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Partial-Wave Production Amplitudes and the Diagonalization of the Unitarity Relations in the Helicity Formalism*

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A general study is made of the partial-wave amplitudes and their unitarity relations for the general N-body-to-N'-body process in the helicity formalism. The partial-wave amplitude is so defined that it is a function of the subenergies that correspond to the noncoincident poles in a tree diagram. The partial-wave unitarity relations introduced here are diagonalized, and may supply the basis for introducing new models or improving others. The Wigner rotations and the crossing angles in the partial-wave expansion are explicitly given in terms of invariant variables. Some related topics such as the partial-wave recoupling condition, recoupling coefficients, etc., are discussed. The expressions obtained here can be applied to multiperipheral models and others. As a simple example, the contributions, elastic and inelastic, to the partial-wave amplitude via unitarity are obtained in a model of direct-channel scalar-meson dominance, with some approximations.

I. INTRODUCTION

Recently, Chew, Goldberger, and Low[1] and Halliday and Saunders[2] have revitalized the unitarity integral approach in the Amati-Bertocchi-Fubini-Stanghellini-Tonin model,[3] and extended this approach to more general multiperipheral calculations such as the multi-Regge model. Chew and collaborators utilized this powerful approach to determine trajectory functions,[4] to generalize it to the spinning-particle cases,[5] and to study cut-pole correlations.[6] Since then many advances in various directions have been made.[7-11] In those papers, multi-Regge expansions were either assumed without explanation, or obtained by expanding the production amplitudes in terms of the irreducible representations of their little groups $O(2,1)$ in the physical region of interest. In the latter case, Regge poles are picked up during successive continuation of the "angular momenta" in the $O(2,1)$ sense; these poles surely control the asymptotic behavior in their respective subenergies. However, there is no confirmation, though one may believe it is true, that these poles are related to the resonances and/or the bound states in the physical region of the proper crossed channel. This assumption underlies the use of the Chew-Frautschi plot,[12] which relates resonances and/or the bound states in the crossed channel to the corresponding Regge poles in the direct channel and vice versa, and thus provides great power of prediction in the analytic S-matrix theory.[13]

In this paper, we investigate partial-wave expansions and related topics which are useful for introducing models and for bootstrap calculations. Assuming[14] that partial-wave expansions for a production process in one channel are continuable through Sommerfeld-Watson transforms into the physical region of its crossed channel, we confirm the Chew-Frautschi interpretation of Regge poles in the production process, since the corresponding quantities in the partial-wave and in the multi-Regge expansions have the same functional form. Encouraged by this correspondence, we can easily see that one may use the technique developed here to study a production process in the physical region of interest, and to expand the production amplitude in terms of the irreducible representations of the little groups, which depend on the unitarity diagram one chooses and can be $O(2,1)$ and/or $O(3)$ groups. In the helicity formalism, we have to remove Wigner rotations in the amplitude before expanding it.

One of our other purposes is to provide a variety of realizations of unitarity. Since no model has been nearly perfect in strong-interaction physics so far, it seems useful to investigate other possibilities. Beginning with the partial-wave expansion in the crossed channel, we have obtained in the direct channel the multi-Regge description of the unitarity with a coupling diagram shown in Fig. 1(a); it turns out that in the forward elastic scattering the poles in CD's version[5] are related to the Regge poles in CF's version[12] through continued Clebsch-Gordan coefficients of $O(2,1)$. With a unitarity diagram in Fig. 1(b), we get diagonalized unitarity relations; the integrations over the little groups have been performed, and thus the number of independent variables reduces to about one third of the original number. Following the CGL model,[1] we have obtained in the whole physical region one-dimensional linear integral equations for the auxiliary functions which are algebraically related to the total absorptive part of the scattering amplitude. This provides a tool for calculating the contributions from the direct-channel resonances in the production process to the total absorptive part of a two-body-to-two-body process. One may also realize unitarity with other coupling diagrams such as in Figs. 1(c) and 1(d). It is emphasized that a choice of various unitarity diagrams depends on the regions of the subenergies in which most of the events in the production processes occur. If one saturates a production process with complete sets of resonances or Reggeons (including background term if any), the choice of diagrams is immaterial. All these applications will be discussed in detail elsewhere.

On the one hand, once a preliminary success of a simple model in which only spinless particles are produced is achieved, the next step is to take baryon-antibaryon pairs and others as intermediate states. On the other hand, one must consider unitarity for at least one production process in order to determine the residue functions of Reggeons or the coupling function of resonances through consistency or bootstrap calculations. Hence, we study unitarity for a production process with intermediate particles of arbitrary mass and spin. We carry out these studies in the helicity formalism.

The helicity partial-wave expansion in the two-body-to-two-body scattering process, introduced by Jacob and Wick,[15] has been widely used by experimentalists to describe data and by theorists to introduce Regge poles. It is natural to generalize this formalism to the N-body-to-N'-body production process. Macfarlane[16] has investigated production amplitudes in the canonical formalism; in particular the two-body-to-three-body amplitude has been obtained explicitly. Chew and collaborators[5,17] have introduced the BCP amplitudes by expanding them in terms of the irreducible representations of the little groups. In the case where all the external particles are spinless, the BCP amplitude and the helicity amplitude are equivalent[18] if the roles of the t and z axes are exchanged. Hartle and Jones[14] have expanded the production amplitudes in the c.m. frame, and they were interested in the latter's high-energy behavior. We shall first derive angular momentum states for the multiparticle system, and then the helicity partial-wave expansions; both depend on the coupling scheme of the system. This gives us a physical or geometrical interpretation of the quantities involved. In the helicity formalism, one can fix the relative normalization of the partial-wave amplitudes for the production processes in the unitarity relations. In addition, the helicity formalism permits the calculation of some interesting properties of scattering amplitudes, such as partial-wave recoupling conditions, recoupling coefficients,[15] etc.

In Sec. I we introduce notation and kinematics. In Sec. I A we define various rest frames for the scattering system, and the Lorentz groups (including the little groups) connecting them. In Sec. I B

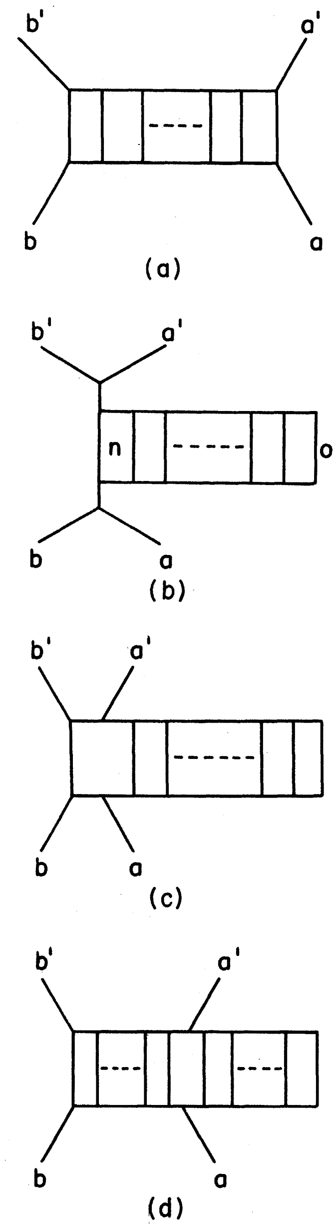

FIG. 1. (a) Unitarity diagram for multiperipheral models. (b) Unitarity diagram for models of resonance dominance. (c) Unitarity diagram for models in which the amplitude is saturated by Regge poles on one line and by resonances in other lines. (d) Unitarity diagram for models in which the amplitude is saturated by Regge poles and resonances.

we express the parameters of the Lorentz group in terms of invariant variables; we particularly discuss how to determine the parameters uniquely, since the invariant variables alone are not enough. In Sec. II C we express the phase-space volume element in terms of the little-group parameters and the invariant mass corresponding to various coupling schemes.

In Sec. III we introduce multiparticle angular momentum states and their corresponding pseudoparticle states, which will be defined later. We calculate the transformation matrix between the plane-wave state and the pseudoparticle state, and the orthogonality relation for the latter, and we point out the difference between the real particle state (a dynamical system) and the pseudoparticle state (a kinematical system). It turns out that these properties and expressions, derived from the standard coupling scheme, hold for other coupling schemes also if a proper replacement and re-interpretation is made.

In Secs. IV A and IV B we derive the helicity partial-wave expansions for general production processes with any coupling scheme. As an example, we derive the helicity partial-wave expansion for the six-particle system with the coupling scheme shown in Fig. 2. This expansion has been discussed in BCP's version by Misheloff,[19] who introduces two different parametrizations of the little groups which depend on the range of subenergy. We instead use one parametrization, which is applicable for all subenergies. In Sec. IV C we calculate the Wigner angles which appear in the partial-wave expansion with an arbitrary coupling scheme. In Sec. IV D we derive the crossing relation for the partial-wave amplitude corresponding to an arbitrary coupling scheme, if that of the helicity amplitudes is given, and vice versa. We infer in general that the partial-wave expansion must contain Wigner rotations or the like in any formalism (including the helicity formalism), if we assume that the scattering amplitude satisfies the substitution rule[20] and analyticity. In the case where the scattering system involves less than five particles, this need not be true in special Lorentz frames. In Sec. IV E we derive the crossing matrices for a general production amplitude.

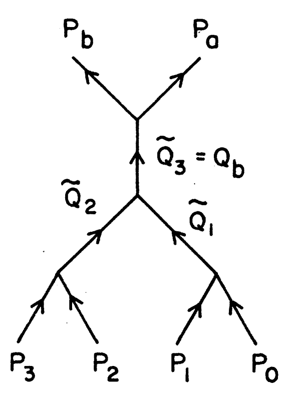

FIG. 2. The coupling scheme for three-Reggeon vertex.

In Sec. V we reduce the recoupling coefficients[15] for multiparticle angular momentum states. We emphasize that the recoupling coefficients relate partial-wave amplitudes with different coupling schemes, and thus they can be used to determine the analytic structure of one partial-wave amplitude if the other is given (partial-wave recoupling condition). In Sec. VI we derive the unitarity relations for the helicity partial-wave amplitudes of the N-body-to-N'-body process in the standard coupling scheme, thereby diagonalizing the unitarity relation. We also discuss some consequences of the Hermitian analyticity of the reaction amplitude.

In Sec. VII we express differential and total cross sections in terms of helicity and partial-wave amplitudes, respectively. These amplitudes should be useful for the phenomenological analysis of production processes. We point out that the unitarity relations here are simpler and thus more manageable for bootstrap or consistency calculations. Viewing[21] the unitarity relations as a counterpart of perturbative expansions in local field theory, we may use it as a basis for proposing new models and possibly for improving others. We illustrate this in a simple (possibly unrealistic) model of direct-channel scalar-meson dominance. Further models for various coupling schemes will be discussed elsewhere.

II. KINEMATICS

We are interested in the process

$$0+1+\cdots+N \to 0'+1'+\cdots+N'. \tag{1}$$

The number of particles is, at least on one side, greater than 1. The kinematics of such processes have been worked out by BCP[17] and CD,[5] in particular for the process $a+b\to 0+1+\cdots+N$. We shall describe these processes by helicity amplitudes and thus have a different parametrization. The group parameters of the momenta and the little groups are different in BCP amplitudes and in helicity amplitudes because the roles of the z and t are interchanged,[18] as mentioned in the Introduction. However, the corresponding parameters of the little groups in these two amplitudes are the same functions of the invariant variables up to a sign; this difference of sign comes from the choice of definition. Hence, we shall use a notation as close to BCP's as possible.

A. The Frames

In the helicity formalism, the 4-momentum p of a particle with mass m and spin σ in the initial system is parametrized by

$$p=m(\cosh\alpha,\ \sinh\alpha\sin\theta\cos\phi,\ \sinh\alpha\sin\theta\sin\phi,\ \sinh\alpha\cos\theta). \tag{2}$$

The parameters α, θ, and ϕ depend on the choice of the frames, as will be discussed later. Symbolically, one may write

$$p=L(p)(m,0,0,0)\equiv L(p)\overline{p}, \tag{3}$$

where the particle boost $L(p)$ is defined by

$$L(p)=u(\vec{p})b(p), \tag{4a}$$

$$b(p)\equiv b(\alpha)\equiv e^{-i\alpha K_3}, \tag{4b}$$

and

$$u(\vec{p})\equiv u_z(\phi)u_y(\theta)\equiv e^{-i\phi J_3}e^{-i\theta J_2}. \tag{4c}$$

We shall sometimes label the momentum p, the rotation u, the boost b, and their group parameters with superscripts and subscripts to indicate, respectively, which frame and which particle they refer to. The operator K_i is the boost generator of the Lorentz group, while J_i is the generator of rotation. The conservation of 4-momentum implies

$$\sum_{i=0}^{n}p_i=\sum_{i=0}^{n}p_i'; \tag{5a}$$

the 4-momentum $-p_i'$ in the final system may be considered as that of the $(n+n'+2-i)$th particle in the initial system and vice versa. We shall take the coupling scheme shown in Fig. 3, unless we specify otherwise. We shall call this the standard coupling scheme.

The 4-momentum Q_i in Fig. 3 is parametrized by the set $(\alpha_{0i}, \theta_{0i}, \phi_{0i})$, and its invariant mass is denoted by s_i:

$$Q_0\equiv p_0,\quad s_i=Q_i^{\,2}, \tag{5b}$$

$$Q_i=p_i+Q_{i-1}=\sum_{j=0}^{i}p_j. \tag{5c}$$

We shall sometimes use a superscript to label the frame in which the 4-momentum is defined.

