\right],\quad x'<x_-<x_+<x''\,. \tag{6.24}$$

All roots in this expression are by convention real and positive. Notice that $G_{\Lambda,N}$ contains the expected barrier penetration factor

$$\exp\left[-\frac{2}{3\kappa\hbar}\int_{x_-}^{x_+}dx\sqrt{3\kappa[V(x)-\Lambda/\kappa]}\right], \tag{6.25}$$

familiar from standard WKB analysis.

VII. DISCUSSION AND MISCELLANY

The semiclassical expression (6.24) for the Green function $G_{\Lambda,N}(x'',x')$ is one of the main results of this paper. It is worth emphasizing once again that $G_{\Lambda,N}$ was defined as a sum over Lorentzian spacetime geometries. All of the integrations converge, and there is no difficulty with the conformal mode as in the Euclidean path integral. Purely as a computational technique, $G_{\Lambda,N}$ was evaluated by distorting the contour of integration for spacetime volume T in the complex plane. In this way, the sum over Lorentzian geometries was approximated by the behavior of the system near a single complex four-geometry.

The complex geometry that dominates the path integral for $G_{\Lambda,N}$ is the classical tunneling solution found in Sec. IV. This may not be entirely obvious from the preceding analysis. On the one hand, the tunneling geometry is a solution to the classical equations of motion that follow from extremizing the action (4.4) with respect to $x(\sigma)$, $p(\sigma)$, and $\alpha(\sigma)$, with the additional identification of $\alpha\,d\sigma$ as the spacetime volume increment dt. On the other hand, the configuration that dominates the path integral (5.14) is an extremum of the phase with respect to $x(t)$, $p(t)$, and T. The difference between these variational problems arises because the action (4.4) is reparametrization invariant, whereas the redundancy associated with this invariance has been eliminated in the path integral. However, for both the classical equations of motion and the extremum of the phase, the equations that follow from varying $x(\sigma),p(\sigma)$ or $x(t),p(t)$ are just Hamilton's equations $\partial F/\partial t=[F,h]$ where F is any function of the canonical variables and h is the Hamiltonian (5.7). These equations alone imply that h is a constant. Now observe that varying the lapse density $\alpha(\sigma)$ in the action (4.4) imposes the Hamiltonian constraint and implies that the constant value of the Hamiltonian h is $-\Lambda/\kappa$. Likewise, varying the total spacetime volume T in the phase of the integral (5.14) shows, using standard results from Hamilton-Jacobi theory,[12] that the value of h at $t=T$ is $-\Lambda/\kappa$; since h is constant, its value is once again determined to be $-\Lambda/\kappa$ for all t. In this way, the condition that the phase is stationary is entirely equivalent to the equations of motion that follow from the classical action. The tunneling solution and the complex geometry that dominates the path integral are both solutions to the same differential equations of motion. Of course, there are many solutions to these equations, corresponding to the freedom of three-space to undergo multiple reflections under the potential barrier, and to the freedom of choosing different signs for the increments dt of spacetime volume. But the solution that dominates the semiclassical evaluation of the path integral is the one whose spacetime volume has the smallest positive imaginary part, and the largest real part. This is precisely the classical tunneling solution written in Eqs. (4.11) and (4.12).

It is of interest to note that the tunneling solution as described in Sec. IV has a lapse density $\alpha(\sigma)$ that is complex and discontinuous. More precisely, $\alpha(\sigma)$ is positive real or positive imaginary, depending on whether σ is in a range such that $x(\sigma)$ is "over" or "under" the potential barrier. However, this character of the lapse density can be changed by transforming to a gauge equivalent history, using the transformation equations (5.1) with a *complex* parameter $\epsilon(\sigma)$. In fact, the only part of the lapse density that cannot be changed, the gauge-invariant part, is the spacetime volume $\int_{\sigma'}^{\sigma''}d\sigma\,\alpha(\sigma)$. For example, the tunneling solution can be gauge transformed to satisfy the "proper time" condition[27] in which α is a constant. That constant equals the (complex) four-volume divided by $\sigma''-\sigma'$. Although attention is often focused on the proper time gauge in constructing the path integral, the final result for the path integral does not depend on any particular choice of gauge.[36,37] As a result, there is no sense in which the extrema of the phase in the path integral must satisfy the proper time condition, or any other gauge condition.

