Unitarization, Castillejo-Dalitz-Dyson Poles, and Scalar Companions*

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It is shown how Castillejo-Dalitz-Dyson poles may be introduced into the effective-range formulas for scalar $(I=J=0)$ $\pi\pi$ and $(I=J=\frac{1}{2})$ $K\pi$ form factors derived from current algebra and a method of pole and cut dominance. The restriction to only two such poles leads to formulas characterized by four free parameters, one of which is conventionally fitted to the zero of the soft-meson s-wave amplitude. In the case of the s-wave $\pi\pi$ phase-shift fit, the remaining three parameters are fixed by requiring the 0^+ resonances, $\epsilon(750)$ and $\epsilon'(1060)$, and the experimentally observed value (in the up-up data) of $\delta_{00}(\pi\pi)\simeq\pi$ at 900 MeV as input. The resulting fit is nearly indistinguishable from the earlier two-parameter fit which uses only the ϵ, over the energy range from threshold to 800 MeV, and is very close to that of the unitarized Lovelace-Veneziano model from threshold to 1060 MeV, although the resonance parameters in the latter fit [$\epsilon(698)$ and $\epsilon'(911)$] differ somewhat. Considerably greater variation in fit is found in the case of the s-wave $K\pi$ phase shift; a qualitative fit to the up-up data of Antich *et al.* is obtained using $\kappa(900)$, $\kappa'(1600)$, and $\delta_0^{(1/2)}(K\pi)\simeq\pi$ at 1000 MeV.

Recently it was shown[1-5] how the conjoint employment of analyticity and hard-pion current algebra might form a basis for dynamical calculations in hadronic interactions. In the process, effective-range formulas for the s-wave $(I=J=0)$ $\pi\pi$ and $(I=J=\frac{1}{2})$ $K\pi$ form factors were derived using one or the other of the several versions[6] of this unitarized hard-meson approach. The phase shift [$\delta_{00}(\pi\pi)$ or $\delta_0^{(1/2)}(K\pi)$, say], which is determined by this procedure, once the simply parametrized form factor [$F(t)$ or $F_\kappa(t)$, say] is given, then provides a few-parameter[7] fit to the experimental phase shifts in either of the two cases of interest. However, in both instances a Chew-Low extrapolation procedure applied to the data[8,9] from the appropriate peripheral process yields a unique set of phase shifts, modulo π. [Indeed, in the case of the $I=0$ s-wave $\pi\pi$ phase shift, this twofold ambiguity[8] leads to *four* possible phase-shift solutions,[10] of which only the crossover (down-up or up-down) solutions have been favored by recent analyses.[11,12]] Some of the earlier treatments[3,5] are characterized by the introduction of a *minimal* number of parameters into the dispersion-theoretic solution for the form factor. (Thus, the data fit in Ref. 3 makes use of one parameter which serves to locate the ϵ at 960 MeV, while in those of Refs. 4 and 5 an additional free parameter is fitted to the s-wave projection of the Adler zero.) These treatments appear to lend some theoretical weight to certain of the possible phase-shift solutions. It may therefore be of interest to show that these preferences are to some extent consequences of this analytic limitation. Furthermore, if one considers the larger class of acceptable solutions to the dispersion equation, following from the introduction of Castillejo-Dalitz-Dyson (CDD) poles, considerable variation in the fit can be effected. However this result, while implying a certain diminution in predictive power, is not entirely a negative one, since we now have the possibility of simple phase-shift interpolations, from threshold over the range of several resonances in a given channel, involving (in the examples we discuss below) no more than four parameters.

It will be sufficient to detail the handling of the CDD ambiguity[13] in only one case, that of the $\pi\pi$ scalar factor $F(t)$, which is defined as in Ref. 4 by

$$\langle\pi^+(p)\pi^-(q)\text{out}|\sigma(0)|0\rangle = F(t), \quad t=-(p+q)^2 \tag{1}$$

with the isoscalar σ field defined by the axial-vector-current equal-time commutation relation

$$\delta(x_0)[A_0^a(x),\, \partial_\mu A_\mu^b(0)] = -i\delta_{ab}\delta(x)\sigma(x). \tag{2}$$

From the unitarized hard-pion approach of Ref. 4 one finds for the absorptive part of $F(t)$

$$F_\pi m_\pi^2 \mathrm{Im}F(t) = \frac{3Q(t)}{16\pi\sqrt{t}}\left[-\frac{m_\pi^2}{F_\pi}+\tfrac{1}{2}m_\pi^2 B(2m_\pi^2-t)\right] \times |F(t)|^2, \quad t\geq 4m_\pi^2, \tag{3}$$

where $Q(t)$, the center-of-mass momentum, is given by

$$Q(t) = (\tfrac{1}{4}t - m_\pi^2)^{1/2}, \tag{4}$$

F_π is 94 MeV, and B is a free parameter. One defines the factored function

$$G(t) = \left[-\frac{1}{F_\pi^2}+\frac{B}{2F_\pi}(2m_\pi^2-t)\right]F(t); \tag{5}$$