The frame $[i,l]$ ($[i,u]$), for $N\geqslant i\geqslant 1$, is defined to be a rest frame of Q_i in which the 3-momentum $\vec{p}_i$ ($\vec{p}_{i+1}$) is along the negative z axis and $\vec{p}_{i+1}$ ($\vec{p}_{i+2}$) in the xz plane; symbolically, we write

$$[i,l]=[Q_i;p_i,p_{i+1}] \tag{5d}$$

and

$$[i,u]=[Q_i;p_{i+1},p_{i+2}]. \tag{5e}$$

The purpose of taking $\vec{p}_i$ ($\vec{p}_{i+1}$) along the negative instead of the positive z axis will be clear later. By definition, the frames $[i,l]$ and $[i,u]$ are related by a rotation u_i:

$$[i,u]=u_i[i,l], \tag{6a}$$

where

$$u_i\equiv u_z(\gamma_i)u_y(\beta_i). \tag{6b}$$

The 4-momentum Q_{i-1} is along the positive z axis in the frame $[i,l]$, and the rotation u_i carries the frame $[i,l]$ to $[i,u]$. Hence, the group parameters of u_i specify the orientation of Q_{i-1} in the frame $[i,u]$; i.e.,

$$\beta_i=\theta_{0i-1}^{(i,u)}\ \text{and}\ \gamma_i=\phi_{0i-1}^{(i,u)}. \tag{6c}$$

The parameters $\theta_{0i-1}^{(i,u)}$ and $\phi_{0i-1}^{(i,u)}$ are, respectively, the polar angles and the azimuthal angles of Q_{i-1} in the frame $[i,u]$. Similarly, the frame $[i,u]$ is related to $[i+1,l]$ by a boost b_i, the parameter of which is determined by the orientation of Q_i in the frame $[i+1,l]$:

$$[i+1,l]=b_i[i,u], \tag{6d}$$

where

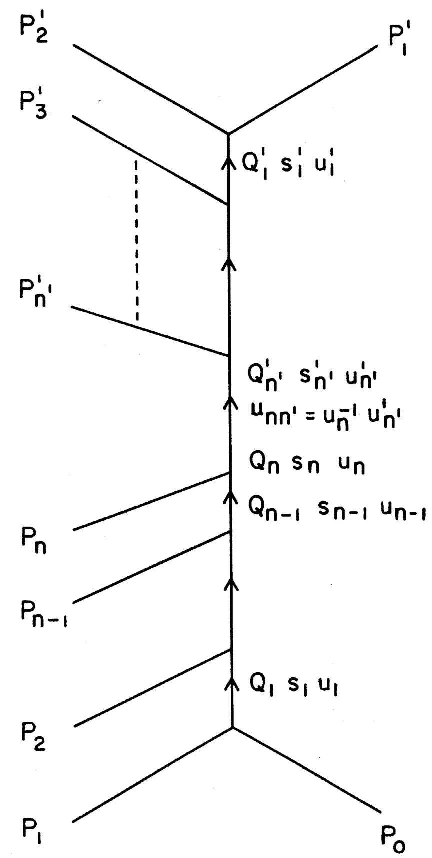

FIG. 3. The standard coupling scheme for the process $0+1+\cdots+N\rightarrow 0'+1'+\cdots+N'$.

$$b_i \equiv b(\delta_i) \text{ and } \delta_i = \alpha_{0i}^{(i+1)}. \tag{6e}$$

Note that $\alpha_{0i}^{(i+1,\,l)} = \alpha_{0i}^{(i+1,\,u)} \equiv \alpha_{0i}^{(i+1)}$, where $[i+1]$ indicates an arbitrary rest frame of Q_{i+1}.

From (6a) and (6d), one can easily see that the frames $[i,l]$ and $[j,l]$ are related by a general Lorentz transformation $\overline{L}_{ij}$:

$$[i,l] = \overline{L}_{ij}[j,l] \quad \text{for } i>j, \tag{7}$$

where

$$\overline{L}_{ij} = \overline{L}_{i-1}\overline{L}_{i-2}\cdots\overline{L}_j \text{ and } \overline{L}_l \equiv b_l u_l. \tag{7$'$}$$

Similar expressions to Eqs. (2)–(7′) can be obtained for the final system. We shall label everything in the final system with a prime in order to distinguish it from that in the initial system (if needed).

B. The Group Parameters

In this subsection, we shall express all the group parameters in terms of 4-momenta of the external particles (or their invariant variables). For convenience, we introduce the following notation:

$$G\begin{bmatrix} x\ y\ \cdots\ w \\ \\ x'\ y'\ \cdots\ w' \end{bmatrix} \equiv \det\begin{vmatrix} k_{xx'} & k_{xy'} & \cdots & k_{xw'} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ k_{wx'} & \cdots & \cdots & k_{ww'} \end{vmatrix}, \tag{8a}$$

$$G(x,y,\ldots,w) \equiv G\begin{bmatrix} x\ y\ \cdots\ w \\ \\ x\ y\ \cdots\ w \end{bmatrix}, \tag{8b}$$

$$k_{xy} \equiv 2p_x\cdot p_y, \quad k_x \equiv k_{xx}, \quad \text{and} \quad p_{ij} = \sum_{l=i}^{j} p_l, \tag{8c}$$

where p_x, etc., can be any linear combinations of the 4-momenta p_i's. With these notations, we may write $2s_i = k_{(0i)}$ and $2m_j^2 = k_j$, for instance. Note that $k_{(0i)}$ $(=2p_{(0i)}^2 = 2Q_i^2)$ and k_{0i} $(=2p_0\cdot p_i)$ are different.

In a rest frame $[w]$ of p_w, in which the 3-momentum $\vec{p}_y$ is along the "negative" z axis, and $\vec{p}_z$ in the xy plane, i.e., $[w] \equiv [p_w; p_y, p_z]$, the group parameters of the 4-momentum p_x are denoted by α_x, θ_x, and ϕ_x. By a straightforward calculation, one can express α_x, θ_x, and ϕ_x in terms of the invariant variables:

$$\cosh\alpha_x = \frac{k_{xw}}{(k_x k_w)^{1/2}}, \quad \sinh\alpha_x = \left[-\frac{G(x,w)}{k_x k_w}\right]^{1/2}, \tag{9a}$$

$$\cos\theta_x = G\begin{bmatrix} x\ w \\ \\ y\ w \end{bmatrix}[G(x,w)G(y,w)]^{-1/2}, \tag{9b}$$

$$\sin\theta_x = [k_w G(x,y,w)/G(x,w)G(y,w)]^{1/2}, \tag{9c}$$

$$\cos\phi_x = G\begin{bmatrix} x\ y\ w \\ \\ z\ y\ w \end{bmatrix}[G(x,y,w)G(z,y,w)]^{-1/2}, \tag{9d}$$

and

$$\sin\phi_x = \epsilon_x^w[G(y,w)G(x,z,y,w)/G(x,y,w)G(z,y,w)]^{1/2}, \tag{9e}$$

where ϵ_x^w equals $+1$ or -1, and is to be determined later.

By convention, the ranges of the group parameters in the physical region are defined[22] as follows:

$$0 \le \alpha_x < \infty, \quad 0 \le \theta_x \le \pi, \quad \text{and} \quad -\pi \le \phi_x \le \pi. \tag{10}$$

Hence, $\sinh\alpha_x$ $(\sin\theta_x)$ is always positive in this region, and thus determined if $\cosh\alpha_x$ $(\cos\theta_x)$ is given. However, $\sin\phi_x$ can be negative, and thus its sign needs to be determined. By direct calculation, one can easily show that

$$\epsilon_i^w\epsilon_j^w = -G\begin{bmatrix} i\ z\ y\ w \\ \\ j\ z\ y\ w \end{bmatrix}. \tag{11}$$

Therefore, only one of ϵ_i^w needs to be determined; it cannot be determined merely by the invariant variables. We can do this in two ways: One is just to assign one of ϵ_i^w to be positive (this is equivalent to the choice of the left- or right-handed coordinate system), while the other is to specify one of ϵ_i^w by the orientation of its 3-momentum in the frame $[w]$. Once the sign of ϵ_i^w in one frame is chosen, then it is determined in any other frame. From now on, we shall not specify the signs for $\sin\phi_i$ since they can be determined as we have stated.

The determinants $G(x,w)$ and $G(x,y,z,w)$ are negative, while $G(x,y,w)$ is positive, if p_x, p_y, p_z, and p_w are real. By convention, we take positive values for the square roots in (9a)–(9e). Since the right-hand sides of Eqs. (9a)–(9e) are functions of invariant variables, one may use 4-momenta of the external particle in any Lorentz frame. We shall show later that all parameters such as those of the little groups, Wigner relations, and crossing matrices can be written in the forms as in Eqs. (9a)–(9e). For convenience, we shall abbreviate the expressions in (9a)–(9e) by

$$[\alpha_x, \theta_x, \phi_x] = [p_x\colon p_w; p_y, p_z]. \tag{9$'$}$$

In the frame $[i,l]$, for example, one has

$$p_w = p_{0i}, \quad p_y = p_i, \quad \text{and } p_z = p_{i+1}. \tag{12}$$

The group parameters of $\alpha_j^{(i,\,l)}$, $\theta_j^{(i,\,l)}$, and $\phi_j^{(i,\,l)}$ for 4-momentum p_j can be obtained by replacing p_x, p_w, p_y, and p_z with p_j, p_{0i}, p_i, and p_{i+1}, respectively, in Eqs. (9a)–(9e); from (12), one may write symbolically

$$[\alpha_j^{(i,\,l)},\, \theta_j^{(i,\,l)},\, \phi_j^{(i,\,l)}] = [p_j\colon p_{0i};\, p_i,\, p_{i+1}]. \tag{13a}$$

One can easily see that

$$\theta_i^{(i,\,l)} = \pi, \quad \phi_i^{(i,\,l)} = 0, \text{ and } \phi_{i+1}^{(i,\,l)} = 0, \tag{13b}$$

as they should. Note that the polar angle $\theta_j^{(i,\,l)}$ differs from the scattering angle between the particles p_i and p_j by a negative sign, since the latter is defined as the angle between two 4-momenta, one in the initial and the other in the final system, while the former is defined as the angle between two 4-momenta in the initial system or in the final system. In the frame $[i, u]$, the group parameters of 4-momentum p_j can be expressed in terms of invariant variables:

$$[\alpha_j^{(i,\,u)},\, \theta_j^{(i,\,u)},\, \phi_j^{(i,\,u)}] = [p_j\colon p_{0i};\, p_{i+1},\, p_{i+2}]. \tag{14}$$

From (6c), (6e), (14), and (9a)–(9e), one can easily obtain the group parameters of u_i and b_i:

$$\cosh\delta_i = k_{(0i)(0i+1)}/[k_{(0i)}k_{(0i+1)}]^{1/2}, \tag{15a}$$

$$\sinh\delta_i = [-G((0i),\,(0i+1))/k_{(0i)}k_{(0i+1)}]^{1/2}, \tag{15b}$$

$$\cos\beta_i = -G\begin{bmatrix} i+1 & (0i) \\ i & (0i) \end{bmatrix}[G(i,\,(0i))G(i+1,\,(0i))]^{-1/2}, \tag{15c}$$

$$\sin\beta_i = [k_{(0i)}G(i+1,\,i,\,(0i))/G(i,\,(0i))G(i+1,\,(0i))]^{1/2}, \tag{15d}$$

$$\cos\gamma_i = -G\begin{bmatrix} i+2 & i+1 & (0i) \\ i & i+1 & (0i) \end{bmatrix}[G(i+1,\,i,\,(0i))G(i+1,\,i+2,\,(0i))]^{-1/2}, \tag{15e}$$

and

$$\sin\gamma_i = \pm[G(i+1,\,(0i))G(i+1,\,i,\,i+2,\,(0i))/G(i+1,\,i,\,(0i))G(i+1,\,i+2,\,(0i))]^{1/2}. \tag{15f}$$

As mentioned before, the sign of the right-hand side of Eq. (15f) is determined if the azimuthal angle of a 4-momentum p_j in one frame is given.

Finally, we quote[23] three useful elementary formulas in the theory of determinants:

$$G\begin{bmatrix} x & y & \cdots & w \\ x' & y' & \cdots & w' \end{bmatrix} = G\begin{bmatrix} x\pm y & y & \cdots & w \\ x' & y'\pm w' & \cdots & w' \end{bmatrix}, \tag{16a}$$

$$G\begin{bmatrix} x & y & z & \cdots & w \\ x' & y' & z' & \cdots & w' \end{bmatrix} G\begin{bmatrix} z & \cdots & w \\ z' & \cdots & w' \end{bmatrix} = G\begin{bmatrix} y & z & \cdots & w \\ y' & z' & \cdots & w' \end{bmatrix} G\begin{bmatrix} x & z & \cdots & w \\ x' & z' & \cdots & w' \end{bmatrix} - G\begin{bmatrix} x & z & \cdots & w \\ y' & z' & \cdots & w' \end{bmatrix} G\begin{bmatrix} y & z & \cdots & w \\ x' & z' & \cdots & w' \end{bmatrix}, \tag{16b}$$

and

$$G\begin{bmatrix} x & y & z & u & \cdots & w \\ x' & y' & z' & u' & \cdots & w' \end{bmatrix} G\begin{bmatrix} u & \cdots & w \\ u' & \cdots & w' \end{bmatrix} = G\begin{bmatrix} x & u & \cdots & w \\ x' & u' & \cdots & w' \end{bmatrix} G\begin{bmatrix} y & z & u & \cdots & w \\ y' & z' & u' & \cdots & w' \end{bmatrix}$$

$$-G\begin{bmatrix} x & u & \cdots & w \\ y' & u' & \cdots & w' \end{bmatrix} G\begin{bmatrix} y & z & u & \cdots & w \\ x' & z' & u' & \cdots & w' \end{bmatrix} + G\begin{bmatrix} x & u & \cdots & w \\ z' & u' & \cdots & w' \end{bmatrix} G\begin{bmatrix} y & z & u & \cdots & w \\ x' & y' & u' & \cdots & w' \end{bmatrix}. \tag{16c}$$

On the other hand, because of the dimensionality of Minkowski space, any five 4-momenta are linearly dependent; in mathematical language, one has

$$G(x, y, z, w, v)=0. \tag{17}$$

We have used these relations before and will use them frequently later.

