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Particle production in inhomogeneous cosmologies

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A perturbative evaluation of the S matrix is used to compute the production of particles in an expanding flat Friedmann-Robertson-Walker universe in the presence of small inhomogeneities. We first consider the production of massless conformally coupled and weakly nonconformally coupled particles, obtaining known results, and then we consider the production of massive particles. The production of massive particles cannot be treated only perturbatively and a method is proposed to compute this in general. The pair-production probability is computed using two different, but related, methods: in one we directly evaluate the number of particles produced and in the other we concentrate mainly on the vacuum-to-vacuum or vacuum persistence amplitude.

I. INTRODUCTION

Quantum field theory in curved spacetime which has been developed during the last 15 years or so[1,2] is a semiclassical theory in which the fields of particles are quantized over unquantized curved spacetime. This theory becomes a testing ground for the, still unknown, exact theory of quantum gravity since one expects that the results of the semiclassical theory will be recovered by the exact theory in appropriate limits. Physically relevant situations where such a semiclassical theory is applicable are those involving strong gravitational tidal forces, near black holes, or those involving very rapid time variations of the gravitational field, near the Planck time (10^{-43} s) in cosmology. For black holes, we have Hawking's celebrated result[3,4] on black-hole evaporation (Hawking radiation) and in cosmology we have Parker's pioneering result[5,6] on particle production in Friedmann-Robertson-Walker (FRW) expanding cosmologies. The theory plays today an important role in our understanding of the early Universe; as an example we may cite that the quantum fluctuations that it predicts for the scalar field in the inflationary cosmology may generate the initial density perturbations[7–9] that will form galaxies.

In the cosmological context, the theory becomes greatly simplified when spacetime is conformally flat, such as in FRW cosmologies, and it has been extensively studied in this context. But spatially homogeneous and isotropic cosmologies are not the most general and, for instance, small deviations of such geometry must have been present in the past, otherwise galaxies would not have been formed.

It is thus natural to consider more general spacetimes and, in fact, Zel'dovich[10] speculated that particle creation near the Planck time in an arbitrary cosmology would result in a homogeneous and isotropic spacetime as a result of the dynamical back reaction of the created matter. This speculation was in agreement with Misner's[11] chaotic cosmology program aimed at providing an explanation, independent of initial conditions, to the homogeneity and isotropy of the present Universe.

This led to the study of quantum field theory in spatially homogeneous but anisotropic cosmologies. The anisotropies break the conformal invariance and massless conformally coupled scalar particles are produced. This has been computed by Zel'dovich and Starobinsky,[12,13] Hu,[14] Birrell and Davies,[15] and Hartle and Hu.[16] Exact results cannot be found when anisotropies are present and different approximation methods have been developed to find the anisotropic contributions, assumed small, to the Bogoliubov coefficients which relate the creation and annihilation operators between the two vacua (usually "in" and "out" vacua) of the theory. These coefficients are directly related to the number of particles produced. In cosmology the rate of particle production is important even for massless conformally coupled particles because the energy for such production is provided by the expansion of the Universe whereas the small anisotropies provide the breaking of the conformal invariance that makes possible such production.[17]

Yet one would expect that the Universe coming from the quantum-gravity era would contain a large number of gravitons and therefore spacetime would have spatial inhomogeneities. In recent years inhomogeneous cosmologies have been intensively studied (see Ref. 18 for a review), especially in connection with the possible existence of a cosmological gravitational-wave background.[19,20]

The purpose of this paper is to study the production of scalar particles that result from the presence of inhomogeneities in an expanding cosmology. The first step is to develop a formalism for such computation. Here, as for the case of anisotropies, exact calculations cannot be carried out and we must appeal to approximate methods assuming that the inhomogeneities are small. For homogeneous spacetimes the time evolution of the modes of the field equation (Klein-Gordon equation for scalar fields) can be separated. Although in general one cannot find exact solutions that give exact Bogoliubov coefficients for small anisotropies, one may use methods to find approximate coefficients.[12,15] For small inhomogeneities we shall develop a method based on the interaction picture and the S matrix. Such a method has been used in the case of small inhomogeneities on Minkowski[21,13] and cosmological[17] backgrounds and recently has been studied intensively in connection with the analysis of self-interacting and mutually interacting fields in curved spacetime[22–26] of such great interest in inflationary cosmology.

In this approach the Lagrangian density for the scalar field is separated into a "free" Lagrangian density, L_0, containing the spatially homogeneous background and an "interaction" Lagrangian density, L_I, containing the inhomogeneities as external fields. Provided that the inhomogeneities vanish, i.e., are switched off, in the "in" and "out" regions, the production of pairs of particles in the presence of external fields can be evaluated with standard S-matrix techniques in a way similar to the electron-positron pair production due to the presence of an external electromagnetic field.[27] To simplify our computation we shall further assume that the background spacetime is conformally flat.

The back-reaction effect so crucial to test the Zel'dovich speculation, for instance, will not be considered in this paper but it will be the subject of further work. An approach based on the effective potential[28] may be of use for this subject.

The plan and summary of the main results of the paper are the following. In Sec. II we define the notation and explain the procedure to be used in the paper. We then compute (Sec. II A) the particle production of massless conformally coupled scalar particles in an expanding spatially flat FRW background in the presence of small inhomogeneities. In this case the conformal symmetry is broken by the inhomogeneities and the production of particles is only due to such perturbations. We recover a recent result by Frieman[17] who obtained it by conformally transforming the result found when there are inhomogeneities on a Minkowski background using the fact that the number density of particles produced is proportional to the square of the (conformally invariant) Weyl tensor due to the inhomogeneities.

In Sec. II B we consider the production of massless nonconformally coupled scalar particles in the same gravitational field of the previous subsection. Conformal invariance is now broken by the coupling and particles are produced also by the expanding background. However, if we consider only a small deviation from conformal coupling the coupling term can be written in the interaction Lagrangian and the computation can be done computing the S matrix perturbatively. Now the nonconformally homogeneous term is seen as an external field responsible for particle production together with the inhomogeneous terms.

If the deviation of conformal coupling is large one cannot treat the corresponding term perturbatively and one can apply the technique to be described in the next section. The results giving the probability of pair production can be written in terms of geometric quantities and agree with those by Birrell and Davies,[15] for anisotropic perturbations, and with those by Frieman.[17]

In Sec. III we consider the production of massive particles in a conformally flat background in the presence of small inhomogeneities. Here the mass breaks conformal invariance and particles are produced by the homogeneous background. The main difference with the previous section is that now it is not possible to treat perturbatively, i.e., to write into the interaction Lagrangian the mass term, even for a small mass. The reason is that such a term cannot be switched off simultaneously in the "in" and "out" regions.

First, in Sec. III A we consider the case in which the conformal factors in the "in" and "out" regions coincide, and the conformal function can be written in the interaction Lagrangian. Then the problem can be treated as the production of massive particles in a Minkowski spacetime with conformal homogeneous and inhomogeneous perturbations. Our computation, in agreement with previous results,[13,17] gives the pair-production probability in terms of geometric quantities only. From the cosmological point of view however this case is not too relevant because the background spacetime has not expanded.

Thus we next treat, in Sec. III B, the problem of an expanding background, i.e., when the conformal "in" factor is smaller than the conformal "out" factor. Now particles are produced by the homogeneous expansion of the background and there is a Bogoliubov transformation between the "in" and "out" vacua. To find the modes that are required in order to compute the S matrix we need to know the specific form of the conformal factor. In general it will not be possible to find an exact solution for these modes. However we can treat the problem in the following way. We may choose a "background" conformal factor such that it allows for an explicit computation of the Bogoliubov transformation connecting the "in" and "out" vacua. The difference between the given conformal factor and the background chosen is then written into the interaction Lagrangian together with the inhomogeneous perturbations. Our election will guarantee that the interaction term is switched off at the "in" and "out" regions on one hand, and on the other it will allow us to give specific formulas for pair-production probability without explicitly knowing the conformal factor, in analogy with the situation in all previous cases studied.