The Green function $G_{\Lambda,N}(x'',x')$ contains an exponential suppression due to tunneling under the potential barrier, but the kernel $K_N(x'',T|x',0)$ in Eq. (6.13) does not. The reason was already discussed in the Introduction, and should be clear from the analysis of the previous section: if the total volume-time T is fixed, then the "energy" $\Lambda_{\rm cl}/\kappa$ is free to adjust and will assume a value greater than the maximum of the potential. Thus, there is always a real classical path between the end points x' and x''. On the other hand, when the "energy" is fixed to a value Λ/κ that is less than the maximum of the potential, the system must tunnel in order to pass from one side of the barrier to the other. Similarly, if the variable φ conjugate to the particle number N is fixed in the path integral, tunneling will not occur. This can be seen by using Eq. (5.13) to obtain the Green function $G_\Lambda(x'',N'';x',N')$, and then Fourier transforming N' and N'' to get $G_\Lambda(x'',\varphi'';x',\varphi')$. One of the N integrations is trivial because of the factor $\delta(N''-N')$, and the other can be done in a semiclassical approximation. The stationary condition is just the integrated form of the equation of motion (4.13) for φ. If $\varphi''-\varphi'$ is chosen to be real, then (4.13) shows that the square-root factor $\sqrt{3\kappa[\Lambda/k-V(x)]}$ is real. This means that the stationary value of particle number N is large enough for the maximum of the potential to be less than Λ/κ. (According to the assumptions made in Sec. IV on the equation of state, the fluid energy density ρ must increase, and therefore $V_{\max}$ must decrease, with increasing N.) So in the path integral, if *either* the total spacetime volume or the cyclic coordinate φ is fixed (to real values), then there is no quantum tunneling. Space can be said to tunnel only when the cosmological constant Λ and the particle number N are fixed, and in regions where $\Lambda/\kappa < V(x)$.

As promised in Sec. VI, I will comment on the effect of including in the potential $V(x)$ an infinite wall at $x=0$. In this case, the WKB approximation to the kernel will include contributions from the direct classical path as well as the classical path that reflects off the wall. This amounts to subtracting from (6.6) a second term of the same form, but where the action is evaluated for the reflected path. By subtracting the second term, that is, including a phase change -1 at the infinite barrier, the resulting kernel vanishes when $x'=0$. As for the Green function, the integral over four-volume T of the "reflection term" in the kernel can be evaluated in the same way as was the "direct term." The result is that the Green function $G_{\Lambda,N}$ consists of two terms, one term just as in Eq. (6.24). The other term is of the same general form as (6.24), but with an extra phase factor

$$-\exp\left[-\frac{4i}{3\kappa\hbar}\int_0^{x'} dx\sqrt{3\kappa[\Lambda/\kappa-V(x)]}\right]. \qquad (7.1)$$

The minus sign is the phase change due to reflection off the barrier at $x=0$; the argument of the exponential comes from the action evaluated along the classical path $x'\to 0\to x'$.

Finally, consider the possibility of computing a transmission coefficient for tunneling between Robertson-Walker universes. Assume that the wave function of the Universe is in an eigenstate of both the fluid particle number and the cosmological constant. In that case, the Wheeler-DeWitt equation is identical to the time-independent Schrödinger equation for a particle in the potential $V(x)$. Using standard results from quantum mechanics, the WKB wave functions can be written down immediately, and the WKB approximation to the transmission coefficient is just the square of the barrier penetration factor (6.25). Of course, the transmission coefficient in quantum mechanics is interpreted as the *probability* for a particle to tunnel through a potential barrier. In the case of the Universe, such a probabilistic interpretation may or may not make sense, depending first on whether the fundamental notion of probability is meaningful in this context, and if so, whether the scalar product $\psi^*\psi$, borrowed from ordinary quantum mechanics, is also valid for quantum gravity.

ACKNOWLEDGEMENTS

I would like to thank C. P. Burgess, G. B. Cook, E. A. Martinez, and B. F. Whiting for their helpful remarks. I am especially grateful to J. W. York for many enlightening discussions, and for his careful review of the manuscript. This work was supported by the National Science Foundation, Grants Nos. PHY-8407492 and PHY-8908741.

