Extended Sugawara construction for the superalgebras $SU(M+1|N+1)$. II. The third-order Casimir algebra

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The machinery developed in paper I is used to compute the operator-product algebra (OPA) of an operator constructed from the third-order Casimir invariant of the superalgebra $SU(m|n)$. The vertex operator construction of $SU(m|n)^{(1)}$ is used to find a realization of the OPA for level $k=1$ in terms of free bosonic fields only. It turns out that in many respects the conformal structure of the affinized Lie superalgebra $SU(m|n)^{(1)}$ is similar to that of the Kač-Moody algebra $SU(m-n)^{(1)}$. An intermediate result suggests the occurrence of extended conformal symmetries in bc systems, to which we will devote a separate discussion.

I. INTRODUCTION

Two-dimensional (2D) conformal field theories (CFT's) play a profound role in a lot of recent theoretical research, both in elementary-particle physics and statistical mechanics, as well as in mathematics. It has been argued that by studying all possible extensions of the conformal symmetry one may come to a classification of these theories. In this paper we investigate the possibility of generalizing a particular construction of extended conformal symmetries, called the Casimir algebra or extended Sugawara construction.[1,2]

There are (at least) two possible routes toward a "super generalization" of Refs. 1 and 2. On the one hand, one can try to "supersymmetrize" the Casimir algebra in the sense of trying to extend a superconformal algebra. A useful starting point in that respect might be the super Wess-Zumino-Witten (WZW) model.[3] (The results of Ref. 4, however, indicate that there exists no associative extension of the spin-3 algebra in terms of superfields of dimension $\frac{3}{2}$ and $\frac{5}{2}$.) Alternatively one might try to generalize by using the Casimir invariants of a Lie superalgebra. This will be the approach taken here. As we will see this construction does not give rise to fields of half-integer dimension. However, as far as a classification of rational conformal field theories is concerned, this is believed not to be a restriction.[5,6] Furthermore, this latter generalization is particularly interesting from the mathematical side where the field of infinite-dimensional superalgebras is only recently being explored in detail.

Let us discuss the idea underlying the construction. Kač-Moody algebras have been successfully applied in understanding and proving properties of CFT's, as well as in their construction. The basic ingredient is the Sugawara construction,[7] which provides a stress-energy tensor (generating the Virasoro algebra) bilinear in the Kač-Moody currents. Moreover, the Virasoro algebra obtained in this way commutes with $\bar{g}$, the horizontal finite-dimensional Lie algebra of the affine Kač-Moody algebra g. Thus we may interpret the Sugawara construction as the construction of the symmetry algebra of the coset pair $(g,\bar{g})$. The Goddard-Kent-Olive (GKO) coset construction[8] generalizes this to a Virasoro algebra associated with every coset pair (g,g') $g\supset g'$ of affine Kač-Moody algebras.

One of the applications of this coset Virasoro algebra $\tilde{\mathcal{V}}$ uses the fact that integrable highest-weight modules of g are finitely reducible in highest-weight modules of the direct sum $g'\oplus\tilde{\mathcal{V}}$ if and only if the central charge c of $\tilde{\mathcal{V}}$ satisfies $c<1$. A motivation for generalizing the Sugawara construction is to extend the coset Virasoro algebra for a general coset (g,g') to a closed algebra $\mathcal{W}$, commuting with g', such that highest-weight modules of g will be finitely reducible with respect to $g'\oplus\mathcal{W}$ also for central charges $c>1$. Clearly, the maximal extension is given by adding to the Virasoro algebra all singlet fields under g'. The question is whether all these singlet fields can be grouped into primary fields and their descendants, and whether there exists a finite set of primary fields which, together with their descendants and normal-ordered products thereof, form a closed operator-product algebra (OPA). In particular, for the coset $(g,\bar{g})$, singlet fields under $\bar{g}$ are easily constructed by contracting the Casimir invariants of $\bar{g}$ with the g currents.

In a previous paper,[9] henceforth referred to as I, we set up the machinery for the actual computation of this paper. We determined the f and d symbols as well as their mutual contractions. We also gave the level-1 vertex-operator realization for the affinized Lie superalgebra $SU(m|n)^{(1)}$, and showed equality of the partition functions at various stages of the bosonization that is involved.