As a background factor we shall take a step-function expansion since it gives the simplest possible background modes. This implies a sudden approximation and the introduction of a time parameter η_0, where the matching between the two Minkowski modes takes place. This also implies a cutoff parameter in the momentum of the modes of the order of the inverse of the transition time, because modes with larger momentum are exponentially damped. From the cosmological point of view, however, this is physically reasonable because to have a well-defined "in" vacuum we match to Minkowski spacetime and the adiabatic "out" vacuum can be approximated by a Minkowskian region too.

As an explicit example we consider the case in which inhomogeneities are not present but we have an arbitrary (but bounded) conformal expansion. Our results are then compared with those of a tanh conformal expansion since exact computation of the particles produced can be given in this case. The computation is done using two different, although related, methods. One is by directly computing the number of particles created, the other is by computing as the fundamental quantity the vacuum persistence probability, and the results are compared.

II. MASSLESS PARTICLES

In this section we compute using the S-matrix approach the production of massless conformally coupled and weakly nonconformally coupled scalar particles in a flat FRW expanding background in the presence of small inhomogeneities. Here we set up the notation to be used throughout the paper.

We start with the Lagrangian density for a scalar field $\phi(x)$ on a gravitational field with a coupling to the curvature scalar

$$\mathcal{L}=\tfrac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-m^2\phi^2-\xi R\phi^2)\ , \tag{2.1}$$

where $g_{\mu\nu}(x)$ is the spacetime metric, $R(x)$ the Ricci scalar curvature, m the mass of the scalar field, and ξ the coupling parameter. We take the signature convention $(+,-,-,-)$, and $R^{\mu}{}_{\nu\rho\sigma}=\Gamma^{\mu}_{\nu\rho,\sigma}-\cdots$. The scalar field equation is then the Klein-Gordon equation

$$(\Box_g+m^2+\xi R)\phi(x)=0\ , \tag{2.2}$$

where $\Box_g$ is the D'Alembert operator constructed with the metric $g_{\mu\nu}$.

We shall assume now small inhomogeneities in the gravitational field and that the spacetime metric can be written as perturbations on a spatially flat FRW background metric:

$$g_{\mu\nu}=\Omega^2(\eta_{\mu\nu}+h_{\mu\nu})\ , \tag{2.3}$$

where $\Omega(\eta)$ is the conformal factor of the FRW metric considered with $\eta=x^0$ the conformal time and $|h_{\mu\nu}|\ll 1$.

Expanding $\mathcal{L}$ in powers of $h_{\mu\nu}$, using (2.3), the Lagrangian density (2.1) can be separated as follows:

$$\mathcal{L}=\mathcal{L}^{(0)}+\mathcal{L}^{(1)}+O(h^2)\ , \tag{2.4}$$

where

$$\begin{aligned}\mathcal{L}^{(0)}&=\frac{\Omega^4}{2}(\Omega^{-2}\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-m^2\phi^2-\xi R^{(0)}\phi^2)\ ,\\ \mathcal{L}^{(1)}&=\frac{\Omega^4}{2}[\tfrac{1}{2}h^{\rho}_{\rho}(\eta^{\mu\nu}\Omega^{-2}\phi_{,\mu}\phi_{,\nu}-\xi R^{(0)}\phi^2)\\ &\qquad -h^{\mu\nu}\Omega^{-2}\phi_{,\mu}\phi_{,\nu}-\xi R^{(1)}\phi^2-\tfrac{1}{2}m^2h\phi^2]\ ,\end{aligned} \tag{2.5}$$

with

$$R^{(0)}=6\Omega^{-3}\ddot{\Omega} \tag{2.6}$$

and

$$\begin{aligned}R^{(1)}=\;&\Omega^{-2}\eta^{\mu\lambda}\eta^{\sigma\nu}(h_{\mu\lambda,\sigma\nu}-h_{\mu\nu,\sigma\lambda})\\ &-6\Omega^{-3}(\eta^{\mu\nu}\Gamma^{\rho}_{\mu\nu}\Omega_{,\rho}+h^{\mu\nu}\Omega_{,\mu\nu})\ ,\end{aligned}$$

where $\Gamma^{\rho}_{\mu\nu}=\frac{1}{2}\eta^{\rho\sigma}(h_{\sigma\mu,\nu}+h_{\sigma\nu,\mu}-h_{\mu\nu,\sigma})$. Indices of $h_{\mu\nu}$ are raised with the Minkowski metric $\eta_{\mu\nu}$, and, from now on, $h=h^{\rho}_{\rho}$.

The Lagrangian $\mathcal{L}^{(1)}$ is greatly simplified with the "synchronous" gauge condition

$$h_{0\nu}=0\ , \tag{2.7}$$

which can always be imposed on the perturbations;[29,30] then it becomes

$$\mathcal{L}^{(1)}=-\frac{\Omega^4}{2}\left[\left(h^{\mu\nu}-\eta^{\mu\nu}\frac{h}{2}\right)\Omega^{-2}\phi_{,\mu}\phi_{,\nu}+\tfrac{1}{2}hm^2\phi^2+\xi\left(R^{(1)}+\frac{h}{2}R^{(0)}\right)\phi^2\right]\ , \tag{2.8}$$

with

$$R^{(1)}=\Omega^{-2}[\eta^{\mu\nu}\eta^{\lambda\sigma}(h_{\mu\nu,\sigma\lambda}-h_{\mu\lambda,\sigma\nu})+3\dot{h}\dot{\Omega}\Omega^{-1}]\ , \tag{2.9}$$

where we have used $h^{\mu\nu}\Omega_{,\mu}=0$ because $\Omega=\Omega(\eta)$, and an overdot means derivation with respect to η.

A. Massless conformally coupled particles

It is well known[5] that massless conformally coupled particles are not produced in conformally flat backgrounds, but here the inhomogeneities break the conformal invariance and particles are produced. Since we assume small inhomogeneities such production can be treated perturbatively by separating the Lagrangian density for the scalar field into an homogeneous part ("free" Lagrangian) and another part containing the inhomogeneities up to the first order ("interaction" Lagrangian). Then the problem is reduced to that of the coupling between the scalar field and an external (unquantized) inhomogeneous gravitational field and a perturbative approach can be used.

Therefore taking $m=0$ and $\xi=\frac{1}{6}$, the Lagrangian density (2.1) may be split into $\mathcal{L}_0$ and $\mathcal{L}_I$:

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I+O(h^2)\ , \tag{2.10a}$$

with

$$\mathcal{L}_0=\mathcal{L}^{(0)}(m=0,\xi=\tfrac{1}{6}),\quad \mathcal{L}_I=\mathcal{L}^{(1)}(m=0,\xi=\tfrac{1}{6})\ , \tag{2.10b}$$

where $\mathcal{L}_0$ may be seen as a "free" Lagrangian density and $\mathcal{L}_I$ as the "interaction" Lagrangian density. This one is linear in the inhomogeneities (unquantized external field) and quadratic in the scalar field. We can now consider particle production in perturbation theory provided that the interaction vanishes in the "in" and "out" regions which we can impose by assuming that the inhomogeneities vanish at least adiabatically in these regions (symbolically, $\eta\to\pm\infty$):

$$\lim_{\eta\to\pm\infty}h_{\mu\nu}=0\ .$$

Now in the interaction picture the states are described by the free modes, i.e., modes that are solutions of the field equation corresponding to the Lagrangian density $\mathcal{L}_0$:

$$(\Box_g+\tfrac{1}{6}R^{(0)})\phi=0\ . \tag{2.11}$$

By making use of the relation $(\Box_g+\frac{1}{6}R^{(0)})\phi=\Omega^{-3}\Box_\eta(\Omega\phi)$, the modes $(\Omega\phi)$ correspond to solutions in flat space. Therefore the positive-definite normal modes normalized in a box of volume L^3 are

$$u_{\mathbf{k}}=\frac{e^{-ik\cdot x}}{\Omega\sqrt{2L^3k^0}}\quad\text{with } k\cdot k=0\ . \tag{2.12}$$

In the asymptotic "in" and "out" regions we assume that spacetime becomes flat:

$$\lim_{\eta\to-\infty} \Omega=\Omega_1, \quad \lim_{\eta\to+\infty} \Omega=\Omega_2 ; \tag{2.13}$$

then the modes become the ordinary flat-space modes. The field operator can be expanded as

$$\phi(x)=\sum_{\mathbf{k}}[a_{\mathbf{k}}u_{\mathbf{k}}(x)+a_{\mathbf{k}}^{\dagger}u_{\mathbf{k}}^{*}(x)] , \tag{2.14}$$

where $a_{\mathbf{k}},a_{\mathbf{k}}^{\dagger}$ are the creation and annihilation operators satisfying the usual commutation relations

$$[a_{\mathbf{k}},a_{\mathbf{k}'}^{\dagger}]=\delta_{\mathbf{k},\mathbf{k}'} . \tag{2.15}$$

The vacuum $|0\rangle$ is defined by

$$a_{\mathbf{k}}|0\rangle=0 .$$

Since there is no mixing of positive- and negative-frequency modes in the "in" and "out" regions [the positive-frequency modes are defined in (2.12) in both regions], there is no particle creation by the background, and the "in" and "out" vacua can be identified.