[1]J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[2]A. Vilenkin, Phys. Rev. D **33**, 3560 (1986); **37**, 888 (1988).
[3]S. B. Giddings and A. Strominger, Nucl. Phys. **B306**, 890 (1988).
[4]R. C. Myers, Phys. Rev. D **38**, 1327 (1988).
[5]J. D. Brown, C. P. Burgess, A. Kshirsagar, B. F. Whiting, and J. W. York, Nucl. Phys. **B328**, 213 (1989).
[6]J. J. Halliwell and R. Laflamme, Santa Barbara/Vancouver Report No. NSF-ITP-89-41, 1989 (unpublished).
[7]B. J. Keay and R. Laflamme, Phys. Rev. D **40**, 2118 (1989).
[8]C. P. Burgess and S. Hagan, McGill University report, 1989 (unpublished).
[9]A. Hosoya and W. Ogura, Phys. Lett. B **225**, 117 (1989).
[10]J. D. Brown and J. W. York, Phys. Rev. D **40**, 3312 (1989).
[11]J. B. Hartle, Phys. Rev. D **38**, 2985 (1988); and report (unpublished).
[12]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1980).
[13]M. Henneaux and C. Teitelboim, Phys. Lett. B **222**, 195 (1989).
[14]These ideas and their implications for quantization are discussed from various perspectives in Refs. 10, 11, 13, and 15–17.
[15]R. D. Sorkin, *History of Modern Gauge Theories,* Conference proceedings, edited by M. Dresden and A. Rosenblum (Plenum, New York, in press).
[16]W. G. Unruh and R. M. Wald, Phys. Rev. D **40**, 2598 (1989).
[17]W. G. Unruh, in *Quantum Gravity,* proceedings of the Fourth Seminar, Moscow, USSR, 1987, edited by M. A. Markov, V. A. Berezin, and V. P. Frolov (World Scientific, Singapore, 1988); Phys. Rev. D **40**, 1048 (1989).
[18]J. B. Hartle, J. Math. Phys. **30**, 452 (1989).
[19]J. J. Halliwell and J. Louko, Phys. Rev. D **39**, 2206 (1989); **40**, 1868 (1989).
[20]B. F. Schutz, Phys. Rev. D **2**, 2762 (1970).
[21]J. D. Brown (unpublished).

[22]C. Teitelboim, Phys. Rev. Lett. **50**, 705 (1983).
[23]D. W. McLaughlin, J. Math. Phys. **13**, 1099 (1972).
[24]J. J. Halliwell, Phys. Rev. D **38**, 2468 (1988).
[25]I. Moss, Ann. Inst. Henri Poincaré **49**, 341 (1988).
[26]The path integral has been constructed by Teitelboim (Ref. 27) for quantum cosmology in general, and by Louko (Ref. 28) for minisuperspace quantum cosmology. It is most easily obtained from a BRST analysis (Ref. 29), as discussed recently by Halliwell (Ref. 24) for the case of minisuperspace models.
[27]C. Teitelboim, Phys. Rev. D **25**, 3159 (1982); **28**, 297 (1983).
[28]J. Louko, Class. Quantum Grav. **4**, 581 (1987).
[29]See, for example, M. Henneaux, Phys. Rep. **126**, 1 (1985), and references therein.
[30]Recently, several other authors have advocated the Lorentzian definition of the path integral. See Refs. 31–33.
[31]E. Farhi, Phys. Lett. B **219**, 403 (1989).
[32]W. M. Suen and K. Young, Phys. Rev. D **39**, 2201 (1989).
[33]M. J. Duncan and L. G. Jensen, Nucl. Phys. **B328**, 171 (1989).
[34]For a review of perfect fluids in the context of general relativity, see C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).
[35]See, for example, B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (The University of Chicago Press, Chicago, 1965).
[36]E. S. Fradkin and G. A. Vilkovisky, CERN Report No. TH.2332, 1977 (unpublished).
[37]I. A. Batalin and G. A. Vilkovisky, Phys. Lett. **69B**, 309 (1977).
[38]K. V. Kuchař, J. Math. Phys. **24**, 2122 (1983).
[39]L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).
[40]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).