This paper is organized as follows. In Sec. II we will give the general result for the operator-product expansions (OPE's) among the fields obtained from the second- and third-order Casimir invariants of the Lie superalgebra SU($m|n$), using the contraction identities in paper I. In Sec. III we restrict to level-1 modules, and employ the vertex-operator construction to realize the Casimir operators of Sec. II in terms of (free) bosonic fields only. Using these reduced expressions we will, in Sec. IV, compute the OPA of Sec. II once again and explore the consequences of putting $m-n=3$, which is the analog of the SU(3) case of Refs. 1 and 2. As a by-product of our calculations we discover an extended symmetry algebra in the $c=-2$ bosonized bc system. Because this is relevant on its own we explore the occurrence of extended symmetry algebras in this context in Sec. V.

II. CASIMIR OPERATORS FOR AFFINIZED LIE SUPERALGEBRAS

In this section we will review how to associate a conformal field theory to a super Kač-Moody algebra and we will show, in the specific example of SU($m|n$), how to discover extensions of the conformal symmetry by exploiting the Casimir invariants of the superalgebra. The mathematics of infinite-dimensional superalgebras is still quite unexplored. Because it is not our purpose here to be "as general as possible" but merely to illustrate some ideas we will restrict ourselves to a convenient subclass; the (untwisted) affinized contragredient Lie superalgebras (see, e.g., Ref. 10).

A contragredient Lie superalgebra $\bar{g}$ is defined by a Cartan matrix which directly leads to a superinvariant supersymmetric bilinear form (,). Choosing a basis $\{T_A, A=1,\ldots,\dim\bar{g}\}$, for $\bar{g}$, we put $\gamma_{AB}=(T_A,T_B)$. The affinization $g\equiv\bar{g}^{(1)}$ of $\bar{g}$ is conveniently encoded in a set of currents $J_A(z)$ with mutual OPE's:

$$J_A(z)J_B(w)=\frac{k\gamma_{AB}}{(z-w)^2}+f_{AB}{}^{C}\frac{J_C(w)}{z-w}+\cdots . \quad (2.1)$$

There exists a Z_2 grading $\bar{g}=\bar{g}_0\oplus\bar{g}_{\bar{1}}$ which determines the bosonic (fermionic) character of the current $J_A(z)$ according to $T_A\in\bar{g}_{\bar{0}}$ ($T_A\in\bar{g}_{\bar{1}}$).

In dealing with operator-product expansions of composite operators there are two important ingredients. (We restrict the following discussion to operators of integer dimension, single values on the complex plane.) First of all, composite operators at coincident points are ill defined unless one specifies a normal ordering. We will choose the point-splitting prescription, which we will denote by parentheses:

$$(AB)(z)=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}\frac{dx}{x-z}A(x)B(z) . \quad (2.2)$$

The contour $\mathcal{C}_z$ surrounds the point z counterclockwise, and operators at noncoincident points are always implicitly assumed to be radially ordered. Another ingredient is the Wick theorem

$$\underbrace{A(z)(BC)}(w)=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}\frac{dx}{x-z}[\underbrace{A(z)B}(x)C(w)+(-)^{AB}B(x)\underbrace{A(z)C}(w)] , \quad (2.3)$$

where $(-)^{AB}$ equals -1 if A and B are both fermonic operators and 1 otherwise. A lot of useful identities and rearrangement lemmas, easily generalized to include fermionic fields, can be found in Ref. 1.

To an affinized Lie superalgebra (2.1) one can associate a conformal structure by the super-Sugawara construction (provided $k\neq-h^\vee$) (at this point we explicitly require that the invariant metric γ_{AB} is in invertible)

$$T(z)=\frac{1}{2(k+h^\vee)}\gamma^{BA}(J_AJ_B)(z) , \quad (2.4)$$

where the dual Coxeter number $h^\vee$ is defined by

$$f^{D}{}_{A}{}^{C}f_{CBD}=2h^\vee\gamma_{AB} \quad (2.5)$$

and the central charge is given by

$$c=\frac{k\,\mathrm{sdim}g}{k+h^\vee} . \quad (2.6)$$

The currents $J_A(z)$ are primary fields of dimension 1 with respect to $T(z)$.