We turn now to the interaction term. In the interaction picture the states $|\Psi\rangle$ evolve in time according to the Schrödinger equation

$$H_I|\Psi\rangle=i\frac{\partial|\Psi\rangle}{\partial\eta} , \tag{2.16}$$

where H_I is the interaction Hamiltonian constructed from the interaction Hamiltonian density $\mathcal{H}_I(x)$. This equation can be formally solved by means of the S-matrix operator connecting the "in" and "out" states, and given by

$$S=\hat{T}\exp\left[-i\int\mathcal{H}_I d^4x\right] , \tag{2.17}$$

where $\hat{T}$ is the time-ordering operator. Then we can expand S perturbatively up to first order in $h_{\mu\nu}$, it becomes $S\simeq\mathbb{1}-i\hat{T}\int\mathcal{H}_I d^4x$.

Now for a nonderivative coupling we have

$$\mathcal{H}_I=-\mathcal{L}_I ; \tag{2.18}$$

in our case, however, the Lagrangian contains a derivative interaction, but it is not hard to see using for instance path-integral methods[31,32] that the substitution (2.18) also applies. Thus we may write

$$S\simeq\mathbb{1}+i\hat{T}\int\mathcal{L}_I d^4x\equiv\mathbb{1}+S^{(1)} . \tag{2.19}$$

Since $\mathcal{L}_I$ is quadratic in the field and its derivatives, to first order in $h_{\mu\nu}$ the particles are produced in pairs, and the probability amplitude for pair creation is given by the S-matrix element

$$\begin{aligned} S_{\mathbf{kp}}^{(1)}&=\langle\mathbf{pk}|S^{(1)}|0\rangle\\ &=\langle\mathbf{pk}|i\hat{T}\int\mathcal{L}_1 d^4x|0\rangle\\ &=i\int d^4x\langle\mathbf{pk}|\hat{T}\mathcal{L}_I|0\rangle . \end{aligned} \tag{2.20}$$

This gives the transition amplitude from the vacuum state $|0\rangle$ to a two-particle state $|\mathbf{kp}\rangle$ with momenta $\mathbf{k}$ and $\mathbf{p}$. The total pair-creation probability is then given by

$$\sum_{\mathbf{kp}}|S_{\mathbf{kp}}^{(1)}|^2 .$$

But before we compute this quantity, since there is no particle production by the background we may follow a procedure which is familiar in flat spacetime for the creation of charged particles in the presence of a classical electromagnetic field.[27] We shall focus our attention in the vacuum-to-vacuum or vacuum persistence probability

$$S_0\equiv|\langle 0,\text{out}|0,\text{in}\rangle|^2 , \tag{2.21}$$

where $|0,\text{in}\rangle$ is the "in" vacuum state (no particles in the "in" region), and $|0,\text{out}\rangle$ is the final evolution of this state as a consequence of the interaction: i.e.,

$$|0,\text{out}\rangle=S|0,\text{in}\rangle . \tag{2.22}$$

S_0 gives the probability of no particle production, that is, the probability of measuring no particles in the region $\eta\to\infty$, if no particles were present in the $\eta\to-\infty$ region in the presence of the interaction. This quantity can be easily computed in our case up to first order in perturbation theory:

$$\begin{aligned} S_0&=\langle 0|S|0\rangle\langle 0|S^{\dagger}|0\rangle\\ &=1-\sum_{n>0}\frac{1}{n!}\langle 0|S|n\rangle\langle n|S^{\dagger}|0\rangle , \end{aligned}$$

where all states are carrying the flag "in," which we drop for simplicity, and we have used the completeness of the Fock-space basis elements. Expanding S perturbatively as in (2.19) and using that $\mathcal{L}_I$ is quadratic in the fields we have

$$\begin{aligned} S_0&\simeq 1-\sum_{\mathbf{kp}}|\langle 0|\int d^4x\,\mathcal{L}_I|\mathbf{kp}\rangle|^2\\ &=1-\sum_{\mathbf{kp}}|S_{\mathbf{kp}}|^2 . \end{aligned} \tag{2.23}$$

On the other hand, since the probability is less than 1, let us write[27]

$$S_0\equiv\exp\left[-\int d^4x\sqrt{-g}\;w(x)\right] , \tag{2.24}$$

where $w(x)\geq 0$ can be interpreted as the "probability density of particle production." This can be justified as follows.[27] Let us divide spacetime into small cells of volume Δv_i and let us call w_i the average value of $w(x)$ in the cell; then

$$\begin{aligned} S_0\simeq\exp\left[-\sum_i w_i\Delta v_i\right]&=\prod_i\exp(-w_i\Delta v_i)\\ &\simeq\prod_i(1-w_i\Delta v_i) . \end{aligned}$$

As $1-w_i\Delta v_i$ is the probability of no particle production in the cell labeled i, $w(x)$ is the probability density of particle production.

Therefore the quantity

$$W\equiv\int d^4x\sqrt{-g}\;w(x) \tag{2.25}$$

is interpreted as the total probability of particle creation. We may now expand (2.24) in powers of $h_{\mu\nu}$ and get, to first order,

$$S_0 \simeq 1 - \int d^4x \sqrt{-g}\, w^{(1)}(x) \ . \tag{2.26}$$

Comparing (2.23) with (2.26) we have thus that, as expected, to first order, the total pair-creation probability is

$$W^{(1)} = \sum_{\mathbf{kp}} |S^{(1)}_{\mathbf{kp}}|^2 \ . \tag{2.27}$$

In order to compute $S^{(1)}_{\mathbf{kp}}$ through formula (2.20), we need to know the expression of $\phi_{,\mu}$. Using the standard decomposition of the field operator in terms of creation and annihilation operators (2.14), and the modes (2.12) for $u_{\mathbf{k}}$, we find $u_{\mathbf{k},\mu} = (-ik_\mu - \Omega_{,\mu}/\Omega)u_{\mathbf{k}}$ and

$$\phi_{,\mu} = -\Omega^{-1}\Omega_{,\mu}\phi - i\sum_{\mathbf{k}} k_\mu (a_{\mathbf{k}} u_{\mathbf{k}} - a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}) \ . \tag{2.28}$$

The brackets of interest are

$$\langle \mathbf{kp}|\hat{T}\phi\phi|0\rangle = 2u^*_{\mathbf{p}} u^*_{\mathbf{k}} \ ,$$

$$\langle \mathbf{kp}|\hat{T}\phi_{,\mu}\phi_{,\nu}|0\rangle = 2u^*_{\mathbf{p}} u^*_{\mathbf{k}} [\Omega_{,\mu}\Omega_{,\nu}\Omega^{-2} - k_{(\mu}p_{\nu)} - i(k+p)_{(\mu}\Omega_{,\nu)}\Omega^{-1}]$$

and it follows immediately that

$$S^{(1)}_{\mathbf{kp}} = i\int d^4x\, \Omega^4 u^*_{\mathbf{p}} u^*_{\mathbf{k}} \left[\Omega^{-2}\left[\eta^{\mu\nu}\frac{h}{2} - h^{\mu\nu}\right][\Omega_{,\mu}\Omega_{,\nu}\Omega^{-2} - k_{(\mu}p_{\nu)} - i(k+p)_{(\mu}\Omega_{,\nu)}\Omega^{-1}] - \frac{1}{6}\left[R^{(1)} + \frac{h}{2}R^{(0)}\right]\right] \ . \tag{2.29}$$