then $G(t)$ satisfies

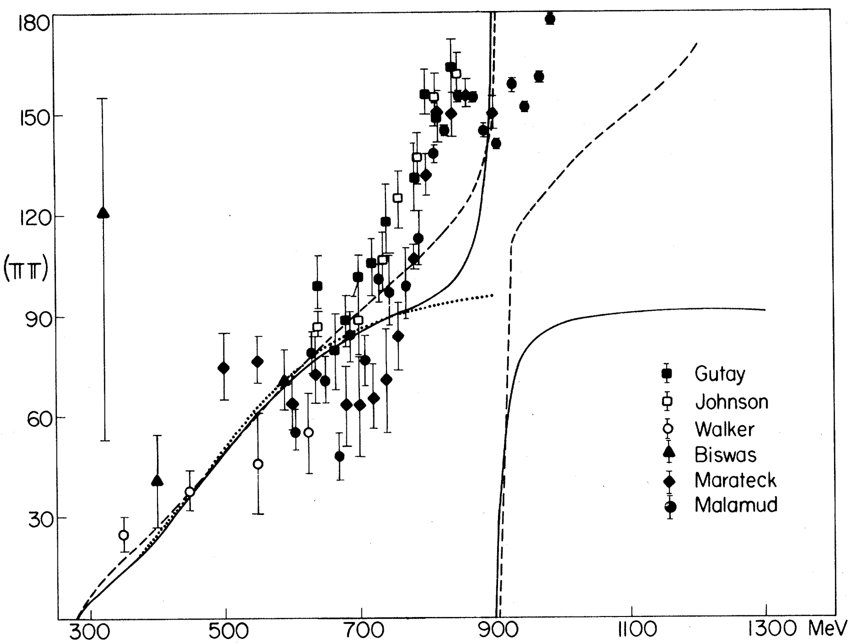

FIG. 1. The fits to $\delta_{00}(\pi\pi)$ using two parameters (Ref. 4) (dotted line) and four parameters (solid line), overlaid on the up-up $\pi\pi$ data (Ref. 15) and the unitarized Lovelace-Veneziano prediction (Ref. 15) (dashed line).

$$\operatorname{Im} G(t)=\frac{3Q(t)}{16\pi\sqrt{t}}|G(t)|^2, \quad t\geq 4m_\pi^{\;2} \tag{6}$$

with solution

$$G(t)=\left\{[G(0)]^{-1}-\frac{t}{\pi}\int_{4m_\pi^{\;2}}^{\infty}\frac{dt'}{t'(t'-t)}\frac{3Q(t')}{16\pi\sqrt{t'}}+\sum_n \frac{\alpha_n t}{t-t_n}\right\}^{-1}, \tag{7}$$

where α_n and t_n are the (real) CDD parameters. (We restrict ourselves to the case $n=1, 2$ for the remainder of this paper.) The unitarity function,[4] $h(t)$, in terms of which the solution [Eq. (7)] is customarily written, is given by

$$\begin{aligned} h(t)&=1-t\int_{4m_\pi^{\;2}}^{\infty}\frac{dt'}{t'(t'-t)}\frac{Q(t')}{\sqrt{t'}}\\ &=\frac{2Q(t)}{\sqrt{t}}\ln\left(\frac{\sqrt{t}+2Q}{2m_\pi}\right)-\frac{i\pi Q}{\sqrt{t}}, \quad t\geq 4m_\pi^{\;2}. \end{aligned} \tag{8}$$

It is natural to remove the spurious singularity in

$$F(t)=\left[-\frac{1}{F_\pi^{\;2}}+\frac{B}{2F_\pi}(2m_\pi^{\;2}-t)\right]^{-1}G(t), \tag{9}$$

at $t=t_0=-2m_\pi^{\;2}[1+1/(BF_\pi m_\pi^{\;2})]$, by requiring $t_0=t_1$. Then $F(t)$ has remaining *four* free parameters (B, α_1, α_2, t_2), which we choose in the following way: (1) We require that the low-energy amplitude $G(t)/F(t)$ have the zero of the soft-pion s-wave amplitude at $t=\frac{1}{2}m_\pi^{\;2}$, thus fixing $B=4/3F_\pi m_\pi^{\;2}$. (2) $\operatorname{Re}[F^{-1}(t_R)]=0$ at $\sqrt{t_R}=750$ and 1060 MeV, the locations of the isoscalar 0^+ resonances, $\epsilon(750)$ and $\epsilon'(1060)$ (the latter called η_0^+ in the Particle Properties Tables[14]). (3) Since requirement (2) indicates a preference for the up-up phase-shift solution, as shown in the data plot of Fig. 1, we take the experimentally indicated zero [of $F(t)$] at about 900 MeV as the remaining item of input. The resulting phase shift, given by