Let us now restrict the discussion to SU($m|n$). As we have shown in I, the contractions between the various super f and d symbols (i.e., at least up to order 3) for SU($m|n$) are of identical structure as those of SU($m-n$). (Similar results were found in Ref. 11.) The same is true for the third-order Casimir algebra. Put (for conventions see I)

$$\begin{aligned} Q_A&=\sum_{B,C}d_A{}^{BC}(J_CJ_B) , \\ Q&=\sum_{A,B,C}d^{ABC}(J_C(J_BJ_A)) , \end{aligned} \quad (2.7)$$

then $Q_A(z)$ and $Q(z)$ are primary with respect to $T(z)$ of (2.4) and in particular we have

$$\begin{aligned} T^{(3)}(z)T^{(3)}(w)=&\frac{c/3}{(z-w)^6}+\frac{2T(w)}{(z-w)^4}+\frac{\partial T(w)}{(z-w)^3}+\frac{1}{(z-w)^2}(2b^2\Lambda+\tfrac{3}{10}\partial^2T+R_4)(w) \\ &+\frac{1}{z-w}(b^2\partial\Lambda+\tfrac{1}{15}\partial^3T+\tfrac{1}{2}\partial R_4)(w)+\cdots , \end{aligned} \quad (2.8)$$

where

$$T^{(3)}(z)=\mathcal{N}(k)Q(z),\quad \mathcal{N}(k)=\frac{1}{3(m-n+k)}\left[\frac{m-n}{2(m-n+2k)[(m-n)^2-4]}\right]^{1/2},$$
$$b^2=\frac{16}{22+5c},\quad c=\frac{k[(m-n)^2-1]}{m-n+k},\quad \Lambda(z)=[(TT)-\tfrac{3}{10}\partial^2T](z)\,, \tag{2.9}$$
$$R_4(z)=-2b^2\Lambda(z)-\tfrac{4}{5}\partial^2T(z)+\frac{1}{2(m-n+k)}(\partial J^A\partial J_A)(z)+9(m-n+k)[\mathcal{N}(k)]^2(Q^AQ_A)(z)\,.$$

To derive (2.8) it suffices to observe that the possible minus signs in (2.3) arrange themselves such that the contractions between indices follow the tensor rules of I. This allows one to apply the results of Ref. 1.

Let us recall that in the "bosonic" case of SU(m) it could be shown that $R_4(z)$ is a null field precisely for $m=3$, $k=1$ (Ref. 1). In that case the algebra closes on $T(z)$ and $T^{(3)}(z)$ only. This result is most easily derived by employing the $k=1$ vertex operator realization for SU(m). Similar issues for the supercase SU($m|n$) will be discussed in the next section.

III. VERTEX-OPERATOR REDUCTION OF THIRD-CASIMIR OPERATORS

In this section and the next we will consider the Casimir algebra of Sec. II for level $k=1$ representation only. In that case we can use the vertex-operator realization to deduce a realization of the OPA (2.8) in terms of scalar fields. In the "bosonic" case of SU(m) this has proved very useful to eliminate explicitly the null fields which could be present in the discussion of the symmetry algebra as in Sec. II. As discussed in I, the vertex-operator realization for SU$(m|n)^{(1)}$ [but also for OSp$(m|n)^{(1)}$, see Ref. 12] can most easily be found by bosonization of the free fermion representation (also called "symplectic boson" representation) of Refs. 13 and 14. This procedure also provides an explicit realization of the various cocycle factors c_α that are involved. For convenience we recall the results of I.

Introduce a set of $m+n$ scalar fields ϕ^I, $I=\{i,\bar{i}\}$, $i=1,\ldots,m$, $\bar{i}=1,\ldots,n$, with the two-point function

$$\langle\phi^I(z)\phi^J(w)\rangle=-(-)^I\delta^{IJ}\ln(z-w)\,, \tag{3.1}$$

where

$$(-)^I=\begin{cases}1\,, & I=1\,,\\ -1\,, & I=\bar{i}\end{cases} \tag{3.2}$$

and a set of n scalar fields (of conformal charge $c=-2$ each) $\chi^{\bar{i}}$, $\bar{i}=1,\ldots,n$ with

$$\langle\chi^{\bar{i}}(z)\chi^{\bar{j}}(w)\rangle=-\delta^{\bar{i}\bar{j}}\ln(z-w)\,. \tag{3.3}$$

Then, in the super-Chevalley basis for SU$(m|n)^{(1)}$ we have

$$J_a=(-)^I(i\partial\varphi^I-i\partial\phi^{I+1})\,,\quad a=I=1,\ldots,m+n-1\,,$$
$$J_{\alpha_{ij}}=e^{i(\phi^i-\phi^j)}c_{\alpha_{ij}}\,,$$
$$J_{\alpha_{i\bar{j}}}=e^{-i\chi^{\bar{j}}}e^{i(\phi^i-\phi^{\bar{j}})}c_{\alpha_{i\bar{j}}}\,, \tag{3.4}$$
$$J_{\alpha_{\bar{i}j}}=\partial(e^{i\chi^{\bar{i}}})e^{i(\phi^{\bar{i}}-\phi^j)}c_{\alpha_{\bar{i}j}}\,,$$
$$J_{\alpha_{\bar{i}\bar{j}}}=\partial(e^{i\chi^{\bar{i}}})e^{-i\chi^{\bar{j}}}e^{i(\phi^{\bar{i}}-\phi^{\bar{j}})}c_{\alpha_{\bar{i}\bar{j}}}\,.$$

(Please note that the conventions are related to those of I by $\phi^{\bar{i}}\to-\phi^{\bar{i}}$ and $\chi^{\bar{i}}\to-\chi^{\bar{i}}$.)