Substituting the expressions for $R^{(0)}$, $R^{(1)}$, the modes $u_{\mathbf{k}}$ and the gauge conditions, we arrive at

$$S^{(1)}_{\mathbf{kp}} = i\int d^4x \frac{e^{i(k+p)\cdot x}}{2L^3\sqrt{k^0p^0}}[-\tfrac{1}{6}\eta^{\mu\nu}\eta^{\lambda\sigma}(h_{\mu\nu,\lambda\sigma} - h_{\mu\lambda,\nu\sigma}) + h^{\mu\nu}k_\mu p_\nu - \tfrac{1}{2}hk\cdot p] - \int d^4x \left[\frac{e^{i(k+p)x}}{2L^3\sqrt{k^0p^0}}\eta^{\mu\nu}h\Omega^{-1}\Omega_{,\nu}\right]_{,\mu} \ . \tag{2.30}$$

The surface term vanishes by the boundary conditions imposed on the inhomogeneous perturbation in the "in" and "out" regions and the condition that the conformal factor goes to a constant in these regions, i.e., the spacetime becomes Minkowskian.

At this point it is natural to introduce the direct and inverse Fourier transforms of the field $h_{\mu\nu}$, defined through

$$\tilde{h}_{\mu\nu}(p) \equiv \int d^4x\, e^{ip\cdot x} h_{\mu\nu}(x), \quad h_{\mu\nu}(x) = (2\pi)^{-4}\int d^4k\, e^{-ik\cdot x}\tilde{h}_{\mu\nu}(k) \ . \tag{2.31}$$

Then $S^{(1)}_{\mathbf{kp}}$ can be rewritten as

$$S^{(1)}_{\mathbf{kp}} = \frac{i\tilde{h}_{\mu\nu}(k+p)}{2L^3\sqrt{k^0p^0}}[k^\mu p^\nu - \tfrac{1}{6}(k+p)^\mu(k+p)^\nu - \tfrac{1}{12}\eta^{\mu\nu}(k+p)^\rho(k+p)_\rho] \ . \tag{2.32}$$

Now, in order to perform the summations and return to normal spacetime (not a box), k^μ should become a continuous variable and the sums be replaced by integrals

$$\sum_k \to \left[\frac{L}{2\pi}\right]^3 \int d^3\mathbf{k} \ ;$$

then (2.27) becomes

$$W^{(1)} = \frac{L^6}{(2\pi)^6}\int \frac{d^3\mathbf{k}\, d^3\mathbf{p}}{4L^6k^0p^0} d^4q\, \delta^4(q-p-k)\tilde{h}_{\mu\nu}(q)(k^\mu p^\nu - \tfrac{1}{6}q^\mu q^\nu - \tfrac{1}{12}q^2\eta^{\mu\nu})\tilde{h}_{\rho\sigma}(-q)(k^\sigma p^\rho - \tfrac{1}{6}q^\sigma q^\rho - \tfrac{1}{12}q^2\eta^{\mu\nu}) \ ,$$

where we have introduced an integral over the four-momentum q^μ and a four-delta $\delta^4(q-p-k)$, so that $q=p+k$. This integral can be written as

$$W^{(1)} = (2\pi)^{-6}\int d^4q\, \tilde{h}_{\mu\nu}(q)\tilde{h}_{\sigma\rho}(-q)[A^{\mu\nu\rho\sigma}(q) - \tfrac{1}{6}q^\mu q^\nu B^{\rho\sigma}(q) - \tfrac{1}{6}q^\rho q^\sigma B^{\mu\nu}(q) + (\tfrac{1}{36}q^\mu q^\mu q^\rho q^\sigma + \tfrac{1}{144}q^4\eta^{\mu\nu}\eta^{\rho\sigma})C(q)] \ , \tag{2.33}$$

where we have introduced the notation

$$A^{\mu\nu\rho\sigma}(q) = \int \frac{d^3\mathbf{p}}{2p^0}\frac{d^3\mathbf{k}}{2k^0}\delta^4(q-p-k)k^\mu p^\nu k^\rho p^\sigma, \quad B^{\mu\nu}(q) = \int \frac{d^3\mathbf{p}}{2p^0}\frac{d^3\mathbf{k}}{2k^0}\delta^4(q-p-k)k^\mu p^\nu \ ,$$

$$C(q) = \int \frac{d^3\mathbf{p}}{2p^0}\frac{d^3\mathbf{k}}{2k^0}\delta^4(q-p-k) \ . \tag{2.34a}$$

All these integrals are the well-known phase-space integrals in Minkowski space and their explicit values in the massive case (substituting $m=0$ we obtain the values for the present case) are

$$A^{\mu\nu\rho\sigma}(q)=\frac{C(q)}{240}[15q^\mu q^\nu q^\rho q^\sigma+5(q^2-4m^2)(P^{\mu\nu}q^\rho q^\sigma+P^{\rho\sigma}q^\mu q^\nu)+(q^2-4m^2)^2(P^{\mu\nu}P^{\rho\sigma}+P^{\mu\rho}P^{\sigma\nu}+P^{\mu\sigma}P^{\rho\nu})]\ ,$$

$$B^{\mu\nu}(q)=\frac{C(q)}{4}[q^\mu q^\nu+\tfrac{1}{3}(q^2-4m^2)P^{\mu\nu}],\quad C(q)=\frac{\pi}{2}\left[1-\frac{4m^2}{q^2}\right]^{1/2}\theta(q^0)\theta(q^2-4m^2)\ , \tag{2.34b}$$

where we have introduced the transverse projector

$$P^{\mu\nu}\equiv\eta^{\mu\nu}-q^\mu q^\nu q^{-2}\ . \tag{2.35}$$

This projector is interesting because it is not difficult to show that for the inhomogeneous metric

$$\bar{g}_{\mu\nu}\equiv\eta_{\mu\nu}+h_{\mu\nu}\ , \tag{2.36}$$

which is conformal to our metric $g_{\mu\nu}=\Omega^2\bar{g}_{\mu\nu}$, the Fourier transform of the scalar curvature to first order in h can be written as

$$\tilde{R}(q)=-P^{\mu\nu}(q)q^2\tilde{h}_{\mu\nu}(q)\ , \tag{2.37a}$$

and the scalar invariant as

$$\tilde{R}^{\mu\nu\rho\sigma}(q)\tilde{R}_{\mu\nu\rho\sigma}(-q)=q^4P^{\mu\nu}P^{\rho\sigma}\tilde{h}_{\mu\rho}(q)\tilde{h}_{\sigma\nu}(-q)\ . \tag{2.37b}$$

After a rather long but straightforward computations we get, finally,

$$\begin{aligned}W^{(1)}&=(2\pi)^{-6}\int d^4q\frac{\theta(q^0-|\mathbf{q}|)}{720}\frac{\pi}{2}[3\tilde{R}^{\mu\nu\rho\sigma}(q)\tilde{R}_{\mu\nu\rho\sigma}(-q)-\tilde{R}(q)\tilde{R}(-q)]\\&=\frac{1}{960\pi}\int\frac{d^4q}{(2\pi)^4}\theta(q^0-|\mathbf{q}|)\tilde{C}^{\mu\nu\rho\sigma}(q)\tilde{C}_{\mu\nu\rho\sigma}(-q)\ ,\end{aligned} \tag{2.38}$$

where $\tilde{C}^{\mu}{}_{\nu\rho\sigma}(q)$ denotes the Fourier transform of the Weyl or conformal tensor of the metric $\bar{g}_{\mu\nu}$. Since this metric is conformally related to the metric (2.3) it is also the Fourier transform of the Weyl tensor of $g_{\mu\nu}$. In the last equality we have used that the quadratic invariant obtained from the Weyl tensor is

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{3}R^2 \tag{2.39a}$$

and that in linearized gravity (i.e., for $\bar{g}_{\mu\nu}$) we have the identity

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2=0\ . \tag{2.39b}$$

Equation (2.38) gives the probability of pair creation of massless conformally coupled scalar particles due to small inhomogeneities (they are included in the geometric term). This expression agrees with the results of Frieman[17] obtained by conformally transforming the energy-momentum tensor of order $(h_{\mu\nu})$, and also with the result by Hartle and Hu[16] and Birrell and Davies[15] for small anisotropies on conformally flat backgrounds.