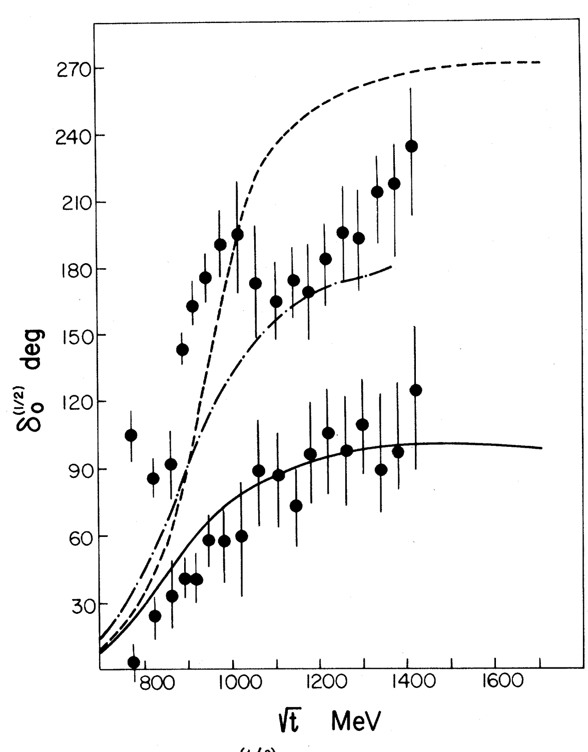

FIG. 2. The fits to $\delta_0^{(1/2)}(K\pi)$ using two parameters (Ref. 5) (solid line) and four parameters (dashed line); the dot-dashed line is the prediction of the unitarized Lovelace-Veneziano model (Ref. 15). The $K\pi$ data are taken from Antich *et al.* (Ref. 9).

$$\cot\delta_{00}(t)=\mathrm{Re}F(t)/\mathrm{Im}F(t), \tag{10}$$

has been plotted in Fig. 1. It is nearly indistinguishable up to 800 MeV from the two-parameter fit [with only the $\epsilon(750)$] given earlier by the author.[4] On the other hand, it compares rather favorably with the Lovelace-Veneziano phase shift[15] over the *extended* range $2m_\pi \leqslant \sqrt{t} \leqslant 1060$ MeV. [To be sure, the resonance locations in the latter[1] fit *are* different ($\epsilon(698)$ and $\epsilon'(911)$), but the qualitative resemblance seems to us quite noteworthy.]

A similar extension of this procedure to $K\pi$ scattering[5] yields for the appropriate scalar form factor, $F_K(t)$, the expression

$$F_K(t)=\left[F_K^{-1}(t_0)+\beta(t-t_0)-\frac{B(t-t_0)}{16\pi^2F_\pi}h_K(t)+\alpha_2\left(\frac{t-t_0}{t-t_2}\right)\right]^{-1}, \tag{11}$$

with

$$t_0=m_K^2+m_\pi^2-\frac{3m_\pi^2}{\sqrt{2}\,F_K(m_\pi^2-m_K^2)B}, \tag{12}$$

$$F_K(t_0)=(m_K^2-m_\pi^2)/\sqrt{2}, \tag{13}$$

and

$$h_K(t)=\frac{2Q_K(t)}{\sqrt{t}}\ln\left(\frac{\sqrt{t}+2Q_K}{m_K+m_\pi}\right)-\frac{i\pi Q_K}{\sqrt{t}}, \qquad t\geqslant(m_K+m_\pi)^2 \tag{14}$$

and with β, B, α_2, t_2 as free parameters. The up-up data of Antich *et al.*[9] are given a qualitative fit with κ(900 MeV), κ'(1600 MeV), and a zero at 1000 MeV. (See Fig. 2.) Note that in this case the divergence between one- and two-resonance fits is considerably greater than in the $\pi\pi$ calculation; the divergence between the two-resonance fit and the Lovelace-Veneziano[15] phase shift can of course be reduced considerably by more judicious choices of the zero (say, at $\sqrt{t}\cong 1400$ MeV) and the far-resonance position.

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[3]J. J. Brehm, E. Golowich, and S. C. Prasad, Phys. Rev. Letters 25, 67 (1970).

[4]R. Rockmore, Phys. Rev. D 2, 2628 (1970).

[5]R. Rockmore, Phys. Rev. D 2, 2693 (1970).

[6]These are the method of Ward identities of Refs. 1 and 3, and the method of pole and cut dominance of Refs. 4 and 5.

[7]We have earlier indicated our objections to *one-parameter* fits in footnote 6 of Ref. 4.

[8]J. H. Scharenguivel *et al.*, Phys. Rev. 186, 1387 (1969).

[9]P. Antich *et al.*, in *Proceedings of a Conference on $\pi\pi$ and $K\pi$ Interactions at Argonne National Laboratory, 1969*, edited by F. Loeffler and E. Malamud (Argonne National Laboratory, Argonne, Ill., 1969), p. 508.

[10]L. J. Gutay, in *Proceedings of a Conference on $\pi\pi$ and $K\pi$ Interactions at Argonne National Laboratory, 1969*, Ref. 9, p. 241; B. Maglić, Rapporteur's talk, in *Proceedings of the Lund International Conference on Elementary Particles, 1969*, edited by G. von Dardel (Berlingska, Lund, Sweden, 1970).

[11]S. Marateck *et al.*, Phys. Rev. Letters 21, 1613 (1968).

[12]W. Deinet *et al.*, Phys. Letters 30B, 359 (1969).

[13]We had already called attention to this problem in footnote 10 of Ref. 2.

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