To avoid a proliferation of i's and partial derivatives later, but also to be able to write the result in an elegant form we introduce

$$H^I=i\partial\phi^I-\frac{1}{m-n}\left[\sum_I(-)^Ii\partial\phi^I\right],$$
$$K^{\bar{i}}=i\partial\chi^{\bar{i}}\,. \tag{3.5}$$

The system is constrained by the equation $\sum_I(-)^IH^I=0$, and we have

$$\langle H^I(z)H^J(w)\rangle=\frac{\gamma^{IJ}}{(z-w)^2}\,,$$
$$\gamma^{IJ}\equiv(-)^I\delta^{IJ}-\frac{1}{m-n}\,, \tag{3.6}$$

and, for instance,

$$J^a=\gamma^{ab}J_b=(-)^I\sum_{J=1}^{I}(-)^JH^J\,,$$
$$I=a=1,\ldots,m+n-1\,. \tag{3.7}$$

Having a bosonic representation of the affine superalgebra SU$(m|n)^{(1)}$ at level $k=1$ in terms of vertex operators, let us compute the vertex-operator reduction of the stress-energy tensor

$$T(z)=\frac{1}{2(m-n+1)}\sum\gamma^{AB}(J_BJ_A)\,.$$

Normal-ordered products $(J_\alpha J_{-\alpha})$ between the off-diagonal generators can be evaluated by applying the Campbell-Baker-Hausdorff formula after which the contour integration of (2.2) can explicitly be performed. The result of this calculation is

$$T(z)=\frac{1}{2}\left[\sum_I(-)^I(H^IH^I)+\sum_{\bar i}(K^{\bar i}K^{\bar i}+\partial K^{\bar i})\right](z)\ . \tag{3.8}$$

For later use we introduce

$$\begin{aligned} T_H^I&=\tfrac{1}{2}(-)^I(H^IH^I)\ ,\\ T_K^{\bar i}&=\tfrac{1}{2}(K^{\bar i}K^{\bar i}+\partial K^{\bar i})\ . \end{aligned} \tag{3.9}$$

Of course, the result (3.8) was to be expected because these are just the energy-momentum tensors of the scalar fields $H^I(z)$ and $K^{\bar i}(z)$. [One might think that there is one scalar field too many in (3.8) to obtain $c=m-n-1$. It should be remembered, however, that the fields H^I are constrained by the equation $\sum_I(-)^IH^I=0$.] Nevertheless it should be emphasized that, contrary to the SU(m) case, the result (3.8) is not proportional to what would be obtained from the "naive" Cartan subalgebra restriction $\sum\gamma^{ab}(J_aJ_b)$, which would only give the first term in (3.8). The so-called "quantum equivalence theorem"[15] breaks down because we are dealing with a nonunitary model.

To obtain the reduction of $Q(z)$ we proceed similarly. During the calculation one makes frequent use of the identities (in particular the rearrangement lemmas) of Appendix A of Ref. 1.

Let us give the intermediate result of $Q_A(z)$ (in the Chevalley basis of I):

$$\begin{aligned} Q_a&=2(m-n+2)(-)^I(T^I-T^{I+1})\ ,\\ &\qquad\qquad I=a=1,\ldots,m+n-1\ ,\\ Q_{\alpha_{ij}}&=(m-n+2)\{(H^i+H^j)J_{\alpha_{ij}}\}\ ,\\ Q_{\alpha_{i\bar j}}&=(m-n+2)\{(H^i+H^{\bar j})J_{\alpha_{i\bar j}}\}\\ &\quad+(m-n)(K^{\bar j}J_{\alpha_{i\bar j}})\ ,\\ Q_{\alpha_{\bar i j}}&=(m-n+2)\{(H^{\bar i}+H^j)J_{\alpha_{\bar i j}}\}\\ &\quad+(m-n)\partial^2(e^{i\chi^{\bar i}})e^{i(\phi^i-\chi^{\bar j})}c_{\alpha_{\bar i j}}\ ,\\ Q_{\alpha_{\bar i\bar j}}&=(m-n+2)[\{(H^{\bar i}+H^{\bar j}+K^{\bar i}+K^{\bar j})J_{\alpha_{\bar i\bar j}}\}\\ &\quad+\partial K^{\bar i}e^{i(\chi^{\bar i}-\chi^{\bar j}+\phi^{\bar i}-\phi^{\bar j})}c_{\alpha_{\bar i\bar j}}]\ , \end{aligned} \tag{3.10}$$