In the particular case in which $|\tilde{C}_{\mu\nu\rho\sigma}(q)|^2=|\tilde{C}_{\mu\nu\rho\sigma}(q)|^2\theta(q^0-|\mathbf{q}|)$ one can make use of Parseval's theorem to change q integration into space integration and one obtains, from (2.25),

$$w^{(1)}(x)=\frac{1}{960\pi}C_{\mu\nu\rho\sigma}(x)C^{\mu\nu\rho\sigma}(x) \tag{2.40}$$

for the probability density of pair creation.

B. Massless nonconformally coupled particles

In this short subsection we consider the production of massless weakly nonconformally coupled scalar particles in the presence of small inhomogeneities. In this case conformal invariance is broken by both the inhomogeneities and the coupling factor. Now the free Lagrangian (2.5) leads to the following equation for the free field:

$$[\Box_\eta+(\xi-\tfrac{1}{6})R^{(0)}]\phi(x)=0 \tag{2.41}$$

which must be solved to find the Bogoliubov transformation relating the "in" positive-definite modes with the "out" modes. Since the background undergoes expansion there will be mixing of positive and negative modes indicating that the "in" and "out" vacua are different. We should find first the "free" field by exactly solving Eq. (2.41) but this requires knowledge of the function $\Omega(\eta)$ and exact solutions cannot be given in general. Thus to simplify matters we shall assume that

$$\epsilon\equiv\xi-\tfrac{1}{6}\ll 1\ , \tag{2.42}$$

i.e., a weakly nonconformal coupling. Then the term with ϵ can be written in the interaction Lagrangian, and the condition that the spacetime becomes flat in the "in" and "out" regions,

$$\lim_{\eta\to\pm\infty}R^{(0)}=0\ , \tag{2.43}$$

guarantees that the interaction is switched off as required. Thus the "free" Lagrangian is now

$$\mathcal{L}_0=\mathcal{L}^{(0)}(m=0,\xi=\tfrac{1}{6})\ , \tag{2.44a}$$

as in (2.10a) and the interaction Lagrangian is

$$\mathcal{L}_I=\mathcal{L}^{(1)}(m=0,\xi=\tfrac{1}{6})+\epsilon\Omega^4R^{(0)}\phi^2\ , \tag{2.44b}$$

where we have used the definitions (2.4)–(2.6) and (2.8) for $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$. Now the background does not create particles and formula (2.27) also applies. Using now the modes (2.12) we can separate

$$S_{\mathbf{kp}}^{(1)}=S_{\mathbf{kp}}^{(1)}(m=0,\xi=\tfrac{1}{6})+S_{\mathbf{kp}}^{(1)}(\epsilon)\ , \tag{2.45a}$$

where the first term is given by (2.32) and the second is

$$S_{\mathbf{kp}}^{(1)}(\epsilon)=-i\epsilon\int d^4x\, e^{i(k+p)x}\Omega^2 R^{(0)}\ . \tag{2.45b}$$

From this we can compute the probability of pair creation which is given by

$$W^{(1)}=\sum_{\mathbf{kp}}|S_{\mathbf{kp}}(m=0,\xi=\tfrac{1}{6})|^2+\sum_{\mathbf{kp}}|S_{\mathbf{kp}}(\epsilon)|^2\ , \tag{2.46}$$

since the crossed term is easily seen to be zero after a partial integration and using that $h_{\mu\nu}$ vanish in the "in" and "out" regions. Thus we concentrate in the computation of the last term of (2.46), which becomes, with obvious notation,

$$W^{(1)}(\epsilon)=\frac{1}{16\pi}\int d^4x\sqrt{-g}\,(\epsilon R^{(0)})^2\ . \tag{2.47}$$

This gives the contribution to pair creation of the weakly nonconformal coupling. We may note that this term is geometrical as the first term in (2.46). Collecting these results we get

$$W^{(1)}=\frac{1}{960\pi}\int\frac{d^4q}{(2\pi)^4}\theta(q^0-|\mathbf{q}|)\tilde{C}^{\mu\nu\rho\sigma}(q)\tilde{C}_{\mu\nu\rho\sigma}(-q)+\frac{1}{16\pi}\int d^4x\sqrt{-g}\,(\epsilon R^{(0)})^2\ . \tag{2.48}$$

In the particular case in which $|\tilde{C}_{\mu\nu\rho\sigma}(q)|^2=|\tilde{C}_{\mu\nu\rho\sigma}(q)|^2\theta(q^0-|\mathbf{q}|)$, using Parseval's theorem, one obtains that the pair-creation probability may be written as spacetime integration of geometrical terms and one may read the probability density of pair creation from (2.25) as

$$w^{(1)}(x)=\frac{1}{960\pi}[C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+60(\epsilon R^{(0)})^2]\ . \tag{2.49}$$

This expression agrees with that of Birrell and Davies[15] when restricted to small anisotropies.

III. MASSIVE PARTICLES

Finally in this section we consider the production of massive nonconformally coupled scalar particles in the presence of inhomogeneities. In this case the term giving the coupling of the mass m to the conformal function $\Omega(\eta)$ cannot be included in the interaction Lagrangian, even for small mass because as remarked in the Introduction the coupling cannot be switched off in the two asymptotic "in" and "out" regions.

However we may distinguish two cases. The first is when the asymptotic "in" and "out" conformal factors are the same,

$$\lim_{\eta\to\pm\infty}\Omega=\Omega_1\ , \tag{3.1}$$

and we can consider that (Ω^2/Ω_1^2-1) is small, then this term can be written in the interaction Lagrangian and the problem is reduced to that of a massive particle in flat space with an homogeneous perturbation due to the conformal factor and with inhomogeneous perturbations. The second case is when

$$\lim_{\eta\to-\infty}\Omega=\Omega_1,\quad \lim_{\eta\to+\infty}\Omega=\Omega_2\ , \tag{3.2}$$

where $\Omega_2>\Omega_1$ (say). This case is relevant in cosmology since it describes a real expanding background and it will be considered in the second part of this section.

A. Minkowski background

In the first case just mentioned we have, for the metric tensor,

$$g_{\mu\nu}=\Omega_1^2(\eta_{\mu\nu}+\hat{h}_{\mu\nu})\ , \tag{3.3}$$

where

$$\hat{h}_{\mu\nu}=f(\eta)\eta_{\mu\nu}+h_{\mu\nu},\quad f(\eta)=\frac{\Omega^2-\Omega_1^2}{\Omega_1^2} \tag{3.4}$$

defines now the perturbations over flat space. Using (2.5)–(2.9) (note that $R^{(0)}=0$ in this case), we can write the following free and interaction Lagrangians:

$$\mathcal{L}_0=\mathcal{L}^{(0)}(m=0,\xi)-m^2\Omega_1^2\Omega^2\phi^2=\frac{\Omega^2}{2}(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-m^2\Omega_1^2\phi^2)\ , \tag{3.5a}$$

$$\mathcal{L}_I=\mathcal{L}^{(1)}(m=0,\xi)-\tfrac{1}{2}m^2(\Omega^2-\Omega_1^2)\Omega^2\phi^2=\frac{\Omega^2}{2}\left[\frac{h}{2}\eta^{\mu\nu}-h^{\mu\nu}\right]\phi_{,\mu}\phi_{,\nu}-\frac{\Omega^4}{2}(\xi R^{(1)}+\tfrac{1}{2}m^2h)\phi^2-\tfrac{1}{2}m^2(\Omega^2-\Omega_1^2)\Omega^2\phi^2\ . \tag{3.5b}$$

Here the interaction is switched off in the two asymptotic regions as required. Note that the coupling constant ξ already appears in the interaction part.