C. Phase Space

Byers and Yang[24] have made an extensive study of the physical regions and the phase-space volume element for the multiparticle system. We are interested rather in expressing the latter in terms of the parameters of the groups u_i and b_i. The phase-space volume element $d\Phi_{N+1}$ for an $(n+1)$-particle system of total 4-momentum Q_n is defined by

$$d\Phi_{n+1}=\prod_{i=0}^{n}[\delta^+(p_i^2-m_i^2)d^4p_i]\delta\left(\sum_{i=0}^{n}p_i-Q_n\right), \tag{18}$$

with

$$\delta^+(p_i^2-m_i^2)=\theta(p_{it}-m_i)\delta(p_i^2-m_i^2), \tag{18$'$}$$

where $\theta(p_{it}-m_i)$ is a step function, and p_{it} is the time component of the 4-momentum p_i. From Eqs. (5b) and (5c), one may make a change of variables from p_0 and p_1 to Q_0 and Q_1, to obtain

$$d\Phi_2=\delta^+(Q_0^2-m_0^2)\delta^+((Q_1-Q_0)^2-m_1^2)$$
$$\times d^4Q_0d^4Q_1\delta(p_0+p_1-Q_1). \tag{19}$$

Since each of the factors in Eq. (19) is Lorentz-invariant, one can calculate each of them separately in some convenient frame. Expressing d^4Q_0 in terms of its group parameters in a rest frame [1] of Q_1, one obtains

$$d\Phi_2=(q_1/4w_1)du_1d^4Q_1\,\theta(\Delta(w_1,m_0,m_1))$$
$$\times\theta(\Delta(w_1,m_1,m_0))\,\delta(p_0+p_1-Q_1), \tag{20}$$

where the triangle function $\Delta(a, b, c)$ is defined by

$$\Delta(a,b,c)=(a-b+c)(a-b-c)/2a. \tag{20$'$}$$

In general, q_i and w_i are, respectively, the relative momentum of p_i and the invariant mass of Q_i in a rest frame $[i]$ of Q_i; they can be expressed in terms of the invariant variables:

$$w_i^2=s_i=\tfrac{1}{2}k_{(0i)},\quad w_0\equiv m_0, \tag{21a}$$

and

$$q_i^2=-G(i,(0i))/2k_{(0i)}. \tag{21b}$$

The rotation u_i specifies the orientation of the 3-momentum $\vec{Q}_{i-1}$ in the frame $[i]$; its solid angle du_i is given by

$$du_i=\sin\beta_i d\beta_i\,d\gamma_i. \tag{22}$$

We have to emphasize that, if one chooses the frame $[i]$ as the frame $[i, u]$, the group element u_i in Eq. (22) equals u_i in Eq. (6a); in general, they are different, even though we have used the same notation for both of them. The freedom of choosing these reference frames will be useful later.

One may further change p_2 to Q_2, express Q_1 in terms of its group parameters in the frame [2], and obtain

$$d\Phi_3=(q_1q_2/16w_1w_2)du_1du_2ds_1d^4Q_2\delta(p_0+p_1+p_2-Q_2)$$
$$\times\theta(\Delta(w_1,m_0,m_1))\theta(\Delta(w_1,m_1,w_0))\theta(\Delta(w_2,m_2,w_1)). \tag{23}$$

It is obvious that one can continue in this manner. Finally, one obtains the $(N+1)$-particle phase-space volume element

$$d\Phi_{n+1}=\rho(n+1)\Theta(n+1)\prod_{i=1}^{n}du_i\prod_{i=1}^{n-1}ds_i\delta\left(\sum_{i=0}^{n}p_i-Q_n\right)d^4Q_n, \tag{24}$$

where

$$\rho(n+1)\equiv\prod_{i=1}^{n}\frac{q_i}{4w_i}, \tag{24$'$}$$

and

$$\Theta(n+1)\equiv\theta(\Delta(w_1,m_0,m_1))\prod_{i=1}^{n}\theta(\Delta(w_i,m_i,w_{i-1})). \tag{24$''$}$$

Note that the δ functions in Eqs. (20), (23), and (24) are important, since they may provide constraints on the ranges of integrations over ds_i.

Should four particles p_0, p_1, p_2, and p_3 be coupled in the scheme shown in Fig. 2, one would define the submomenta $\tilde{Q}_i$ such that

$$\tilde{Q}_0\equiv p_0,\quad \tilde{Q}_1\equiv\tilde{Q}_0+p_1, \tag{25a}$$

$$\tilde{Q}_0'\equiv p_2,\quad \tilde{Q}_2=\tilde{Q}_0'+p_3, \tag{25b}$$

and

$$\tilde{Q}_3=\tilde{Q}_2+\tilde{Q}_1=p_0+p_1+p_2+p_3. \tag{25c}$$

From Eqs. (25a)–(25c), one may change, in the phase-space volume element, the variables p_0, p_1, p_2, and p_3 to the variables $\tilde{Q}_0$, $\tilde{Q}_3$, $\tilde{Q}_0'$, and $\tilde{Q}_2$, respectively, and then express $d\tilde{Q}_0$, $d\tilde{Q}_0'$, and $d\tilde{Q}_2$ in terms of the group parameters in the rest frames of $\tilde{Q}_1$, $\tilde{Q}_2$, and $\tilde{Q}_3$ separately. Following the steps leading to Eq. (24), one obtains

$$d\tilde{\Phi}_4=\tilde{\rho}(4)\tilde{\Theta}(4)\prod_{i=1}^{3}d\tilde{u}_i\prod_{i=1}^{2}d\tilde{s}_i\,\delta(p_0+p_1+p_2+p_3-\tilde{Q}_3)d^4\tilde{Q}_3, \tag{26a}$$

where

$$\tilde{\rho}(4)=\prod_{i=1}^{3}\frac{\tilde{q}_i}{4\tilde{w}_i},\quad \tilde{w}_i{}^2\equiv\tilde{s}_i\equiv\tilde{Q}_i{}^2,\quad i=1,2,3 \tag{26b}$$

and

$$\tilde{\Theta}(4)\equiv\theta(\Delta(\tilde{w}_1,m_0,m_1))\theta(\Delta(\tilde{w}_1,m_1,m_0))\times\theta(\Delta(\tilde{w}_2,m_2,m_3))\theta(\Delta(\tilde{w}_2,m_3,m_2)). \tag{26c}$$

The relative 3-momentum $\tilde{q}_1$ and the solid angle $d\tilde{u}_1$ are defined in the rest frame of $\tilde{Q}_1$; similarly for $\tilde{q}_2$, $\tilde{q}_3$, $d\tilde{u}_2$, and $d\tilde{u}_3$. One can easily find, by comparing Eq. (26a) with Eq. (24), that they have the same form except that (a) the step functions $\Theta(n+1)$ may be different in order to fit individual coupling schemes, and (b) the quantities such as q_i, w_i, Q_i and u_i have different interpretations; each refers to its own coupling scheme.

Calculating the phase-space volume element in various coupling schemes (such as, for example, those shown in Fig. 4), one can convince himself that the phase-space volume element in Eq. (24) is applicable to any scheme, if the quantities q_i, w_i, and s_i are properly interpreted, and if some of the step functions are properly replaced. This extension of the range of validity for Eq. (24) is important for calculating the recoupling coefficient[15] (and thus the partial-wave recoupling condition) between two different coupling schemes, and also for deriving the partial-wave unitarity relations of various coupling schemes.

III. MULTIPARTICLE ANGULAR MOMENTUM STATE

An angular momentum state (a.m. state) can be projected out from either canonical states[25] or helicity states.[15] In order to avoid Clebsch-Gordan coefficients in a partial-wave expansion, we shall deal with helicity states. The 2-particle and 3-particle helicity a.m. states have been worked out by JW.[15] As a preparation for more general cases, we shall briefly review their work.

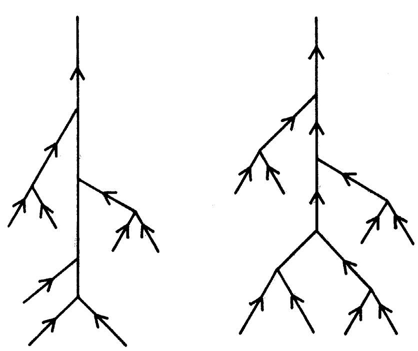

FIG. 4. Some coupling schemes for the multiparticle angular momentum states.

A. Two- and Three-Particle a.m. States

The helicity state of a particle of spin σ, helicity λ, mass m, and 4-momentum p is defined[15] by

$$|p\sigma\lambda\rangle=L(p)|\overline{p}\sigma\lambda\rangle, \tag{27}$$

where the rest state $|\overline{p}\sigma\lambda\rangle$ transforms as

$$g|\overline{p}\sigma\lambda\rangle=\sum_{\nu}D^{\sigma}_{\nu\lambda}(g)|\overline{p}\sigma\nu\rangle \tag{27$'$}$$

under the rotation g. The particle boost $L(p)$ was defined in Eqs. (4a)–(4c) and (2). The helicity state transforms under the Lorentz transformation Λ as

$$\Lambda|p\sigma\lambda\rangle=\sum_{\nu}D^{\sigma}_{\nu\lambda}[\Lambda,p]|\Lambda p\sigma\nu\rangle, \tag{28a}$$

where $D^{\sigma}_{\nu\lambda}[\Lambda,p]$ is a matrix element of the Wigner rotation $[\Lambda,p]$:

$$[\Lambda,p]\equiv L^{-1}(\Lambda p)\Lambda L(p). \tag{28b}$$

We choose the normalization of a helicity state with

$$\delta(p^2-m^2)\langle p'\sigma'\lambda'|p\sigma\lambda\rangle=\delta^4(p-p')\delta_{\sigma'\sigma}\delta_{\lambda\lambda'}. \tag{28c}$$

Let us first consider the standard coupling scheme shown in Fig. 3. In the frame [1], one can project out from the two-particle plane-wave state an a.m. state, which plays the same role as $|\overline{p}\sigma\lambda\rangle$ does for a one-particle state:

$$|Q_1^{(1)}j_1m_1\lambda_1\lambda_0\rangle=N_1[\rho(2)]^{1/2}\int du_1 D^{j_1}_{\mu_1 m_1}(u_1{}^{-1})u_1\times|p_0^{(1,l)}\sigma_0\lambda_0;p_1^{(1,l)}\sigma_1\lambda_1\rangle, \tag{29}$$

where

$$|p_0\sigma_0\lambda_0;p_1\sigma_1\lambda_1\rangle\equiv|p_0\sigma_0\lambda_0\rangle\otimes|p_1\sigma_1\lambda_1\rangle.$$

Our definition of the two-particle plane-wave state differs[15] from JW's by a phase factor $(-1)^{\sigma_1-\lambda_1}$. The first particle in the 2-particle state on the right-hand side of Eq. (29) has its momentum along the positive z axis. We still have freedom in the choice of the reference frame. Here, we have chosen the frame $[1,l]$. In general, the normalization constant N_i and the helicity index μ_i are defined by

$$N_i\equiv[(2j_i+1)/4\pi]^{1/2}\text{ and }\mu_i\equiv m_{i-1}-\lambda_i, \tag{30}$$

where $m_0=\lambda_0$, and m_i $(i>0)$ is to be defined later.

As for the one-particle states, we can define a helicity state of 4-momentum Q_1 corresponding to the a.m. state in Eq. (29):

$$|Qj_1m_1\lambda_1\lambda_0\rangle\equiv L(Q_1)|Q_1^{(1)}j_1m_1\lambda_1\lambda_0\rangle. \tag{31}$$

Substituting Eq. (29) into Eq. (31), and using Eq. (28a) and Eq. (19), one obtains

$$|Qj_1m_1\lambda_1\lambda_0\rangle = N_1[\rho(2)]^{1/2}\int d\Phi_2 D^{j_1}_{\mu_1 m_1}(u_1^{-1})\sum_{\nu_i} D^{\sigma_1}_{\nu_1\lambda_1}\times[L_{11}, p_1(1,l)]D^{\sigma_0}_{\nu_0\lambda_0}[L_{11}, p_0(1,l)]\times|p_0\sigma_0\nu_0; p_1\sigma_1\nu_1\rangle, \quad (32)$$

where the summation is over all ν_i [i.e., ν_0 and ν_1 in Eq. (32)], and

$$L_{11} \equiv L(Q_1)u_1, \quad (32')$$

$$p_i = L_{11}p_i^{(1,l)}, \quad i=0,1. \quad (32'')$$

The particle boost $L(Q_1)$ is determined by Q_1, while the group parameters of the rotation u_1 are dummy variables. Note that for convenience, we have replaced $p_i^{(1,l)}$ by $p_i(1,l)$ in the Wigner rotations. From Eqs. (28c) and (32), one can easily obtain

$$\langle p_0\sigma_0\nu_0; p_1\sigma_1\nu_1|Q_1j_1m_1\lambda_1\lambda_0\rangle = [\rho(2)]^{-1/2}N_1\delta(p_0+p_1-Q_1)D^{j_1}_{\mu_1 m_1}(u_1^{-1})\times\prod_{i=0}^{1} D^{\sigma_i}_{\nu_i\lambda_i}[L_{11}, p_i(1,l)] \quad (33a)$$

and

$$\langle Q_1'j_1'm_1'\lambda_1'\lambda_0'|Q_1j_1m_1\lambda_1\lambda_0\rangle = \delta(Q_1-Q_1')\delta_{j_1j_1'}\delta_{m_1m_1'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_0\lambda_0'}. \quad (33b)$$

Given p_0 and p_1, the particle boost $L(Q_1)$ and the rotation u_1 can be determined. In our choice of $[1,l]$ as a reference frame the group parameters of u_1 are given by Eqs. (15c)–(15f), in which p_2 and p_3 may be viewed as constant 4-vectors. Thus all quantities in Eq. (33a) are known [the Wigner rotations in Eq. (33a) can be calculated by Wick's method[15]].