where $\{\ \}$ stands for the weighted symmetrization $\{AB\}=\frac{1}{2}(AB+BA)$. After another contraction with J^A one finally arrives at the result

$$Q(z)=(m-n+2)(m-n+1)\left[\sum_I(-)^I(H^I(H^IH^I))+6\sum_{\bar i}(T_K^{\bar i}H^{\bar i})+2\sum_{\bar i}\Psi_K^{\bar i}\right](z)\ , \tag{3.11}$$

where we have defined

$$\Psi_K^{\bar i}=(K^{\bar i}(K^{\bar i}K^{\bar i}))+\tfrac{3}{2}(\partial K^{\bar i}K^{\bar i})+\tfrac{1}{4}\partial^2K^{\bar i}\ . \tag{3.12}$$

As a check on the calculations we have verified that $Q(z)$ as given above is indeed primary with respect to the total stress-energy tensor $T(z)=T_H(z)+T_K(z)$ of (3.8). In fact, both $\sum_I(-)^I(H^I(H^IH^I))+6\sum_{\bar i}(T_K^{\bar i}H^{\bar i})$ and $\Psi_K^{\bar i}$ turn out to be primary fields under $T(z)$. In Sec. V we will argue that $\Psi_K^{\bar i}$ is very interesting in its own right, but let us now first discuss some features of the reduced $Q(z)$ in (3.11).

The corresponding result for SU(m) is easily read off from (3.11) by putting $n=0$, to get

$$\begin{aligned} Q^{(n=0)}(z)&\sim\sum_i(H^i(H^iH^i))\\ &\sim\sum_{\text{CSA}}d^{abc}(J_a(J_bJ_c))\ . \end{aligned} \tag{3.13}$$

One sees that the reduced expression is proportional to the "naive" restriction of $Q(z)$ to the Cartan subalgebra [1](CSA). This is related to the statement that the d symbols restricted to the Cartan subalgebra satisfy similar contraction identities as the full d symbols. In particular, the fact that $Q^{(n=0)}(z)$ is primary is a consequence of the fact that the restricted d symbol is also traceless.

In this context the use of the "constrained" fields $H^i(z)$ now has a very simple interpretation. For U(m) a basis of the Cartan subalgebra can be chosen for which $d_{ijk\cdots}=1$ if $i=j=k=\cdots$ and $d_{ijk\cdots}=0$ otherwise. Therefore the Casimir operators of U(m) have very simple expressions, which are in fact exactly the SU(m) Casimir operators if we restrict to the subspace given by the constraint $\sum_I(-)^IH^I=0$. The reason for mentioning this is that representing the Casimir operators in the U(m) context, in which *all* the d symbols are simple, might be convenient to perform similar computations for higher-order Casimir operators.

Let us now return to the full SU($m|n$) expression. In this case the corresponding "naive" Cartan subalgebra restriction of Q would only give rise to the first term on the right-hand side of (3.11). This is clearly not the correct result, since $\sum_I(-)^I(H^I(H^IH^I))$ is not even primary. The reason is that, contrary to the SU(m) case, the Cartan subalgebra restriction of the d symbol is *not* traceless, while the full d symbol is. [This can be traced back to the fact that there is a U(1) factor present in the even part of the Lie superalgebra SU($m|n$).] In fact, the abstract proof for the proportionality (3.13) (and similar ones for the higher-order Casimir operators) breaks down for nonunitary models.[16] It is an intriguing question whether one can still predict the outcome (3.11) of the vertex operator reduction without doing the reduction explicitly.