The wave equation for the free modes is now

$$(\Box_\eta+m^2\Omega_1^2)\phi=0 \tag{3.6}$$

which has plane-wave solutions

$$u_{\mathbf{k}}=\frac{e^{-ik\cdot x}}{\sqrt{2L^3k^0}},\quad k^2=m^2\Omega_1^2\ , \tag{3.7}$$

describing massive particles in flat spacetime. Therefore we can identify the "in" and "out" vacua as $|0\rangle$.

Following Sec. II the amplitude for pair production can now be written to first order in perturbation theory. For the probability of pair production we finally get

$$W^{(1)}=\int \frac{d^4q}{(2\pi)^4}\frac{\theta(q^0)\theta(q^2-4m^2)}{960\pi}\left[1-\frac{4m^2}{q^2}\right]^{1/2}\left[\tilde{C}_{\mu\nu\rho\sigma}(q)\tilde{C}^{\mu\nu\rho\sigma}(-q)\left[1-\frac{4m^2}{q^2}\right]^2 +\tilde{R}(q)\tilde{R}(-q)\left[60(\xi-\tfrac{1}{6})^2-40\frac{m^2}{q^2}(\xi-\tfrac{1}{6})+\frac{20}{3}\frac{m^4}{q^4}\right]\right] \quad (3.8)$$

after using definition (2.34a) and relations (2.39).

As expected the probability of pair creation depends on local geometrical quantities. Here $|\tilde{R}(q)|^2$ includes the effect due to the conformal factor as well as inhomogeneities, i.e., $\hat{h}_{\mu\nu}$, whereas $\tilde{C}_{\mu\nu\rho\sigma}(q)\tilde{C}^{\mu\nu\rho\sigma}(-q)$ takes into account the inhomogeneities only, i.e., $h_{\mu\nu}$.

The result (3.8) is in agreement with a recent result by Frieman,[17] except for the sign of the $\frac{20}{3}(m/q)^4$ coefficient, and reduces to that of Zel'dovich and Starobinsky[13] when $\xi=\frac{1}{6}$ and $m=0$. It may be used to find particle production in flat space with perturbations of gravitational origin. Note that since all the geometrical quantities can be written via Einstein's equations in terms of the energy-momentum tensor of the source that produce the gravitational perturbations there is no need to know the gravitational field explicitly; all we need is the energy-momentum tensor. In fact using (2.39) and Einstein's equations $R_{\mu\nu}=8\pi G(T_{\mu\nu}-\frac{1}{2}Tg_{\mu\nu})$ we can rewrite (3.8) as

$$W^{(1)}=\frac{G^2}{15(2\pi)^3}\int d^4q\,\theta(q^0)\theta(q^2-4m^2)\left[1-\frac{4m^2}{q^2}\right]^{1/2}\left\{\tilde{T}_{\mu\nu}(q)\tilde{T}^{\mu\nu}(-q)\left[1-\frac{4m^2}{q^2}\right]^2 +\tilde{T}(q)\tilde{T}(-q)\left[-\frac{1}{3}\left[1-\frac{4m^2}{q^2}\right]+30(\xi-\tfrac{1}{6})^2 -20\frac{m^2}{q^2}(\xi-\tfrac{1}{6})+\frac{10}{3}\frac{m^4}{q^4}\right]\right\},$$

and of course the same is true for the pair-creation probabilities of Sec. II which are also expressed in geometrical terms. This fact has been used by Garriga, Harari, and Verdaguer[33] to compute particle production due to cosmic strings. Note also that static sources do not produce particles at this perturbative order.

B. Cosmological background

We shall now consider the case when the background spacetime expands and the conformal factor satisfies (3.2). The coupling between the mass and the conformal factor has to be taken into account in the free field equation. There is now particle production by the background, besides we must give an explicit form for the conformal factor in order to compute such particle production. This means that in general it will not be possible to give an exact solution for the modes and consequently it will be difficult to go a step further in order to take the inhomogeneities into account.

To overcome such difficulties we may proceed as follows. We first define a background conformal factor $\Omega_b(\eta)$ such that

$$\lim_{\eta\to-\infty}\Omega_b=\Omega_1,\quad \lim_{\eta\to+\infty}\Omega_b=\Omega_2 \quad (3.9)$$

and assume that (Ω^2/Ω_b^2-1) is small. This last term can then be written into the interaction Lagrangian and we are left in the free Lagrangian with $m^2\Omega_b^2$ which we may choose so that a simple explicit solution to the free field equation can be found. Thus we consider

$$\mathcal{L}_0=\frac{\Omega^2}{2}(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-m^2\Omega_b^2\phi^2-\tfrac{1}{6}\Omega^2R^{(0)}\phi^2)\,, \quad (3.10a)$$

$$\mathcal{L}_I=\frac{\Omega^2}{2}\left[\left[\eta^{\mu\nu}\frac{h}{2}-h^{\mu\nu}\right]\phi_{,\mu}\phi_{,\nu}-\frac{h}{2}m^2\Omega^2\phi^2 -\frac{1}{6}\left[R^{(1)}+\frac{h}{2}R^{(0)}\right]\Omega^2\phi^2\right] -\frac{m^2}{2}(\Omega^2-\Omega_b^2)\Omega^2\phi^2-(\xi-\tfrac{1}{6})\frac{\Omega^4}{2}R^{(0)}\phi^2\,. \quad (3.10b)$$

The interaction Lagrangian is switched off in the "in" and "out" regions as required, and the free Lagrangian defines the free modes which we need in order to compute the pair-creation probability due to the interaction as

$$(\Box_\eta+m^2\Omega_b^2)(\Omega\phi)=0\,. \quad (3.11)$$

To carry out explicit computations we shall take as a background conformal function a step expansion

$$\Omega_b(\eta)=\Omega_1\theta(\eta_0-\eta)+\Omega_2\theta(\eta-\eta_0)\,, \quad (3.12)$$

which is the simplest function we can take and gives the simplest modes as solutions of the field equation (3.11), namely, the plane-wave modes

$$u_{\mathbf{k}}=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\Omega}\frac{\chi_{\mathbf{k}}(\eta)}{\sqrt{2L^3\omega_k^{\rm in}}},\quad \chi_{\mathbf{k}}(\eta)=\begin{cases} e^{-i\omega_k^{\rm in}\eta}, & \eta\leq\eta_0\,,\\ \dfrac{\omega_k^+}{k_k^{\rm out}}e^{-2i\omega_k^-\eta_0}e^{-i\omega_k^{\rm out}\eta}-\dfrac{\omega_k^-}{\omega_k^{\rm out}}e^{-2i\omega_k^+\eta_0}e^{i\omega_k^{\rm out}\eta}, & \eta>\eta_0\,,\end{cases} \tag{3.13}$$

where

$$\omega_k^{\rm in}=\sqrt{\mathbf{k}^2+m^2\Omega_1^2},\quad \omega_k^-=\tfrac{1}{2}(\omega_k^{\rm in}-\omega_k^{\rm out})\,,$$
$$\omega_k^{\rm out}=\sqrt{\mathbf{k}^2+m^2\Omega_2^2},\quad \omega_k^+=\tfrac{1}{2}(\omega_k^{\rm in}+\omega_k^{\rm out})\,.$$