One can easily show that the state $|Q_1j_1m_1\lambda_1\lambda_0\rangle$ transforms under the Lorentz group as the helicity state of a real particle with spin j_1, but it has a continuous mass spectrum. Comparing Eq. (33b) with Eq. (28c), one sees that the normalization of this state differs from that of a real particle. In this sense, we call it the helicity state of a pseudoparticle or a kinematical system.[26]

In the frame [2], one can project out an a.m. state from the pseudo-two-particle state:

$$|Q_2^{(2)}j_2m_2\lambda_2; s_1j_1m_1\lambda_1\lambda_0\rangle = N_2\left(\frac{q_2}{4w_2}\right)^{1/2}\int du_2 D^{j_2}_{\mu_2m_2}(u_2^{-1})u_2\,|Q_1^{(2,l)}j_1m_1\lambda_1\lambda_0; p_2^{(2,l)}\sigma_2\lambda_2\rangle, \quad (34)$$

where

$$|Q_1j_1m_1\lambda_1\lambda_0; p_2\sigma_2\lambda_2\rangle \equiv |Q_1j_1m_1\lambda_1\lambda_0\rangle\otimes|p_2\sigma_2\lambda_2\rangle. \quad (34')$$

We have chosen the frame $[2,l]$ as the reference frame. As in the two-particle case, one can define a helicity state of a pseudoparticle, substitute Eq. (34) into the latter, and then express it in terms of the three-particle plane-wave state:

$$|Q_2j_2m_2\lambda_2; s_1j_1m_1\lambda_1\lambda_0\rangle \equiv L(Q_2)|Q_2^{(2)}j_2m_2\lambda_2; s_1j_1m_1\lambda_1\lambda_0\rangle = N_1N_2[\rho(2)]^{1/2}\int du_1du_2D^{j_2}_{\mu_2m_2}(u_2^{-1})D^{j_1}_{\mu_1m_1}(u_1^{-1})\sum_{\nu_i}\prod_{i=0}^{2}D^{\sigma_i}_{\nu_i\lambda_i}[L_{2i}, p_i(i,l)]|p_0\sigma_0\nu_0; p_1\sigma_1\nu_1; p_2\sigma_2\nu_2\rangle, \quad (35)$$

where

$$L_{22} = L(Q_2)u_2, \quad L_{20} \equiv L_{21} = L(Q_2)u_2L(Q_1^{(2)})u_1, \quad (35')$$

and

$$p_i = L_{2i}p_i^{(i,l)}, \quad p_0^{(0,l)} \equiv p_0^{(1,l)}. \quad (35'')$$

Once again, $L(Q_2)$ is determined by Q_2, and by construction $L(Q_1^{(2)}) = b_1$; the parameter δ_1 of b_1 can be expressed in terms of s_1 and s_2 [see Eqs. (15a) and (15b)]. The group parameters of u_1 and u_2 are dummy variables.

Using the relations

$$\int du\, D^{j*}_{\mu m}(u^{-1})D^{j'}_{\mu m'}(u^{-1}) = 4\pi\delta_{jj'}\delta_{mm'}/(2j+1) \quad (36a)$$

and

$$\sum_\nu D^{\sigma}_{\nu\lambda}[L,p]D^{\sigma*}_{\nu\lambda'}[L,p] = \delta_{\lambda\lambda'}, \quad (36b)$$

and replacing du_1du_2 in Eq. (35) by $d\Phi_3$ in Eq. (23), one can easily calculate the transformation matrix and the orthogonality relation:

$$\langle p_0\sigma_0\nu_0; p_1\sigma_1\nu_1; p_2\sigma_2\nu_2|Q_2j_2m_2\lambda_2; s_1j_1m_1\lambda_1\lambda_0\rangle$$
$$= N_1N_2[\rho(3)]^{-1/2}\delta(p_0+p_1+p_2-Q_2)\delta((p_0+p_1)^2-s_1)D^{j_2}_{\mu_2m_2}(u_2^{-1})D^{j_1}_{\mu_1m_1}(u_1^{-1})\prod_{i=0}^{2}D^{\sigma_i}_{\nu_i\lambda_i}[L_{2i}, p_i(i,l)]\Theta(3) \tag{37a}$$

and

$$\langle Q_2'j_2'm_2'\lambda_2'; s_1'j_1'm_1'\lambda_1'\lambda_0'|Q_2j_2m_2\lambda_2; s_1j_1m_1\lambda_1\lambda_0\rangle = \delta(Q_2-Q_2')\delta(s_1-s_1')\prod_{i=0}^{2}(\delta_{j_ij_i'}\delta_{m_im_i'})\prod_{i=0}^{2}\delta_{\lambda_i\lambda_i'}\Theta(3). \tag{37b}$$

All quantities on the right-hand side of Eq. (37a) can be determined from the 4-momenta p_0, p_1, and p_2. In our choice of the reference frames, the 4-momenta p_3 and p_4 in the expressions for group parameters of u_1 and u_2 may be considered to be constant. The orthogonality relation for the three-particle a.m. state is known.[15]

B. $(N+1)$-Particle a.m. State

The $(N+1)$-particle a.m. state can be defined step by step beginning from the two-particle a.m. state, or by mathematical induction. Let us project out in the frame $[n]$ the $(N+1)$-particle a.m. state from the pseudo-two-particle plane-wave state, including the particle p_n and the N-particle a.m. state:

$$|Q_n^{(n)}j_nm_n\lambda_n; s_{n-1}j_{n-1}m_{n-1}\lambda_{n-1}; \cdots; s_1j_1m_1\lambda_1\lambda_0\rangle$$
$$= N_n\left(\frac{q_n}{4w_n}\right)^{1/2}\int du_nD^{j_n}_{\mu_nm_n}(u_n^{-1})u_n|Q_{n-1}^{(n,l)}j_{n-1}m_{n-1}\lambda_{n-1}; \cdots; s_1j_1m_1\lambda_1\lambda_0; p_n^{(n,l)}\sigma_n\lambda_n\rangle. \tag{38}$$

Above we have chosen $[n,l]$ as the reference frame. Once again, one may define a pseudoparticle state, substitute Eq. (38) into the latter, and replace the N-particle a.m. state by the $(N-1)$-particle a.m. state, and so on. One then obtains

$$|Q_nj_nm_n\lambda_n; \cdots; s_1j_1m_1\lambda_1\lambda_0\rangle \equiv L(Q_n)|Q_n^{(n)}j_nm_n\lambda_n;\cdots; s_1j_1m_1\lambda_1\lambda_0\rangle$$
$$= N(n+1)[\rho(n+1)]^{1/2}\sum_{\nu_i}\int\prod_{i=1}^{n}[du_iD^{j_i}_{\mu_im_i}(u_i^{-1})]\prod_{i=0}^{n}D^{\sigma_i}_{\nu_i\lambda_i}[L_{ni}, p_i(i,l)]|p_0\sigma_0\nu_0; \cdots; p_n\sigma_n\nu_n\rangle, \tag{39}$$

where

$$N(n+1) \equiv \prod_{i=1}^{n}N_i \tag{39$'$}$$

and

$$P_i \equiv L_{ni}p_i^{(i,l)}, \quad L_{n0} \equiv L_{n1}, \quad p_0^{(0,l)} \equiv p_0^{(1,l)}, \tag{39$''$}$$

and

$$L_{ni} = L(Q_n)u_nb_{n-1}\cdots b_iu_i, \quad i \geq 1. \tag{39$'''$}$$

It is interesting to note that no matter what the group parameters of u_i's are, the 4-momentum Q_n is kept unchanged. This property will be important in performing unitarity integrals later.

From Eqs. (24), (28c), (36a), (36b), and (39) we can calculate the transformation matrix and the orthogonality relation:

$$\langle p_0\sigma_0\nu_0; \cdots; p_n\sigma_n\nu_n|Q_nj_nm_n\lambda_n; \cdots; s_1j_1m_1\lambda_1\lambda_0\rangle$$
$$= N(n+1)[\rho(n+1)]^{-1/2}\delta\left(\sum_{i=0}^{n}p_i - Q_n\right)\prod_{i=1}^{n-1}\delta\left(\left[\sum_{j=0}^{i}p_j\right]^2 - s_i\right)\prod_{i=1}^{n}D^{j_i}_{\mu_im_i}(u_i^{-1})\prod_{i=0}^{n}D^{\sigma_i}_{\nu_i\lambda_i}[L_{ni}, p_i(i,l)]\Theta(n+1), \tag{40a}$$

and

$$\langle Q_n' j_n' m_n' \lambda_n'; \cdots; s_1' j_1' m_1' \lambda_1' \lambda_0' | Q_n j_n m_n \lambda_n; \cdots; s_1 j_1 m_1 \lambda_1 \lambda_0 \rangle = \delta(Q_n - Q_n') \prod_{i=1}^{n-1} \delta(s_i - s_i') \prod_{i=1}^{n} [\delta_{j_i j_i'} \delta_{m_i m_i'}] \prod_{i=0}^{n} \delta_{\lambda_i \lambda_i'} \Theta(n+1). \tag{40b}$$

From Eqs. (7′) and (39‴), the Lorentz transformation L_{ni} is related to $b_{n-1} u_y(\beta_{n-1}) \overline{L}_{n-1i}$ as follows:

$$L_{ni} = [L(Q_n) u_n u_z(\gamma_{n-1})][b_{n-1} u_y(\beta_{n-1}) \overline{L}_{n-1\,i}]. \tag{41}$$

The group parameters of the last factor in Eq. (41) can be expressed in terms of products of the 4-momenta p_i through Eqs. (15a)–(15f). The Lorentz transformation $L(Q_n) u_n u_z(\gamma_{n-1})$ carries the frame $b_{n-1} u_y(\beta_{n-1})[n-1, l]$ to the working frame in which the 4-momenta p_i are defined, and thus can be determined by the 4-momenta p_i. The group parameters of u_n and $u_z(\gamma_{n-1})$ are given by Eqs. (15c)–(15f) also, if one identifies p_{n+1} and p_{n+2} properly. For example, in an $(N+1)$-to-$(N'+1)$ process with the coupling scheme shown in Fig. 3, one must take $-p_{n'}'$ and $-p_{n'-1}'$ as p_{n+1} and p_{n+2}, respectively. The particle boost $L(Q_n)$ is determined directly by the total 4-momentum of the initial system. From Eq. (40b), one can easily obtain the completeness relation of the pseudoparticle states:

$$\int d^4 Q_n \prod_{i=1}^{n-1} ds_i \Theta(n+1) \sum_{j_i m_i \lambda_i} |Q_n j_n m_n \lambda_n; \cdots \rangle \langle Q_n j_n m_n \lambda_n; \cdots | = 1. \tag{42}$$

The ranges of integrations over ds_i are determined by the step functions $\Theta(n+1)$, which in general depend on the coupling schemes.

C. Other Coupling Schemes

We shall show that Eqs. (40a) and (40b) can be applied to other coupling schemes. For example, we consider a four-particle system with a coupling scheme shown in Fig. 2. One can project out a two-particle a.m. state, and define a pseudoparticle state for the particles p_0 and p_1 (p_2 and p_3). Then one may couple these two pseudoparticle states in a rest frame $[3\sim]$ of $\tilde{Q}_3$, in which the 3-momentum $\vec{\tilde{Q}}_2$ is along the negative z axis, and obtain a four-particle a.m. state:

$$\begin{aligned} |\tilde{Q}_3 j_3 m_3; \tilde{s}_2 j_2 m_2 \lambda_3 \lambda_2; \tilde{s}_1 j_1 m_1 \lambda_1 \lambda_0 \rangle &\equiv L(Q_3) |\tilde{Q}_3^{(3\sim)} j_3 m_3; \cdots \rangle \\ &= N(4)[\tilde{\rho}(4)]^{1/2} \int \prod_{i=1}^{3} [d\tilde{u}_i D^{j_i}_{\tilde{\mu}_i m_i}(\tilde{u}_i^{-1})] \\ &\quad \times \sum_{\nu_i} \prod_{i=0}^{3} D^{\sigma_i}_{\nu_i \lambda_i} [\tilde{L}_{3i}, p_i(i, l\sim)] |p_0 \sigma_0 \nu_0; p_1 \sigma_1 \nu_1; p_2 \sigma_2 \nu_2; p_3 \sigma_3 \nu_3 \rangle, \end{aligned} \tag{43}$$

where

$$\tilde{\mu}_1 = \lambda_0 - \lambda_1, \quad \tilde{\mu}_2 = \lambda_2 - \lambda_3, \quad \tilde{\mu}_3 = m_1 - m_2, \tag{43′}$$

and

$$p_i = \tilde{L}_{3i} p_i^{(i, l\sim)}, \quad p_0^{(0, l\sim)} \equiv p_0^{(1, l\sim)}, \quad p_3^{(3, l\sim)} \equiv p_3^{(2, l\sim)}. \tag{43″}$$

The frames $[i, l\sim]$, $i = 1, 2, 3$, are defined to be

$$[1, l\sim] \equiv [\tilde{Q}_1; p_1, \tilde{Q}_2], \tag{43‴}$$

$$[2, l\sim] \equiv [\tilde{Q}_2; p_3, \tilde{Q}_1], \tag{43IV}$$

and

$$[3, l\sim] \equiv [\tilde{Q}_3; \tilde{Q}_2, p_4], \tag{43V}$$

where the 4-momentum p_4 may be considered as free parameters in the four-particle system. With these definitions, one must impose a consistency condition on $[1, l\sim]$ and $[2, l\sim]$:

$$\tilde{b}_1 \tilde{u}_1 [1, l\sim] = [3, l\sim] = \tilde{b}_2 \tilde{u}_2 [2, l\sim]. \tag{44}$$

One can easily show that such a constraint can be satisfied. Hence, the Lorentz transformations $\tilde{L}_{3i}$'s in Eqs. (43) and (43″) are given by

$$\tilde{L}_{31} = L(Q_3)\tilde{u}_3\tilde{b}_1\tilde{u}_1, \quad \tilde{L}_{30} \equiv \tilde{L}_{31} \tag{45a}$$

and

$$\tilde{L}_{32} = L(Q_3)\tilde{u}_3\tilde{b}_2\tilde{u}_2, \quad \tilde{L}_{33} \equiv \tilde{L}_{32}. \tag{45b}$$

We shall explicitly express the group parameters of $\tilde{u}_1$, $\tilde{u}_2$, and $\tilde{u}_3$ in terms of the invariant variables later. As in the case of the standard coupling scheme (see Fig. 3), we have already made a specific choice of the frames $[i, l\sim]$; our choice will be seen to be the most convenient and suitable for our purpose.