IV. THE LEVEL-1 THIRD-ORDER CASIMIR OPERATOR-PRODUCT ALGEBRA

In the previous section we computed the reduction of $Q(z)=\sum d^{ABC}(J_C(J_BJ_A))$ in the $(k=1)$ vertex-operator realization of the affinized Lie superalgebra SU$(m|n)$. After normalization $T^{(3)}(z)=\mathcal{N}Q(z)$, where $\mathcal{N}=\mathcal{N}(k=1)$ is given by (2.9), the operator product $T^{(3)}(z)T^{(3)}(w)$ should reproduce the result (2.8) with $c=m-n-1$. This we have explicitly verified. [An intermediate result in this computation is the operator product $\Psi_K(z)\Psi_K(w)$, which is treated in more detail in Sec. V.] As a by-product we find the expression for $R_4(z)$ in this realization

$$\begin{aligned}R_4(z)=&\left[\frac{-2}{m-n-2}-2b^2\right](TT)(z)+\tfrac{3}{10}(2b^2-1)\partial^2T(z)+\left[\frac{1}{2}+\frac{1}{m-n-2}\right]\sum_I(-)^I(H^I\partial^2H^I)(z)\\&+\frac{m-n}{2(m-n-2)}\sum_I(-)^I(H^I(H^I(H^IH^I)))(z)+\frac{6(m-n)}{m-n-2}\left[\sum_{\bar i}(T_K^{\bar i}T_K^{\bar i})-2\sum_{\bar i}(T_K^{\bar i}T_H^{\bar i})\right](z)\\&+\frac{3(m-n)}{2(m-n-2)}\left[-\partial^2T_K+\tfrac{8}{3}\sum_{\bar i}(H^{\bar i}\Psi_K^{\bar i})\right](z)\;.\end{aligned}\tag{4.1}$$

One clearly sees the appearance of the field $\sum_I(-)^I(H^I(H^I(H^IH^I)))$, which would be present in the vertex-operator reduction of the fourth-order Casimir operator of SU$(m|n)$. In the bosonic case $(n=0)$ we know that this fact picks out $m=3$ as a special case. Since we have seen that all the d-symbol contractions, etc., depend only on $m-n$, this suggests that for the case $m-n=3$ we might find special properties.

By using the rearrangement lemma [see Eq. (A15) in Ref. 1]

$$\sum_{I,J}(-)^I(-)^J((H^IH^I)(H^JH^J))=\sum_{I,J}(-)^I(-)^J(H^I(H^I(H^JH^J)))+2\sum_I(-)^I(H^I\partial^2H^I)\tag{4.2}$$

we can write, for $m-n=3$,

$$\begin{aligned}R_4(z)=\tfrac{3}{2}\Bigg[&\sum_I(-)^I(H^I(H^I(H^IH^I)))-\tfrac{1}{2}\sum_{I,J}(-)^I(-)^J(H^I(H^I(H^JH^J)))-2(T_KT_K)-4(T_KT_H)\\&+12\sum_{\bar i}(T_K^{\bar i}T_K^{\bar i})-24\sum_{\bar i}(T_K^{\bar i}T_H^{\bar i})-3\partial^2T_K+8\sum_{\bar i}(H^{\bar i}\Psi_K^{\bar i})\Bigg]\;.\end{aligned}\tag{4.3}$$

Before we discuss the general situation, let us put $m=3$, $n=0$, which effectively brings us back to the "bosonic" Lie algebra SU(3). In this case the fields ϕ^i and χ^i are absent, so we just have

$$R_4(z)=\tfrac{3}{2}\left[\sum_i(H^i(H^i(H^iH^i)))-\tfrac{1}{2}\sum_{i,j}(H^i(H^i(H^jH^j)))\right],\tag{4.4}$$

which vanishes identically because of the constraint $\sum_{i=1}^3H^i=0$. [This is just the statement that $\mathrm{Tr}A^4=\frac{1}{2}(\mathrm{Tr}A^2)^2$, for traceless 3×3 matrices.] For SU(m), $m\geq4$ the two terms in (4.4) are never proportional. Of course, this is all equivalent to the statement that in SU(3) the fourth-order Casimir operator is proportional to the square of the second-order Casimir operator. So we have reproduced the result of Ref. 1 which states that the field $R_4(z)$ in the SU(m) third-order Casimir algebra is a null field precisely for $m=3$, $k=1$, and actually identically vanishes in the vertex-operator realization.

For general $m-n=3$ the expression (4.3) clearly does not vanish identically. In fact (for $m\neq3$) one easily constructs states $|\chi\rangle$ such that $\langle\chi|R_4\rangle\neq0$ where $|R_4\rangle=\lim_{z\to0}R_4(z)|0\rangle$ is the state corresponding to the field $R_4(z)$. This proves that $R_4(z)$ is not a null field. Hence, for general $m-n=3$, $k=1$ the Casimir algebra does not close on the second- and third-order Casimir operators only. Nevertheless, there is something special about $m-n=3$; it turns out that $R_4(z)$ is a norm-zero field (i.e., $\langle R_4|R_4\rangle=0$), or equivalently has a vanishing two-point function $\langle R_4(z)R_4(w)\rangle=0$, for all $m-n=3$. This can straightforwardly be proved by showing that the central term in the OPE $R_4(z)R_4(w)$ vanishes.