From this we can read directly the Bogoliubov coefficients. The positive-definite modes in the $\eta\to-\infty$ region ($u_{\mathbf{k}}$) are those proportional to $e^{-i\omega_k^{\rm in}\eta}$ whereas the positive-definite modes in the $\eta\to+\infty$ region ($\bar{u}_{\mathbf{k}}$) are those proportional to $e^{-i\omega_k^{\rm out}\eta}$; they are related by

$$\bar{u}_{\mathbf{k}}=\alpha_{\mathbf{k}}u_{\mathbf{k}}+\beta_{\mathbf{k}}u_{\mathbf{k}}^* \tag{3.14}$$

and the corresponding vacua is denoted by $|0\rangle$ and $|\bar{0}\rangle$, respectively. Thus from (3.13) we read the Bogoliubov coefficients as

$$\alpha_{\mathbf{k}}=\frac{\omega_k^+}{\sqrt{\omega_k^{\rm out}\omega_k^{\rm in}}}e^{2i\omega_k^-\eta_0},\quad \beta_{\mathbf{k}}=\frac{\omega_k^-}{\sqrt{\omega_k^{\rm out}\omega_k^{\rm in}}}e^{-2i\omega_k^+\eta_0}\,. \tag{3.15}$$

Although the step expansion that we are using gives the simplest possible modes, it is not continuous and high-frequency modes are not well represented. Thus we must introduce a physically motivated transition time parameter $\Delta\eta$ between the "in" and "out" regions. Then the modes with frequencies $k^0\geq(\pi\Delta\eta)^{-1}$ can be excluded since in a more realistic smooth transition they are exponentially damped, i.e., $|\beta_{\mathbf{k}}|\sim\exp(-\pi k^0\Delta\eta)$. In cosmology particle production takes place mainly at the beginning of the semiclassical approach, close to the Planck time when one matches the cosmological metric to Minkowski spacetime.[1]

Of course the main contribution to the production of massive particles comes from the nonperturbative terms on the right-hand side of Eq. (3.10a) but we are interested in the evolution of the perturbative term here. Since we now have particle creation by the background and furthermore we have the interaction, we will treat the problem in a somewhat different way than previously. We shall consider two different but related approaches. In the first (i) we shall directly compute the total number of physical out particles created starting with no physical particles; in the second (ii) we shall use again the vacuum persistence amplitude as the fundamental quantity modifying it to the present situation.

1. *Number of particles created*

Let us consider a physical state $|\Psi\rangle$, in the interaction picture it evolves in time according to the Schrödinger equation (2.16), let us assume that

$$\lim_{\eta\to-\infty}|\Psi\rangle=|0,\text{in}\rangle\;;$$

that is, we have no particles initially. Because of the interaction it will finally evolve to

$$\lim_{\eta\to+\infty}|\Psi\rangle=S|0,\text{in}\rangle=\mathcal{N}\left[|0,\text{in}\rangle+\sum_n\frac{1}{n!}|n,\text{in}\rangle\langle n,\text{in}|S^{(1)}|0,\text{in}\rangle\right],$$

where $\mathcal{N}$ is a normalizing factor in order to have $\langle\Psi|\Psi\rangle=1$, the vector $|n\rangle$ symbolizes any vector containing n particles, and we have used the expansion of S to first order in powers of $h_{\mu\nu}$.

Since the interaction is quadratic in the quantum field, the first-order contribution will come from the matrix elements $\langle\mathbf{kp}|S^{(1)}|0\rangle$, i.e., the transition amplitude to two-particle states. It is easy to see that $\mathcal{N}=1+O(h^2)$.

Now in the "out" region the physical vacuum is given by $|\bar{0},\text{out}\rangle$. The Bogoliubov transformation (3.14) can be used to relate the respective annihilation and creation operators as

$$\bar{a}_{\mathbf{k}}=\alpha_{\mathbf{k}}a_{\mathbf{k}}+\beta_{\mathbf{k}}^*a_{-\mathbf{k}}^\dagger\,.$$

Thus we may now compute the number density per unit volume of out particles in the mode $\mathbf{k}$ in the final state, $\lim_{\eta\to+\infty}|\Psi\rangle$,

$$\lim_{\eta\to+\infty}(L\Omega)^{-3}\langle\Psi|\bar{a}_{\mathbf{k}}^\dagger\bar{a}_{\mathbf{k}}|\Psi\rangle\,.$$

The average total number density of created particles is then[22,17]

$$N=\lim_{\eta\to+\infty}(2\pi\Omega)^{-3}\int d^3\mathbf{k}\langle\Psi|\bar{a}^{\dagger}_{\mathbf{k}}\bar{a}_{\mathbf{k}}|\Psi\rangle$$
$$=(2\pi\Omega)^{-3}\left[\int d^3\mathbf{k}|\beta_{\mathbf{k}}|^2+\int d^3\mathbf{k}\,d^3\mathbf{p}\,\delta^3(\mathbf{k}+\mathbf{p})\,\mathrm{Re}[\langle\mathbf{kp}|S^{(1)}|0\rangle(\alpha_{\mathbf{k}}\beta_{\mathbf{k}}+\alpha_{\mathbf{p}}\beta_{\mathbf{p}})]\right]. \quad (3.16)$$

Note that whereas the first term contains the background contribution only, the second is due to the combined effect of the interaction and the background. If $\beta_{\mathbf{k}}=0$, i.e., the background does not create particles the second term is also zero, and thus we must go to the next order to find the production of particles due to the interaction. This is consistent with our previous results where the relevant terms were of type $\sum_{\mathbf{kp}}|S^{(1)}_{\mathbf{kp}}|^2$.

Example. As an example we shall consider an arbitrary (but bounded) conformal factor $\Omega(\eta)$ and assume no inhomogeneities for simplicity, then we have

$$\langle\mathbf{kp}|S^{(1)}|0\rangle=-i\int d^4x\;\Omega^2u^*_{\mathbf{k}}u^*_{\mathbf{p}}m^2\delta\Omega^2\,,$$

where $\delta\Omega^2\equiv(\Omega^2-\Omega_b^2)$.

We may now check (3.16) for an arbitrary $\Omega(\eta)$ with the particle creation when

$$\Omega(\eta)=\tfrac{1}{2}[(\Omega_2+\Omega_1)+(\Omega_2-\Omega_1)\tanh(\rho\eta)]\,, \quad (3.17)$$

which can be computed exactly. Here ρ is the inverse transition time, when $\rho\to\infty$ this reduces to the step expansion. The Bogoliubov coefficient $\tilde{\beta}_{\mathbf{k}}$ corresponding to the tanh expansion is given by[1]

$$|\tilde{\beta}_{\mathbf{k}}|^2=\frac{\sinh^2\left[\dfrac{\pi\omega_k^-}{\rho}\right]}{\sinh\left[\dfrac{\pi\omega_k^{\rm in}}{\rho}\right]\sinh\left[\dfrac{\pi\omega_k^{\rm out}}{\rho}\right]}\,; \quad (3.18)$$

therefore, from (3.16),

$$N=\frac{1}{(2\pi\Omega)^3}\int d^3\mathbf{k}|\tilde{\beta}_{\mathbf{k}}|^2$$

because there is no interaction term and $|\Psi\rangle=|0,\text{in}\rangle$. Assuming that ρ^{-1} is small (an expansion "close" to the step expansion), we can expand $|\tilde{\beta}_{\mathbf{k}}|^2$ to first order. The zeroth-order term is, as expected, $|\beta_{\mathbf{k}}|^2$, the average number density of particles created by the step expansion. The interesting term is the first-order term, which is easily seen to be $-(\pi^3/3\rho)m^4(\Omega_1^2-\Omega_2^2)^2$, so, the contribution to N of this first-order correction is

$$N^{(1)}=-\frac{m^4}{24}\frac{(\Omega_1^2-\Omega_2^2)^2}{\Omega^3\rho}\,. \quad (3.19)$$

On the other hand, we can compute the particles created by the "interaction," i.e., the difference between the real expansion (tanh) and the background expansion (step), using formula (3.16), and obtain

$$N^{(1)}=-\frac{m^4}{4\rho^2}\frac{(\Omega_1^2-\Omega_2^2)^2}{(2\pi\Omega)^3}\int d^3\mathbf{k}\frac{1}{\omega_k^{\rm in}\omega_k^{\rm out}}$$
$$=-\frac{m^4}{8\pi}\frac{(\Omega_1^2-\Omega_2^2)^2}{\Omega^3\rho}\,, \quad (3.20)$$

where we have introduced a cutoff in the momenta, due to the sudden approximation we are using. The cutoff has been chosen to be $k^0_{\max}=\pi\rho$ as previously discussed. The result obtained agrees with the exact result in its dependence in the variables and the small numerical discrepance between (3.20) and (3.19) is due to the fact that we had to choose a cutoff parameter. Had we introduced a smooth background conformal factor such a problem would not have arose.