From Eqs. (26a) and (43), one can easily obtain the transformation matrix and the orthogonality relation:

$$\langle p_0\sigma_0\nu_0; p_1\sigma_1\nu_1; p_2\sigma_2\nu_2; p_3\sigma_3\nu_3 | \tilde{Q}_3 j_3 m_3; \bar{s}_2 j_2 m_2 \lambda_3\lambda_2; \bar{s}_1 j_1 m_1 \lambda_1 \lambda_0\rangle$$

$$= N(4)[\tilde{\rho}(4)]^{-1/2}\delta\left(\sum_{i=0}^{3} p_i - \tilde{Q}_3\right)\delta((p_0+p_1)^2 - \bar{s}_1)\delta((p_2+p_3)^2 - \bar{s}_2)\tilde{\Theta}(4)\prod_{i=0}^{3} D^{\sigma_i}_{\nu_i\lambda_i}[\tilde{L}_{3i}, p_i(i, l\sim)]\prod_{i=1}^{3} D^{j_i}_{\mu_i m_i}(u_i^{-1}) \tag{46a}$$

and

$$\langle \tilde{Q}_3' j_3' m_3'; \bar{s}_2' j_2' m_2' \lambda_3'\lambda_2'; \bar{s}_1' j_1' m_1' \lambda_1'\lambda_0' | \tilde{Q}_3 j_3 m_3; \bar{s}_2 j_2 m_2 \lambda_3\lambda_2; \bar{s}_1 j_1 m_1 \lambda_1\lambda_0\rangle$$

$$= \delta(\tilde{Q}_3 - \tilde{Q}_3')\prod_{i=1}^{2}\delta(\bar{s}_i - \bar{s}_i')\prod_{i=1}^{3}(\delta_{j_i j_i'}\delta_{m_i m_i'})\prod_{i=0}^{3}\delta_{\lambda_i\lambda_i'}\tilde{\Theta}(4). \tag{46b}$$

The particle boost $L(\tilde{Q}_3)$ can be determined by the total 4-momentum $\sum p_i$ of the four-particle system. From the definitions of $[i, l\sim]$ and Eqs. (45a), (45b), and (9a)–(9e), one can calculate the group parameters $\tilde{\delta}_i$, $\tilde{\beta}_i$, and $\tilde{\gamma}_i$ of $\tilde{b}_i$ and $\tilde{u}_i$:

$$\cosh\tilde{\delta}_1 = k_{(01)(03)}/[k_{(01)}k_{(03)}]^{1/2}, \tag{47a}$$

$$\cosh\tilde{\delta}_2 = k_{(23)(03)}/[k_{(23)}k_{(03)}]^{1/2}, \tag{47b}$$

$$\cos\tilde{\beta}_1 = G\begin{bmatrix} 1 & (01) \\ (23) & (01)\end{bmatrix}[G(1,(01))G((23),(01))]^{-1/2}, \tag{47c}$$

$$\cos\tilde{\beta}_2 = G\begin{bmatrix} 3 & (23) \\ (01) & (23)\end{bmatrix}[G(3,(23))G((01),(23))]^{-1/2}, \tag{47d}$$

$$\cos\tilde{\beta}_3 = G\begin{bmatrix} (23) & (03) \\ 4 & (03)\end{bmatrix}[G((23),(03))G(4,(03))]^{-1/2}, \tag{47e}$$

$$\cos\tilde{\gamma}_1 = G\begin{bmatrix} 1 & (23) & (01) \\ 4 & (23) & (01)\end{bmatrix}[G(1,(23),(01))G(4,(23),(01))]^{-1/2}, \tag{47f}$$

$$\cos\tilde{\gamma}_2 = G\begin{bmatrix} 3 & (01) & (23) \\ 4 & (01) & (23)\end{bmatrix}[G(3,(01),(23))G(4,(01),(23))]^{-1/2}, \tag{47g}$$

and

$$\cos\tilde{\gamma}_3 = G\begin{bmatrix} (23) & 4 & (03) \\ 5 & 4 & (03)\end{bmatrix}[G((23),4,(03))G(5,4,(03))]^{-1/2}. \tag{47h}$$

The expressions for $\sinh\tilde{\delta}_i$ and $\sin\tilde{\beta}_i$ are uniquely determined by Eqs. (47a)–(47e). The expressions for $\sin\tilde{\gamma}_i$ are determined by Eqs. (47f)–(47h) up to a sign,[27] which can be fixed by the prescription stated below Eq. (11). For production processes, the choice of p_4 and p_5 depends on the coupling scheme of the total system. For example, for the coupling scheme shown in Fig. 2 (Fig. 5), the momenta p_4 and p_5 may be taken to be $-p_b$ and $-p_a$ ($-\tilde{Q}_2'$ and $-\tilde{Q}_1'$).

Comparing Eqs. (46a) and (46b) with Eqs. (40a) and (40b), one can easily see that they are formally equal to each other respectively, if the rules (a) and (b) stated below Eq. (26c) are observed (the reinterpreted quantities should include L_{ni}). One may assert that the above statement holds for any coupling scheme. Similarly, the completeness relation (42) can be applied to the case of any coupling scheme if one observes the above-mentioned rules.

IV. PARTIAL-WAVE AMPLITUDES

The canonical partial-wave expansions for general production processes has been discussed by Macfarlane[16]; in particular, the expansion for the process $a+b \to 1+2+3$ has been given explicitly. As is well known, the helicity partial-wave expansion (h.p.w. expansion) avoids complications resulting from Clebsch-Gordan coefficients. The h.p.w. expansions for the processes $a+b \to a'+b'$, and $a+b \to 1+2+3$ has been worked out, respectively, by JW,[15] and others.[28] We shall treat the most general production process $0+1+\cdots+N \to 0'+1'+\cdots+N'$ and suppress all internal quantum numbers such as strangeness, hypercharge, isospin, etc.

A. Process $a+b \to 0+1+\cdots+N$

We shall first derive the h.p.w. expansions for all the processes involved in the unitarity relation for the process $a+b \to a'+b'$; a typical production process is $a+b \to 0+1+\cdots+N$. We shall deal with the h.p.w. amplitudes for arbitrary coupling schemes.

We begin with the standard coupling scheme shown in Fig. 6. Let us define the helicity amplitudes[29] as follows:

$$T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'})\delta(p_a+p_b-p_{a'}-p_{b'}) \equiv \langle p_{a'}\sigma_{a'}\nu_{a'}; p_{b'}\sigma_{b'}\nu_{b'}|T|p_a\sigma_a\nu_a; p_b\sigma_b\nu_b\rangle \tag{48a}$$

and

$$T_{\nu_i\nu_a\nu_b}(s_i, u_i; s_b, u_{bn}; N+1, 2)\delta\left(p_a+p_b-\sum_{i=0}^{n} p_i\right) \equiv \langle p_0\sigma_0\nu_0; \cdots; p_n\sigma_n\nu_n|T|p_a\sigma_a\nu_a; p_b\sigma_b\nu_b\rangle, \tag{48b}$$

where the T-matrix operator T is related to the scattering operator S by

$$S = 1 + iT. \tag{49}$$

In Eq. (48a) [or Eq. (48b)], one has $s_b = s_{b'}$ (or $s_b = s_n$), and $u_{bb'}$ (or u_{bn}) is to be determined. In an arbitrary frame, we define the h.p.w. amplitudes T^{j_b} and $T^{j_i j_b}$ as

$$T^{j_b}_{\lambda_{a'}\lambda_{b'}\lambda_a\lambda_b}(s_b)\delta(Q_b - Q_{b'})\delta_{j_b j_{b'}}\delta_{m_b m_{b'}} \equiv \langle Q_{b'}j_{b'}m_{b'}\lambda_{b'}\lambda_{a'}|T|Q_b j_b m_b \lambda_b \lambda_a\rangle \tag{50a}$$

and

$$T^{j_i j_b}_{\lambda_i m_i \lambda_a \lambda_b}(s_i, s_b; N+1, 2)\delta(Q_b - Q_n)\delta_{j_b j_n}\delta_{m_b m_n} \equiv \langle Q_n j_n m_n \lambda_n; \cdots; s_1 j_1 m_1 \lambda_1 \lambda_0|T|Q_b j_b m_b \lambda_b \lambda_a\rangle. \tag{50b}$$

In defining these h.p.w. amplitudes, we have used conservation of 4-momentum and angular momentum,

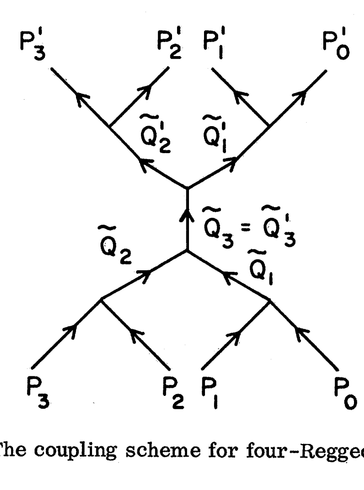

FIG. 5. The coupling scheme for four-Reggeon vertex.

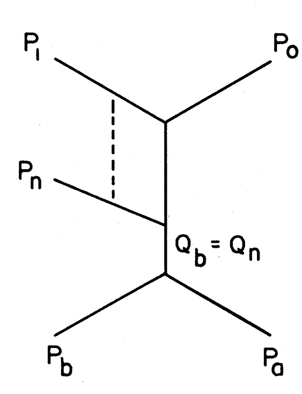

FIG. 6. The standard coupling scheme for the process $a+b \to 0+1+\cdots+N$.

and the Wigner-Eckart theorem.

In an arbitrary frame, we insert two complete sets (42) of the pseudoparticle states into the right-hand side of Eqs. (48a) and (48b): one for each side of the T-matrix operator. We then substitute Eqs. (33a) and (40a) into Eqs. (48a) and (48b), respectively, and after a straightforward calculation obtain

$$T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'}) = [\rho^*(b)\rho(b')]^{-1/2} \sum_{j_b\lambda_i} [N(b)]^2 D^{j_b}_{\mu_{b'}\mu_b}(u_{bb'}{}^{-1}) \prod_{i=a',b'} D^{\sigma_i}_{\nu_i\lambda_i}[L_{b'i}, p_i(i,l)]$$
$$\times \prod_{i=a,b} D^{\sigma_i *}_{\nu_i\lambda_i}[L_{bi}, p_i(i,l)] T^{j_b}_{\lambda_{a'}\lambda_{b'}\lambda_a\lambda_b}(s_b)\Theta(b')\Theta(b) \tag{51a}$$

and

$$T_{\nu_i\nu_a\nu_b}(s_i, u_i; s_b, u_{bn}; N+1, 2) = [\rho^*(b)\rho(n+1)]^{-1/2} \sum_{j_b j_i m_i \lambda_i} N(b)N(n+1) D^{j_b}_{\mu_n\mu_b}(u_{bn}{}^{-1}) \prod_{i=1}^{n-1} D^{j_i}_{\mu_i m_i}(u_i{}^{-1})$$
$$\times \prod_{i=0}^{n} D^{\sigma_i}_{\nu_i\lambda_i}[L_{ni}, p_i(i,l)] \prod_{i=a,b} D^{\sigma_i *}_{\nu_i\lambda_i}[L_{bi}, p_i(i,l)] T^{j_i j_b}_{\lambda_i m_i \lambda_a\lambda_b}(s_i, s_b; N+1, 2)\Theta(n+1)\Theta(b), \tag{51b}$$

where j_n in $N(n+1)$ is equal to j_b. Note that Eqs. (51a) and (51b) contain step functions.[30] All quantities in Eqs. (51a) and (51b), except $u_{bb'}$ and u_{bn}, are defined or given in Sec. II and Sec. III, if one counts the particle in the order (a, b) for the initial system and (a', b') for the final system. The group elements $u_{bb'}$ and u_{bn} are defined by

$$u_{bb'} \equiv u_b{}^{-1}u_{b'} \quad \text{and} \quad u_{bn} \equiv u_b{}^{-1}u_n. \tag{52}$$

One can easily show that u_{bn} ($u_{bb'}$) is a rotation along the y axis, which carries the frame $[n, l]$ ($[b', l]$) to the frame $[b, l]$:

$$u_{bn} \equiv u_y(\beta_{bn}) \quad \text{and} \quad u_{bb'} \equiv u_y(\beta_{bb'}). \tag{53}$$

If one takes $-p_b$ and $-p_a$ as p_{n+1} and p_{n+2}, respectively, the parameter β_{bn} is given by Eqs. (15c) and (15d), and similarly for $\beta_{bb'}$.

It is easy to check that if one considers the parameters β_i and γ_i and the invariant mass s_i as independent variables, one has $3N-1$ independent variables as expected $[3(N+3)-10=3N-1]$. This gives rise to a question: Which is it better to take, the above set or the invariant variables (products among the momenta), as independent variables? The advantage of choosing invariant variables as independent variables is that the Wigner rotations and crossing angles can be neatly expressed in terms of them. However, in case the number of the particles in the total system (including the initial and final systems) is equal to or larger than six, some of the invariant variables are quadratically related and then the analytic structure of the helicity amplitude becomes more complicated. On the other hand, if one takes group parameters and invariant masses, both depending on the choice of coupling schemes, as independent variables, the Wigner rotations in the h.p.w. expansion will be very complicated functions of the latter (in the case where all particles in the total system are spinless, this difficulty disappears). Besides, it is not easy to relate two sets of these independent variables corresponding to different coupling schemes. This difficulty of choosing independent variables may be inherited from the complicated analytic structure of the helicity amplitudes for production processes (this may be true for any other formalism). We shall not discuss it further in this paper.