As an independent check on this computation we also showed that

$$T^{(3)}(z)R_4(w)=O\left[\frac{1}{(z-w)^3}\right],\tag{4.5}$$

which, in particular, implies that that OPA structure constant $\mathcal{C}_{T^{(3)}R_4}{}^{T^{(3)}}$ vanishes, such that the three-point function $\langle T^{(3)}(z_1)T^{(3)}(z_2)R_4(z_3)\rangle$ vanishes. This would be in contradiction with $\mathcal{C}_{T^{(3)}T^{(3)}}{}^{R_4}\neq0$ [see (2.8)] unless $R_4(z)$ has a vanishing two-point function.

To summarize, the "super" generalization of the results of Ref. 1 is that the field $R_4(z)$ has a vanishing norm for all $m-n=3$, $k=1$. (As a side remark let us mention that for unitary representations zero-norm states $|\chi\rangle$ are automatically null states, since if for some state $|\phi\rangle$ we would have $\langle\phi|\chi\rangle\neq0$ then the norm of $|\phi\rangle+\epsilon|\chi\rangle$ could be made negative for $|\epsilon| \to \infty$.) To answer the question if it would nevertheless be possible to have a singlet algebra based on a finite number of primary fields would require an explicit computation of operator products $R_4(z)R_4(w)$ and so forth. Clearly, to extend the previous analysis of the third-order Casimir operator-product expansion to operators of higher dimensions would be notoriously complicated. To make some progress one would certainly need more powerful machinery, as for instance the theory of quantum groups (see Ref. 17 in this context). Still, the more straightforward analysis of this paper (though admittedly not complete), is useful in the sense that it sheds light onto what kind of structures one might expect to emerge.

V. EXTENDED CONFORMAL SYMMETRIES IN *bc* SYSTEMS

In the reduction of the third-order Casimir operator $Q(z)$ for $SU(m|n)$ we have seen that the following field appears [see Eq. (3.12), we restrict to one index $\bar{i}$]:

$$\Psi_K=(K(KK))+\tfrac{3}{2}(\partial KK)+\tfrac{1}{4}\partial^2 K \ , \tag{5.1}$$

where $K=i\partial\chi$ corresponds to a bosonized spin-0 *bc* system of central charge $c=-2$. Its stress-energy tensor is given by

$$T_K=\tfrac{1}{2}((KK)+\partial K) \ . \tag{5.2}$$

With respect to this stress-energy tensor, Ψ_K turns out to be a primary field of dimension 3.

By invoking the free field Wick theorem, or repeatedly using (2.3), one shows that the normalized field $\tilde{\Psi}_K=\sqrt{2/27}\,\Psi_K$ satisfies

$$\begin{aligned}\tilde{\Psi}_K(z)\tilde{\Psi}(w)=&\frac{-\frac{2}{3}}{(z-w)^6}+\frac{2T_K(w)}{(z-w)^4}+\frac{\partial T_K(w)}{(z-w)^3}\\&+\frac{1}{(z-w)^2}[\tfrac{8}{3}(T_KT_K)-\tfrac{1}{2}\partial^2 T_K](w)\\&+\frac{1}{z-w}[\tfrac{4}{3}\partial(T_KT_K)-\tfrac{1}{3}\partial^3 T_K](w)\\&+\cdots \ .\end{aligned} \tag{5.3}$$

This is precisely the spin-3 algebra (2.8), without the field $R_4(z)$, for $c=-2$. If we would have known this beforehand we might have anticipated the occurrence of the field $(\sum_{\bar{i}}\Psi_K^{\bar{i}})(z)$ in the vertex-operator reduction of $Q(z)$. [Of course, the sum $\Psi_K=\sum_{\bar{i}}\Psi_K^{\bar{i}}$ does not satisfy a spin-3 algebra with $c=-2n$. The difference, which occurs at $O(1/(z-w)^2)$ because this term is nonlinear both in T_K as well as in c, is absorbed in the field $R_4(z)$.]

It is well known that one can find a realization of the spin-3 algebra (for every c value) in terms of two scalar fields coupled to a background charge.[18] This corresponds to the Feigin-Fuchs construction for the level-1 vertex-operator representation of the Kač-Moody algebra $A_2^{(1)}=SU(3)^{(1)}$ (Ref. 2). It is remarkable however that one can also find a realization, though nonunitary, of the spin-3 algebra in terms of one scalar field only.