2. *Vacuum persistence amplitude*

Here we present an alternative computation, closer in some sense to the way in which interactions are handled in flat spacetime. The central concept in this discussion is the vacuum persistence amplitude or

$$S_0=|\langle\bar{0},\text{out}|0,\text{in}\rangle|^2$$
$$\equiv\exp\left[-\int d^4x\;w(x)\right]\equiv e^{-W}\,, \quad (3.21)$$

which gives the probability of measuring no "out" particles if no "in" particles were present.

When the background does not create particles, we have $|\bar{0},\text{out}\rangle=|0,\text{out}\rangle$ (but $|0,\text{out}\rangle\neq|0,\text{in}\rangle$), and in this case

$$S_0=|\langle 0,\text{out}|0,\text{in}\rangle|^2=e^{-W_I}\,,$$

as discussed in Sec. II A (subindex I means interaction).

Here the background creates particles and the concept of a particle is ill defined during the transition period. Thus, it will make no sense to try to define a probability density of particle creation out of this expression as we did in Sec. II.

In the absence of interaction, the creation of particles is only due to the background, so $|0,\text{out}\rangle=|0,\text{in}\rangle$ (but $|\bar{0},\text{out}\rangle\neq|0,\text{out}\rangle$) and in this case

$$S_0=|\langle\bar{0},\text{out}|0,\text{out}\rangle|^2=e^{-W_b}\,.$$

We know since Parker,[32] that $W_b=\sum_{\mathbf{k}}\ln|\alpha_{\mathbf{k}}|^2$ and when $|\beta_{\mathbf{k}}|\ll 1$ one has

$$W_b=\sum_{\mathbf{k}}|\beta_{\mathbf{k}}|^2\,,$$

that is, the total number of particles created by the background expansion. This shows again the relationship between S_0 and the number of particles created.

As S_0 contains also information on the background production, and we are mainly interested in the effect of the interaction, let us define the relative ratio

$$R\equiv\frac{|\langle\bar{0},\text{out}|0,\text{in}\rangle|^2}{|\langle\bar{0},\text{out}|0,\text{out}\rangle|^2}\equiv e^{-W^{(1)}}\,. \quad (3.22)$$

Note that $R=1$ when there is no interaction ($\Longrightarrow W^{(1)}=0$), and when the background does not create particles it agrees with S_0. Then $W^{(1)}$ is a genuine effect of the interaction. A rather similar argument as that of Sec. II A shows that $W^{(1)}$ can be interpreted as the probability of pair production minus the nonperturbative background effect. (In fact $W^{(1)}$ can be negative, when there is background contribution W_b to the probability; then $W^{(1)}$ should be regarded as a correction to W_b, and the total sum $W_b+W^{(1)}$ is positive.)

Let us now compute R. Since

$$\langle\bar{0},\mathrm{out}|0,\mathrm{in}\rangle=\langle\bar{0},\mathrm{out}|S^{\dagger}|0,\mathrm{out}\rangle\ ,$$

because $|0,\mathrm{out}\rangle=S|0,\mathrm{in}\rangle$, and introducing a complete basis of vectors,

$$\langle\bar{0},\mathrm{out}|0,\mathrm{in}\rangle=\sum_n\frac{1}{n!}\langle\bar{0},\mathrm{out}|\mathrm{n},\mathrm{out}\rangle\langle n,\mathrm{out}|S^{\dagger}|0,\mathrm{out}\rangle\ .$$

As the S matrix is quadratic in the quantum fields, the lowest-order contribution is

$$\begin{aligned}\langle\bar{0},\mathrm{out}|0,\mathrm{in}\rangle&=\langle\bar{0},\mathrm{out}|0,\mathrm{out}\rangle\langle 0,\mathrm{out}|S^{\dagger}|0,\mathrm{out}\rangle+\frac{1}{2!}\sum_{\mathbf{kp}}\langle\bar{0},\mathrm{out}|\mathbf{kp},\mathrm{out}\rangle\langle\mathbf{kp},\mathrm{out}|S^{\dagger}|0,\mathrm{out}\rangle\\&=\langle\bar{0},\mathrm{out}|0,\mathrm{out}\rangle\left[\langle 0,\mathrm{out}|S^{\dagger}|0,\mathrm{out}\rangle+\frac{1}{2!}\sum_{\mathbf{kp}}\Lambda_{\mathbf{kp}}\langle\mathbf{kp},\mathrm{out}|S^{\dagger}|0,\mathrm{out}\rangle\right]\end{aligned}\tag{3.23}$$

where

$$\Lambda_{\mathbf{kp}}\equiv\frac{\langle\bar{0}|\mathbf{kp}\rangle}{\langle\bar{0}|0\rangle}=2\beta_{\mathbf{k}}\alpha_{\mathbf{k}}^{-1}\delta_{\mathbf{k},-\mathbf{p}}=2\delta_{\mathbf{k},-\mathbf{p}}\frac{\omega_k^-}{\omega_k^+}e^{-2i\omega_k^{\mathrm{in}}\eta_0}\ .$$

Then

$$\begin{aligned}R&=\left|\langle 0,\mathrm{out}|S^{\dagger}|0,\mathrm{out}\rangle+\frac{1}{2!}\sum_{\mathbf{kp}}\Lambda_{\mathbf{kp}}\langle\mathbf{kp},\mathrm{out}|S^{\dagger}|0,\mathrm{out}\rangle\right|^2\\&\simeq 1+2\mathrm{Re}\left[\frac{1}{2!}\sum_{\mathbf{kp}}\Lambda_{\mathbf{kp}}\langle\mathbf{kp},\mathrm{out}|S^{(1)\dagger}|0,\mathrm{out}\rangle\right]\end{aligned}\tag{3.24}$$

to first order in the interaction. Thus we can read off the expression for $W^{(1)}$ from (3.22) as

$$W^{(1)}=-\mathrm{Re}\left[\sum_{\mathbf{kp}}\Lambda_{\mathbf{kp}}\langle\mathbf{kp},\mathrm{out}|S^{(1)\dagger}|0,\mathrm{out}\rangle\right]\ .\tag{3.25}$$

Example. As previously, we consider now an arbitrary conformal factor $\Omega(\mu)$ and assume no inhomogeneities. Then Eq. (3.25) becomes

$$W^{(1)}=\sum_{\mathbf{k}}\frac{\omega_k^-}{\omega_{\mathbf{k}}^+}\int d\eta\, m^2\delta\Omega^2\frac{\sin 2\omega_{\mathbf{k}}^a(\eta_0-\eta)}{\omega_{\mathbf{k}}^a}\ ,$$

where $a=$in when $\eta<\eta_0$ but $a=$out when $\eta>\eta_0$, and the density per unit volume is

$$N^{(1)}=\frac{1}{8\pi^3\Omega^3}\int d^3\mathbf{k}\frac{\omega_k^-}{\omega_k^+}\int d\eta\, m^2\delta\Omega^2\frac{\sin 2\omega_{\mathbf{k}}^a(\eta_0-\eta)}{\omega_{\mathbf{k}}^a}\ .\tag{3.26}$$

As we did in the previous approach, we shall apply formula (3.26) in the tanh expansion, and compare with the exact and the perturbative results previously found. Using the conformal function given by expression (3.17), and the same cutoff, we find, as the leading term in ρ^{-1},

$$N^{(1)}=-\frac{m^4}{8\pi}\frac{(\Omega_1^2-\Omega_2^2)^2}{\Omega^3\rho}\ ,\tag{3.27}$$

which agrees with our previous result (3.20).

The more interesting case including inhomogeneities is more complicated and is now under consideration by the authors. This may be relevant for applications such as those of Ref. 33 when one cannot ignore the cosmological background.

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