Finally, one can easily project out the h.p.w. amplitudes from Eqs. (51a) and (51b). One should first remove the Wigner rotations by multiplying the h.p.w. expansion with their inverses and by summing over the intermediate helicities. Then, one can immediately obtain the h.p.w. amplitude by projection method as used in the two-to-two process.

B. Other Amplitudes

For a general process $0+1+\cdots+N \to 0'+1'+\cdots+N'$ with the standard coupling scheme shown in Fig. 3, we can define the helicity amplitude and the h.p.w. amplitude as follows:

$$T_{\nu_{i'}\nu_i}(s_i', u_i'; s_{nn'}, u_{nn'}; s_i, u_i; N'+1, N+1)\,\delta\left(\sum_{i=0}^{n} p_i - \sum_{i=0}^{n'} p_i'\right) \equiv \langle p_0'\sigma_0'\nu_0; \cdots; p_{n'}'\sigma_{n'}'\nu_{n'}' | T | p_0\sigma_0\nu_0; \cdots; p_n\sigma_n\nu_n\rangle \tag{54a}$$

and

$$T^{j_i' j_{nn'} j_i}_{\lambda_i' m_i' \lambda_i m_i}(s_i', s_{nn'}, s_i; N'+1, N+1)\delta_{j_n j_n'}\delta_{m_n m_n'}\delta(Q_n - Q_{n'}') \equiv \langle Q_n' j_n' m_n' \lambda_n'; \cdots; s_1' j_1' m_1' \lambda_1' \lambda_0' | T | Q_n j_n m_n \lambda_n; \cdots; s_1 j_1 m_1 \lambda_1 \lambda_0 \rangle. \tag{54b}$$

Proceeding as in the derivation of Eqs. (51a) and (51b), we obtain

$$\begin{aligned} T_{\nu_i'\nu_i}&(s_i', u_i'; s_{nn'}, u_{nn'}; s_i, u_i; N'+1, N+1) \\ &= [\rho(N'+1)\rho^*(N+1)]^{-1/2} \sum_{j_{nn'} j_i\, m_i \lambda_i j_i' m_i' \lambda_i'} N'(n'+1)N(n+1) \prod_{i=1}^{n'-1} D^{j_i'}_{\mu_i' m_i'}(u_i'^{-1}) D^{j_{nn'}}_{\mu_{nn'} m_{nn'}}(u_{nn'}^{-1}) \prod_{i=1}^{n-1} D^{j_i *}_{\mu_i m_i}(u_i^{-1}) \\ &\quad\times \prod_{i=0}^{n'} D^{\sigma_i'}_{\nu_i' \lambda_i'}[L_{n'i}', p_i'(i,l)'] \prod_{i=0}^{n} D^{\sigma_i *}_{\nu_i \lambda_i}[L_{ni}, p_i(i,l)]\, T^{j_i' j_{nn'} j_i}_{\lambda_i' m_i' \lambda_i m_i}(s_i', s_{nn'}, s_i; N'+1, N+1)\Theta(n'+1)\Theta^*(n+1). \end{aligned} \tag{55}$$

All the quantities in Eq. (55) have been previously defined or given, except for $u_{nn'}$ ($\equiv u_n^{-1}u_{n'}$), which carries the frame $[n', l]'$ to the frame $[n, l]$. The group element $u_{nn'}$ is a pure rotation $u_y(\beta_{nn'})$ around the y axis and its parameter $\beta_{nn'}$ is given by Eqs. (15c) and (15d) if one views $-p_n$ and $-p_{n-1}$ as p_{n+1}' and p_{n+2}'.

Suppose we couple the particles in a scheme shown in Fig. 2 for the process $a+b \to 0+1+2+3$. Proceeding as before, one can obtain the h.p.w. expansion for this coupling scheme. This expansion has exactly the same form as Eq. (15b), if the proper replacements and re-interpretations are made (see the latter part of Sec. III C). The $O(2, 1)$ version of the BCP amplitude for this case has been discussed by Misheloff[19]; he made two different parametrizations for the little groups, depending on the values of $\bar{s}_1$, $\bar{s}_2$, and $\bar{s}_3$ $[\equiv (p_a+p_b)^2]$. The helicity amplitude here has only one parametrization for all s_i's, and can be continued to the physical region of any crossed channel. In general, the transformation matrix between the multiparticle plane-wave state and the pseudoparticle state has the same form for different coupling schemes, and thus so does the h.p.w. expansion. Therefore, the h.p.w. expansions (51a), (51b), and (55) hold for any coupling scheme.

The decay amplitude for the process $a \to 0+1+\cdots+N$ is useful for the calculation of form factors via unitarity relations. We can define the helicity decay amplitude and its h.p.w. amplitude in the same manner as for general processes. One can easily obtain the h.p.w. expansion:

$$\begin{aligned} T_{\nu_i \nu_a}&(s_i, u_i; N+1, 1) \\ &= [\rho(n+1)]^{-1/2} \sum_{j_i m_i \lambda_i} N(n+1) \prod_{i=1}^{n-1} D^{j_i}_{\mu_i m_i}(u_i^{-1}) \prod_{i=0}^{n} D^{\sigma_i}_{\nu_i \lambda_i}[L_{ni}, p_i(i,l)] D^{\sigma_a *}_{\nu_a \lambda_a}[L_{nn}, p_a(n,l)]\, T^{j_i}_{\lambda_i m_i \lambda_a}(s_i; N+1, 1)\Theta(n+1). \end{aligned} \tag{56}$$

Once again, all the quantities in Eq. (56), which depend on the choice of coupling schemes, have previously been given or defined. In this special case, the rotation u_n equals the identity and u_{n-1} is merely a rotation around the y axis. In the standard coupling scheme (as shown in Fig. 3), the frame $[n, l]$ is defined to be $[p_a; p_n, p_{n-1}]$, i.e., $\gamma_{n-1} = 0$. Finally, we mention that the h.p.w. amplitudes in Eqs. (55) and (56) can easily be projected out as in Eqs. (51a) and (51b).

C. Wigner Rotations

The h.p.w. expansions in general have matrix elements of Wigner rotations. We shall express these rotations in terms of the invariant variables.

The matrix element of the Wigner rotations can be decomposed into a product of two parts; for example, for the process $0+1+\cdots+N \to 0'+1'+\cdots+N'$, one has

$$D^{\sigma_i}_{\nu_i \lambda_i}[L_{ni}, p_i(i,l)] = \sum_{\lambda} D^{\sigma_i}_{\nu_i \lambda}[L(Q_n)u_n, p_i(n,l)] D^{\sigma_i}_{\lambda \lambda_i}[\bar{L}_{ni}, p_i(i,l)] \tag{57a}$$

and

$$D^{\sigma_i'}_{\nu_i' \lambda_i'}[L_{n'i}', p_i'(i,l)'] = \sum_{\lambda} D^{\sigma_i'}_{\nu_i' \lambda}[L(Q_n)u_n, p_i'(n,l)] D^{\sigma_i'}_{\lambda \lambda_i'}[u_{nn'}\bar{L}_{n'i}', p_i'(i,l)']. \tag{57b}$$

The Lorentz transformation $L(Q_n)u_n$ carries the frame $[n, l]$ to the working frame in which the reaction process is considered, and can be determined by the 4-momenta of the external particles in the latter frame; the group parameters of $L(Q_n)u_n$ may be viewed as free variables, independent of the invariant variables. With these group parameters given, one can calculate the first Wigner rotation in the right-hand side of Eqs. (57a) and (57b) by means of Wick's method.[15] In the analytic S-matrix theory, one is usually interested in the amplitudes which can be expressed[31] in terms of only the invariant variables, i.e., one must choose some special frame (such as the frames $[n, l]$, $[n', l]'$, etc.) such that the Lorentz transformation with free parameters becomes the identity. In the following, we shall deal with the amplitudes in the frame $[n, l]$.

Let us define a rest frame $[p_i]$ of the particle p_i, such that $[p_i] \equiv [p_i; Q_i, p_{i+1}]$. The frames $[n, l]$ and $[p_i]$ are related by a general Lorentz transformation

$$[n, l] = \overline{L}_{ni} L(p_i^{(i,l)})[p_i], \tag{58a}$$

where

$$\overline{L}_{ni} L(p_i^{(i,l)}) \equiv u_z(\phi^i) u_y(\theta^i) b(\alpha^i) u_z(\overline{\psi}^i) u_y(\overline{\theta}^i) u_z(\overline{\phi}^i). \tag{58b}$$

The group parameters α^i, θ^i, and ϕ^i are already determined by the orientation of p_i in the frame $[n, l]$, while the parameters $\overline{\theta}^i$ and $\overline{\phi}^i$ are determined by that of Q_n in the frame $[p_i]$:

$$[\alpha^i, \theta^i, \phi^i] = [p_i: Q_n; p_n, p_{n+1}], \tag{59a}$$

$$\cos\overline{\theta}^i = -\cos\theta_{0n}^{(p_i)} = -G\begin{bmatrix} (0n)\ i \\ \\ (0i)\ i \end{bmatrix} [G((0n), i) G((0i), i)]^{-1/2}, \tag{59b}$$

$$\sin\overline{\theta}^i = \sin\theta_{0n}^{(p_i)} = [k_i G((0n), (0i), i) / G((0n), i) G((0i), i)]^{1/2}, \tag{59c}$$

$$\cos\overline{\phi}^i = \cos\phi_{0n}^{(p_i)} = G\begin{bmatrix} (0n)\ (0i)\ i \\ \\ i+1\ (0i)\ i \end{bmatrix} [G((0n), (0i), i) G(i+1, (0i), i)]^{-1/2}, \tag{59d}$$

and

$$\sin\overline{\phi}^i = -\sin\phi_{0n}^{(p_i)}$$

$$= -\epsilon_{0n}^{(p_i)} [G((0i), i) G((0n), i+1, (0i), i) / G((0n), (0i), i) G(i+1, (0i), i)]^{1/2}, \tag{59e}$$

where $\epsilon_{0n}^{(p_i)}$ ($= +1$ or -1) can be determined as before.

The calculation of the parameter $\overline{\psi}^i$ is more complicated. First, one may introduce two intermediate frames $[n, i]$ and $[n, i]'$, defined by

$$[n, i] \equiv u_y^{-1}(\theta^i) u_z^{-1}(\phi^i)[n, l] \equiv u_z(\overline{\psi}^i)[n, i]', \tag{60a}$$

$$= u_z(\overline{\psi}^i) b(\alpha^i) u_y(\overline{\theta}^i) u_z(\overline{\phi}^i)[p_i], \tag{60b}$$

where $u_y^{-1} u_z^{-1}$ may be rewritten[22] as

$$u_y^{-1}(\theta^i) u_z^{-1}(\phi^i) = u_z(\pi) u_y(\theta^i) u_z(\pi - \phi^i). \tag{60c}$$

From the definition in Eq. (60a), we have $[n, i] = [Q_n; Q_n - p_i, p_n]$, and then the group parameters of the 4-momenta p_i in this frame are given by Eqs. (9a)–(9c) and (9′):

$$[\alpha_i^{(n,i)}, \theta_i^{(n,i)}, \phi_i^{(n,i)}] = [p_i: Q_n; Q_n - p_i, p_n]. \tag{61}$$

On the other hand, the corresponding group parameters in the frame $[n, i]'$ can be obtained by applying $b(\alpha^i) u_y(\overline{\theta}^i) u_z(\overline{\phi}^i)$ on the momentum p_i in the frame $[p_i]$. The rotation $u_z(\overline{\psi}^i)$ carries the frame $[n, i]'$ to $[n, i]$, and the parameter $\overline{\psi}^i$ can be obtained by comparing the group parameters of Q_i in these two frames. After a tedious but straightforward calculation, one obtains

$$\cos\overline{\psi}^i = -G\begin{bmatrix} (0i)\ i\ (0n) \\ \\ n\ \ i\ (0n) \end{bmatrix} [G((0i), i, (0n)) G(n, i, (0n))]^{-1/2}, \tag{59f}$$

and

$$\sin\overline{\psi}^i = -\epsilon_{0i}^{(n,i)} [G(i, (0n)) G((0i), n, i, (0n)) / G((0i), i, (0n)) G(n, i, (0n))]^{1/2}. \tag{59g}$$

From Eqs. (4a), (28a), (28b), (55), (57a), (58a), and (58b), one can then obtain

$$D^{\sigma_i}_{\nu_i\lambda_i}[\bar{L}_{ni}, p_i(i,l)] = e^{-i\nu_i\bar{\psi}^i} d^{\sigma_i}_{\nu_i\lambda_i}(\cos\bar{\theta}^i) e^{-i\lambda_i\bar{\phi}^i}. \tag{62a}$$

The Wigner rotations for the particles p_i' in the final system are given by Eqs. (59b)–(59g) and (62a) with Q_i, p_i, and p_{i+1} replaced by Q_i', p_i', and p_{i+1}', respectively. Obviously, if one works in the frame $[n', l]'$, one obtains the Wigner rotations, as in the working frame $[n, l]$, by interchanging the roles of the particles in the initial system and in the final system. It is interesting to note that the Wigner rotations depend on the five momenta only, irrespective of the number of particles in the total system. If the total system involves only four particles, the Wigner rotations become multiples of the identity, a fact which is well known.

Suppose that a general coupling scheme is defined such that the reference frame for p_i (p_i'), corresponding to the frame $[i, l]$ ($[i, l]'$) in the standard coupling scheme, is $[p_{ci}; p_i, p_{ei}]$ ($[p_{ci}'; p_i', p_{ei}']$). The working frame, in which the amplitudes can be expressed in terms of only the invariant variables, is defined to be $[Q_n; p_g, p_h]$, where the 4-vectors p_{ci}, p_{ci}', etc., can be any linear combinations of p_i's and p_i''s, respectively. Then, the Wigner rotation for the particle p_i (p_i') can be obtained by Eqs. (59b)–(59g) and (62) with p_n, Q_i, and p_{i+1} replaced by p_g, p_{ci} (p_{ci}') and p_{ei} (p_{ei}'), respectively.