To investigate in general the possibility of having an extended conformal symmetry in a bosonized *bc* system, we investigated the question whether it is possible to construct polynominals in $K,\partial K,\partial^2 K,\ldots$ which are primary of spin-Δ with respect to

$$T_K=\tfrac{1}{2}((KK)+Q\partial K),\quad c=1-3Q^2 \ . \tag{5.4}$$

By writing down the most general polynomial and requiring it to be primary under T_K we found (up to an overall constant factor) the following possibilities for $\Delta\le 5$:

$$\begin{aligned}&\Delta=1,\quad \Psi=K,\quad c=1\ ,\\&\Delta=2,\quad \Psi=(KK),\quad c=0\ ,\\&\Delta=3,\quad \Psi=(K(KK))+\tfrac{3}{2}Q(\partial KK)+\tfrac{1}{4}\partial^2K,\quad c=-2\ ,\\&\Delta=4,\quad \Psi=(K(K(KK)))+\tfrac{3}{2}(\partial K\partial K)-(\partial^2KK),\quad c=1\ ,\\&\qquad\quad\ \Psi=(K(K(KK)))+2Q(\partial K(KK))+\tfrac{3}{5}(\partial K\partial K)+\tfrac{4}{5}(\partial^2KK)+\tfrac{1}{15}Q\partial^3K,\quad c=-\tfrac{22}{5}\ ,\\&\Delta=5,\quad \Psi=(K(K(K(KK))))+\tfrac{5}{2}Q(\partial K(K(KK)))+\tfrac{5}{2}(\partial K(\partial KK))+\tfrac{5}{3}(\partial^2K(KK))+\tfrac{5}{12}Q(\partial^2K\partial K)\\&\qquad\qquad+\tfrac{5}{24}Q(\partial^3KK)+\tfrac{1}{36}\partial^4K,\quad c=-7\ .\end{aligned} \tag{5.5}$$

For every allowed c value the two roots $Q=\pm\frac{1}{3}\sqrt{1-c}$ give rise to primary fields Ψ that are related by the parity transformation $K\to -K$.

The $\Delta=1$ and $\Delta=4$ solutions are well known from the Virasoro representation theory at $c=1$ [they correspond to the Schur polynomials $\chi_{\square}$ and $\chi_{\boxplus}$ (Ref. 19)]. The other solutions all correspond to nonunitary models. As we have seen the $\Delta=3$, $c=-2$ solution turns up in the Casimir operators of affinized Lie superalgebras. It might also be relevant with regard to string theory because the same $c=-2$ system is used in the construction of the fermion emission vertex.[20] The $\Delta=4$, $c=-\frac{22}{5}$ solution might also have interesting applications because this is the c value describing the Lee-Yang edge singularity in the Ising model with an imaginary magnetic field.[21]

Additional information is obtained if one notices that the nonunitary solutions for c are present in the list of minimal models $c(r,s)=1-6(r-s)^2/rs$ and that the field Ψ_Δ corresponds to the field $\Psi_{(3,1)}$ in the grid of conformal dimensions

$$h^{(r,s)}(p,q)=\frac{(pr-qs)^2-(r-s)^2}{4rs}\,,$$
$$1\le p\le s-1,\ 1\le q\le r-1\ . \qquad (5.6)$$

From the known fusion rules of the minimal models[22] one immediately concludes that for $\Delta=3,4,5$ no additional fields appear in the singular terms of the operator-product algebra of $T(z)$ and $\Psi_\Delta(z)$, so that T and Ψ_Δ form a closed symmetry algebra by itself, as we had found for $\Delta=3$ by explicit computation. Let us remark that for generic c values the spin-5 operator-product algebra is not associative. The value $c=-7$ is however precisely one of the c values found in Ref. 23 for which there does exist an associative spin-5 algebra.

It should be emphasized that the presence of an integer-dimension field Ψ_Δ in the column $h^{(r,s)}(p,1)$ of the grid of conformal dimensions of a minimal model in general signifies the presence of an extended algebra.[23] What is remarkable however, is that for the cases discussed here the fields $\Psi_\Delta(z)$ turn out to be polynomial expressions in $K,\partial K,\ldots$. If this is not just a coincidence one could anticipate the existence of a polynomial in $K,\partial K,\ldots$ primarily of spin-Δ for $c=1-3(\Delta-1)^2/(\Delta+1)$, i.e., the c value for which $h^{(r,s)}(3,1)=\Delta$.

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