D. Crossing Relations

Assuming that the scattering amplitude satisfies the substitution rule[20] and also analyticity, one can show that the amplitude for one channel is proportional to that of the crossed channel in the same frame, provided the helicities of the crossed particles change sign. We shall prove the crossing relation between the corresponding h.p.w. amplitude, and determine the proportionality constant, which depends on the helicities of the crossed particles only.

The h.p.w. amplitudes of the process $0+1+\cdots+N \to 0'+1'+\cdots+N'$ (process A) and the process $0+1 \to 0'+1'+\cdots+N'+2+\cdots+N$ (process B) in the standard coupling scheme are both functions of the subenergies s_i (in addition to the helicities and angular momenta). On the other hand, the matrix elements $\prod D^{j_i}(u_i^{-1})$ for the initial system in the h.p.w. expansion of process A can be expressed in terms of $\prod D^{j_i}(u_i'^{-1})$ for the corresponding particles in the h.p.w. expansion of process B. One then finds that the h.p.w. expansions for process A and process B differ somewhat in form; the difference, which depends on the subenergies as well as other invariant variables, must be compensated by those Wigner rotations which have different reference frames in process B.

We first extend definitions of the frames $[i, l]$, $[i, u]$, $[p_i]$, etc., to the cases in which some of the 4-momenta have negative time components and are also timelike (the latter will be referred to as negative timelike). In the definitions of such frames, the z component of a real 4-momentum will be defined to be along the "negative" z axis if its t component and z component have opposite signs. This extension automatically satisifies the crossing condition in the sense that since a "negative" timelike momentum in one channel is positive timelike in its crossed channel, this definition is consistent with the previous one for the positive timelike 4-momentum.

Let us work in the physical region of process A. The 4-momenta p_i ($i = 2, 3, \ldots, N$) in process B are negative timelike, and hence we need the extended definitions for the reference frames. One can easily show (see Fig. 7) that

$$[i, l] = u_y(\beta_{n'+n-i}')[n'+n-i, l]' \tag{62b}$$

and

$$[n'+n-i, l]' = u_y(\beta_i)[i, l], \tag{62c}$$

i.e.,

$$u_y(\beta_i) = u_y^{-1}(\beta_{n'+n-i}'). \tag{62d}$$

Then, one can transform the frame $u_y(\beta_i)[i, l]$ to $[i+1, l]$ along the left line in Fig. 7, and then go back to the original frame via the right path; one has

$$u_z(\gamma_i)b(\delta_i)u_z(\gamma_{n'+n-i-1}')b(\delta_{n'+n-i-1}') = 1, \tag{63a}$$

i.e.,

$$u_z(\gamma_i) = u_z^{-1}(\gamma_{n'+n-i-1}') \tag{63b}$$

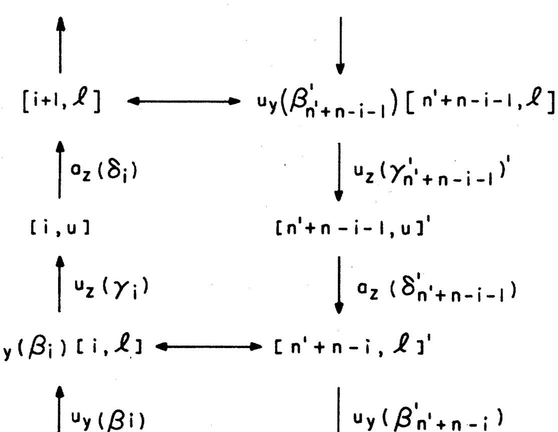

FIG. 7. Relations between the frames for process A and process B.

and

$$b(\delta_i)=b^{-1}(\delta'_{n'+n-i-1}). \tag{63c}$$

In the physical region of process A the boost $b(\delta_i)$ $(i=2, \ldots, N)$ transform $Q_i^{(i,u)}$ to a 4-momentum with a positive z component in the frame $[i+1, l]$, while $b(\delta'_{n'+n-i-1})$ $(i=2,\ldots,N)$ boosts Q_{i+1} in the frame $[n'+n-i-1, u]'$ to that with a negative z component in the frame $[n'+n-i, l]'$. From Eqs. (62b) and (62c), one also knows that the magnitudes of δ_i and $\delta'_{n'+n-i-1}$ are equal. Hence, the decompositions Eqs. (63b) and (63c) are unique. In our convention,[22] $b(\delta'_{n'+n-i})$ and $u_y(\beta'_{n'+n-i})$ $(i=2,\ldots,N)$ are not a pure z boost and a y rotation, respectively, but are in fact inverses of the latter. And conversely, $b(\delta_i)$ and $u_y(\beta_i)$ $(i=2,\ldots,N)$ must be considered as inverses of a pure z boost and a y rotation, respectively, in the physical region of process B. From Eqs. (62d) and (63b), one obtains

$$\prod_{i=1}^{n-1} D^{j_i *}_{\mu_i m_i}(u_i^{-1})=\prod_{i=1}^{n-1} D^{j_i}_{m_i'\mu_i}(u'^{-1}_{n'+n-i})\prod_{i=2}^{n}\exp(i\lambda_i\gamma'_{n'+n-i}). \tag{64}$$

Here, the relations $\gamma_n=0$ and $\gamma'_{n'+n-1}=0$ have been used.

Without loss of generality, we assign $j'_{n'+n-i+1}=j_i$. In Eq. (55), the summation over m_i can be replaced by the summation over μ_{i+1} since they are related by $\mu_{i+1}=m_i-\lambda_{i+1}$ [see Eq. (30)]. From the range of m_i, one can obtain the range of μ_{i+1}:

$$j'_{n'+n-i}\geq \mu_{i+1}+\lambda_{i+1}\geq -j'_{n'+n-i} \tag{65a}$$

and

$$j'_{n'+n-i-1}\geq \mu_{i+1}\geq -j'_{n'+n-i-1}. \tag{65b}$$

Equations (65a) and (65b) imply restrictions for the range of $m'_{n'+n-i-1}$ in process B if one replaces μ_{i+1} by $m'_{n'+n-i-1}$ and assigns

$$\lambda_i=-\lambda'_{n'+n-i+1}, \quad i=2,\ldots,n. \tag{65c}$$

Substituting Eq. (64) into Eq. (55), and changing the dummy variables m_i $(i=2,\ldots,n)$ to μ_{i+1} $(=m'_{n'+n-i-1})$, one obtains a h.p.w. expansion for process A; the matrix elements of the little groups in the expansion become those for process B. Hence, the crossing relation for the scattering amplitude,

$$T_{\nu_i'\nu_0\nu_1\nu_2\cdots\nu_n}(s_i', u_i'; s_{nn'}, u_{nn'}; s_i, u_i; N'+1, N+1)=K_{\nu_i}(2,\cdots,N)\,T_{\nu_i'\nu_0\nu_1}(s_i', u_i'; s_{nn'+n-1}, u_{nn'+n-1}; N'+N, 2) \tag{66a}$$

implies that for the h.p.w. amplitude,

$$[(2j_n+1)/s_n]^{1/2}\,T^{j_i'j_{nn'}j_i}_{\lambda_i'm_i'\lambda_0\lambda_1\lambda_2\cdots\lambda_n m_i}(s_i', s_{nn'}, s_i; N'+1, N+1)=[(2j_1+1)/s_1]^{1/2}$$

$$\times\,T^{j_i'j_{1n+n'-1}}_{\lambda_i'm_i'\lambda_0\lambda_1}(s'_{1n+n'-1}; N'+N, 2)K_{\lambda_i}(2,\ldots,n) \tag{66b}$$

if the following relation is satisfied:

$$K_{\nu_i}(2,\ldots,n)\prod_{i=n'+1}^{n'+n-1}\left[D^{\sigma_i'}_{\nu_i'\lambda_i'}[L'_{ni}, p_i'(i,l)']e^{i\lambda_i'\nu_{i-1}'}\right]=K_{\lambda_i}(2,\ldots,n)\prod_{i=2}^{n}D^{\sigma_i *}_{\nu_i\lambda_i}[L_{ni}, p_i(i,l)], \tag{66c}$$

where

$$\sigma_i=\sigma_{n'+n-i+1}, \quad j_1=j_{1n+n'-1}, \quad j_{nn'}=j_{n'}, \tag{66d}$$

and

$$\nu_i=-\nu'_{n'+n-i+1}, \quad \lambda_i=-\lambda'_{n'+n-i+1}. \tag{66e}$$

Note that the first index n in $s_{nn'+n-1}$, $u_{nn'+n-1}$ and L'_{ni} means that we work in the frame $[n,l]$. From Eq. (62b), one can relate the frames $[i,l]$ and $[n'+n-i+1, l]'$ by

$$[n'+n-i+1, l]'=b(\delta'_{n'+n-i})u_z(\gamma'_{n'+n-i})[i,l],$$

$$i=2,3,\ldots,n \tag{67a}$$

i.e.,

$$L'_{nn'+n-i+1}b(\delta'_{n'+n-i})u_z(\gamma'_{n'+n-i})=L_{ni}. \tag{67b}$$

From Eqs. (67b) and (66e), one can easily show that

$$D^{\sigma_i}_{\nu_i'\lambda_i'}[L'_{ni}, p_i'(i,l)']e^{i\lambda_i'\gamma_{i-1}'}\Big|_{i\to n'+n-i+1}$$

$$=D^{\sigma_i}_{-\nu_i-\lambda_i}[L_{ni}, p_i(i,l)] \tag{67c}$$

$$=D^{\sigma_i *}_{\nu_i\lambda_i}[(L_{ni}, p_i(i,l)](-1)^{\nu_i-\lambda_i}. \tag{67d}$$

Substituting Eq. (67c) into Eq. (66c), one can easily see that the condition (66c) is satisfied if one chooses

$$K_{\nu_i}(2,\ldots,N)=\prod_{i=2}^{N}(-1)^{\nu_i}. \tag{67e}$$

Literally, each crossed particle changes the sign of its helicity ν and introduces a phase factor $\pm(-1)^{\nu}$ in both helicity amplitudes and h.p.w. amplitudes. One can convince himself that this statement holds for any coupling scheme and any production process.

It is likely that in any formalism a scattering amplitude, with particles having definite helicities or z components of spin, must have Wigner rotations or the like in its partial-wave expansion if it satisfies the substitution rule[20] and analyticity. If the number of particles in the scattering system is less than five, the Wigner angles may be zero in some special frames.

E. Crossing Angles

The crossing matrices, which relate the helicity amplitude in one channel to those in the crossed channel, have been worked out for the two-to-two process[32] and the two-to-N process.[33] We shall calculate explicit expressions of the angles of the matrices in these and more general processes.

Let us first calculate the crossing matrices, which relate the helicity amplitude for process A in the frame $[n, l]$ to that of process B in the frame $[1, l]$. These two frames are related by

$$[n, l] = \overline{L}_{n1}[1, l]. \tag{7'}$$

It is interesting to note that in the physical region of process A the Lorentz transformation $\overline{L}_{n1}$ is orthochronous. The helicity amplitudes for process A in the frames $[n, l]$ and $[1, l]$ are related by

$$T_{\nu_i'\nu_i}(\ldots; N'+1, N+1; [n, l]) = \sum_{\overline{\nu}_i\overline{\nu}_i'} \prod_{i=0}^{n} D^{\sigma_i *}_{\nu_i\overline{\nu}_i}[\overline{L}_{n1}, p_i(1, l)] \prod_{i=0}^{n'} D^{\sigma_i'}_{\nu_i'\overline{\nu}_i'}[\overline{L}_{n1}, p_i'(1, l)] \times T_{\overline{\nu}_i'\overline{\nu}_i}(\ldots; N'+1, N+1; [1, l]), \tag{68a}$$

where $[n, l]$ or $[1, l]$ in the helicity amplitude T indicates the working frame in which the amplitude is defined. Suppose that the crossing relation holds true. Then, the amplitude on the right-hand side of Eq. (68a) can be replaced by the amplitude for process B in the frame $[1, l]$. In Eq. (68a), we change the dummy variable ν_i $(i = 2, \ldots, N)$ to $-\nu_i$, and obtain the crossing matrices:

$$(-1)^{\nu_i - \overline{\nu}_i} D^{\sigma_i}_{-\nu_i, -\overline{\nu}_i}[\overline{L}_{n1}, p_i(1, l)] \tag{68b}$$

for the particles p_0 and p_1,

$$D^{\sigma_i'}_{\nu_i'\overline{\nu}_i'}[\overline{L}_{n1}, p_i'(1, l)] \tag{68c}$$

for the particles $p_0', \ldots, p_{n'}'$, and

$$\pm(-1)^{\nu_i} D^{\sigma_i}_{-\nu_i\overline{\nu}_i}[\overline{L}_{n1}, p_i(1, l)] \tag{68d}$$

for the particles $p_2, \ldots, p_n$, where the choice of the sign in Eq. (68d) does not affect measurable quantities. In deriving Eqs. (68b) and (68d), we have used the following relation for any real rotation group element g:

$$D^{\sigma *}_{\nu\lambda}(g) = (-1)^{\nu - \lambda} D^{\sigma}_{-\nu - \lambda}(g). \tag{69}$$

One can convince himself that the crossing matrix in Eqs. (68b)–(68d) can be applied to more general processes by proper replacements.

Suppose we want to calculate the crossing matrices which relate the helicity amplitude in the frame $[w']$ to its crossed-channel amplitude in the frame $[w]$. The frames $[w]$ and $[w']$ are defined to be

$$[w] \equiv [p_w; p_y, p_z] \text{ and } [w'] \equiv [p_{w'}; p_{y'}, p_{z'}]. \tag{70a}$$

Let the Lorentz transformation, carrying the frame $[w']$ to $[w]$, be $L_{ww'}$:

$$[w] = L_{ww'}[w']. \tag{70b}$$

Then the crossing matrices are as follows:

$$(-1)^{\nu - \lambda} D^{\sigma}_{-\nu - \lambda}[L_{ww'}, p_x(w')] \tag{71a}$$

for uncrossed particle p_x,

$$\pm(-1)^{\lambda} D^{\sigma}_{\nu - \lambda}[L_{ww'}, p_x(w')] \tag{71b}$$

for crossed particle p_x in the initial system, and

$$D^{\sigma}_{\nu\lambda}[L_{ww'}, p_x(w')] \tag{71c}$$

for uncrossed particle p_x,

$$\pm(-1)^{\nu} D^{\sigma}_{-\nu\lambda}[L_{ww'}, p_x(w')] \tag{71d}$$

for crossed particle p_x in the final system.

Now, the calculation of the crossing matrices reduces to calculating the matrix element $D^{\sigma}_{\nu\lambda}[L_{ww'}, p_x(w')]$. Since we have already calculated the Wigner rotations in Sec. IV C, we prefer to introduce an auxiliary frame $[x]$ such that the crossing rotation is the product of two Wigner rotations which have been calculated in Sec. IV C:

$$[L_{ww'}, p_x(w')] = [L_{wx}, p_x(x)][L_{w'x}, p_x(x)]^{-1} \tag{71e}$$

$$\equiv e^{-i\Psi_x J_3} e^{-i\Omega_x J_2} e^{-i\Phi_x J_3}. \tag{71f}$$

The frame $[x]$ is defined to be a rest frame of p_x such that $[x] = [p_x; p_u, p_v]$. The Wigner rotation $[L_{wx}, p_x(x)]$ ($[L_{w'x}, p_x(x)]$) is given by Eqs. (59b)–(59g) if one replaces Q_n, p_n, Q_i, p_i, and p_{i+1} by p_w ($p_{w'}$), p_y ($p_{y'}$), p_u ($p_{u'}$), p_x, and p_v ($p_{v'}$), respectively. Combining these two Wigner rotations in Eq. (71e), one obtains after a tedious but straightforward calculation

$$\cos\Omega_x = -G\begin{bmatrix} w & x \\ w' & x \end{bmatrix}[G(w, x)G(w', x)]^{-1/2}, \tag{72a}$$

$$\cos\Phi_x = G\begin{bmatrix} w & w' & x \\ y' & w' & x \end{bmatrix}[G(w, w', x)G(y', w', x)]^{-1/2}, \tag{72b}$$

and

$$\cos\Psi_x = -G\begin{bmatrix} w' & w & x \\ & & \\ y & w & x \end{bmatrix}[G(w, w', x)G(y, w, x)]^{-1/2}. \tag{72c}$$

The signs for $\sin\Psi_x$ and $\sin\Phi_x$ can be determined by the procedures outlined in the paragraph following Eq. (10).

Even though those two Wigner rotations depend on the auxiliary frame, the crossing angles are independent of it as they should be. It is interesting to note that the crossing angles depend on the frames $[w]$ and $[w']$ "partially," in the sense that the former depends only on p_w, p_w', p_y, and p_y', while the latter depends on p_z and p_z' also. For example, the angles of the crossing matrices between the helicity amplitudes of process A and process B can be obtained from Eqs. (72a)–(72c) by replacing p_x, p_w, $p_{w'}$, p_y, and $p_{y'}$ with p_i, Q_n, Q_1, p_{n+1} $(-p'_{n'})$, and p_2, respectively.

V. PARTIAL-WAVE RECOUPLING CONDITIONS

The h.p.w. amplitude for one coupling scheme provides enough information to obtain the full helicity amplitude. We shall show how the h.p.w. amplitude for another coupling scheme can be obtained from that for a given coupling scheme. This can be done with the aid of recoupling coefficients, which for the three-particle a.m. state have been worked out by Wick[15] and others.[34] In this section, we shall investigate the general case, and use it to express the partial-wave recoupling condition for the h.p.w. amplitudes which govern the dynamics.

The $(N+1)$-particle recoupling coefficient I_{n+1} between two coupling schemes is defined as the contraction of two $(N+1)$-particle pseudoparticle states, and can be expressed in terms of the transformation matrices

$$\begin{aligned} I_{n+1} &\equiv \langle Q_n' j_n' m_n' \cdots | Q_n j_n m_n \cdots\rangle \\ &= \sum_{\nu_i}\int\prod_{i=0}^{n}[dp_i\,\delta^+(p_i^2 - m_i^2)] \\ &\quad\times\langle Q_n' j_n' m_n'\cdots|p_0\sigma_0\nu_0;\ \cdots;\ p_n\sigma_n\nu_n\rangle \\ &\quad\times\langle p_0\sigma_0\nu_0;\ \cdots;\ p_n\sigma_n\nu_n|Q_n j_n m_n\cdots\rangle. \end{aligned} \tag{73}$$

Here, the ith particle in the first coupling scheme is assumed to be the i'th one in the second coupling scheme, while the i'th particle in the second is the i''th in the first. We shall label all the quantities in the second scheme with a prime.

Let us first define the two frames $[n, d]$ and $[n, d]'$ by

$$[n, d] \equiv b(\delta_{n-1})u_y(\beta_{n-1})[n-1, l] \tag{74a}$$

and

$$[n, d]' \equiv b(\delta'_{n-1})u_y(\beta'_{n-1})[n-1, l]'. \tag{74b}$$

By definition, they are both c.m. frames of the $(N+1)$-particle system, and thus are related by a rotation g, i.e.,

$$[n, d] = g[n, d]'. \tag{74c}$$

The frames $[i, l]$ and $[i', l]'$ are the reference frames of the same particle in these two coupling schemes, respectively. From Eqs. (74c) and (7), we can easily show that

$$[i, l] = \overline{L}_{ni}^{\,-1}u_z(\gamma_{n-1})g\,u_z^{-1}(\gamma'_{n-1})\overline{L}'_{ni'}[i', l]'. \tag{75}$$

The dependence of the recoupling coefficient on the group parameters in Eq. (73) comes from the transformation matrices. All these parameters, except the two sets $(\gamma_n, \beta_n, \gamma_{n-1})$ and $(\gamma_n', \beta_n', \gamma'_{n-1})$, can be expressed in terms of the momenta p_i. However, one cannot take both sets as independent parameters. One can easily show from Eq. (74c) that they are related by

$$L(Q_n)g_n g = L(Q_n')g_n' \quad\text{or}\quad g_n g = g_n', \tag{75'}$$

where

$$g_n = u_z(\gamma_n)u_y(\beta_n)u_z(\gamma_{n-1}) \tag{75''}$$

and

$$g_n' = u_z(\gamma_n')u_y(\beta_n')u_z(\gamma'_{n-1}). \tag{75'''}$$

After substituting Eq. (40a) into Eq. (73), one may rewrite some of the factors in the following form (note that $\sigma_i = \sigma'_{i'}$, and $\sigma_{i''} = \sigma_i'$):

$$D^{j_n}_{\mu_n m_n}(u_n^{-1})D^{j_{n-1}}_{\mu_{n-1}m_{n-1}}(u_{n-1}^{-1}) = D^{j_n}_{\mu_n m_n}(g_n^{-1})D^{j_{n-1}}_{\mu_{n-1}m_{n-1}}(u_y^{-1}(\beta_{n-1}))e^{-i\gamma_{n-1}\lambda_n}, \tag{76a}$$

$$\sum_{\nu_i} D^{\sigma_i}_{\nu_i\lambda_i}[L_{ni}, p_i(i, l)]D^{\sigma_i *}_{\nu_i\lambda'_{i'}}[L'_{ni'}, p'_{i'}(i', l)'] = D^{\sigma_i}_{\lambda'_{i'}\lambda_i}[U_{i'i}, p_i(i, l)] \quad\text{for } i \neq n, n'' \tag{76b}$$

$$\sum_{\nu_n} D^{\sigma_n}_{\nu_n\lambda_n}[L_{nn}, p_n(n,l)]D^{\sigma_n *}_{\nu_n\lambda'_n}[L'_{nn'}, p'_{n'}(n',l)'] = D^{\sigma_n}_{\lambda'_n\lambda_n}[U_{n'n}, p_n(n,l)]e^{i\gamma_{n-1}\lambda_n}. \tag{76c}$$

Here, the Lorentz transformation $U_{i'i}$ ($U_{n'n}$) carries the frame $[i,l]$ ($[n,d]$) to the frame $[i',l]'$ ($[n,l]'$); similarly for the second scheme. It is important to note that $U_{i'i}$, $U_{n'n}$, and $U_{nn''}$ do not depend on the group element g_n. For example, if both coupling schemes have the form as shown in Fig. 3 but differ in the ordering of the particles, one can show that

$$U_{i'i} = [b'_{n-1}u_y(\beta'_{n-1})\overline{L}'_{n-1i'}]^{-1}g^{-1}b_{n-1}u_y(\beta_{n-1})\overline{L}_{n-1i}, \quad i\neq n \text{ and } i'\neq n \tag{76d}$$

$$U_{n'n} = [b'_{n-1}u_y(\beta'_{n-1})\overline{L}'_{n-1n'}]^{-1}g^{-1}, \quad n'\neq n \tag{76e}$$

$$U_{nn''} = g^{-1}b_{n-1}u_y(\beta_{n-1})\overline{L}_{n-1n''}, \qquad n''\neq n \tag{76f}$$

and

$$U_{nn} = g^{-1}. \tag{76g}$$

Since only $D^{j_n}(g_n^{-1})$ and $D^{j'_n}(g_n'^{-1})$ depend on the group element g_n, one can perform the integration over dg_n:

$$\frac{N_{j_n}}{2\pi}\int dg_n D^{j_n}_{\mu_n m_n}(g_n^{-1})D^{j'_n *}_{\mu'_n m'_n}(g_n'^{-1}) = D^{j_n}_{\mu'_n\mu_n}(g^{-1})\delta_{j_n j'_n}\delta_{m_n m'_n}. \tag{76h}$$

Substituting the phase-space volume element (24) corresponding to the first coupling scheme into Eq. (73), and performing the procedures which lead to Eqs. (76a)–(76h), one obtains the recoupling coefficient for the $(N+1)$-particle pseudoparticle states:

$$\begin{aligned} I_{n+1} = {}& 2\pi\delta(Q_n - Q'_n)\delta_{j_n j'_n}\delta_{m_n m'_n}D^{j_n}_{\mu'_n\mu_n}(g^{-1})\rho(n+1)N(n)N'(n) \\ &\times\int d\cos\beta_{n-1}\prod_{i=1}^{n-2}du_i\prod_{i=1}^{n-1}\delta\left(\left(\sum_j^i p'_j\right)^2 - s'_i\right)\rho'(n+1)\prod_{i=1}^{n-2}[D^{j'_i}_{\mu'_i m_i}(u_i^{-1})D^{j'_i}_{\mu'_i m'_i}(u_i'^{-1})] \\ &\times D^{j_{n-1}}_{\mu_{n-1}m_{n-1}}[u_y^{-1}(\beta_{n-1})]D^{j'_{n-1}}_{\mu'_{n-1}m'_{n-1}}[u_y^{-1}(\beta'_{n-1})]\prod_{i=0}^{n}D^{\sigma_i}_{\lambda'_i\lambda_i}[U_{i'i}, p_i(i,l)]. \end{aligned} \tag{77}$$

Now, all the integrations in Eq. (77) depend on the choice of coupling schemes. All the parameters of the little groups and all the Wigner angles in Eq. (77) have been worked out in Sec. IV. One must, however, make a proper replacement for the 4-momenta in the expressions there.

For three-particle states, the integration can be performed easily since there is only one integration variable and one δ function. One can easily see that adding one more particle results in an increase of two in the number of integration variables, and of one in the number of δ functions, giving a net increase of one in integration variables. Therefore, the larger the number of particles in the system, the more complicated the expression. When the number of the particles becomes five or more, some of the relations between the invariant variables even become quadratic. This makes the integrations more difficult. Nevertheless, for particular cases, the calculations are not too complicated. For the three-particle case, one can obtain Wick's result[15] immediately from Eq. (77). For spinless external particles, Eq. (77) reduces to a much simpler form. For the four-particle case, there is only one nontrivial integration. We shall not investigate these cases further in this paper.

The h.p.w. amplitudes with different coupling schemes in the initial systems can be related through the recoupling coefficient:

$$T^{j'_i j j_i}_{\lambda'_i m'_i \lambda_i m_i}(s'_i, s, s_i; N'+1, N+1) = \int d^4Q_n\prod_{i=1}^{N-1}ds''_i\sum_{j''_i m''_i\lambda''_i}T^{j'_i j j''_i}_{\lambda'_i m'_i\lambda''_i m''_i}(s'_i, s, s''_i, N', N)\langle Q''_n j''_n m''_n\cdots|Q_n j_n m_n\cdots\rangle. \tag{78}$$

This is the partial-wave recoupling condition. If the h.p.w. amplitude for a particular coupling scheme is given, one can check by means of Eq. (78) whether the h.p.w. amplitudes for the other coupling schemes develop poles in the subenergies. Equation (78) is very useful for checking whether the models introduced via partial-wave unitarity relations have duality properties, and also to study the singularity structure of the h.p.w. amplitude corresponding to other coupling schemes. If the coupling schemes for the final system are also different, one may introduce another recoupling coefficient in Eq. (78).

VI. UNITARITY FOR PARTIAL-WAVE AMPLITUDES

The unitarity relation, $S^\dagger S = 1$, can be expressed in terms of the T-matrix operator [see Eq. (49)]:

$$iT^\dagger - iT = T^\dagger T. \tag{79}$$

Bilinearity in $T^\dagger$ and T forces us to introduce a complete set of intermediate states on the right-hand side of Eq. (79). In this paper, we suppress all the internal quantum numbers, such as strangeness, isospin, hypercharge, etc., and deal with number space only.

Let us first consider the scattering process $a+b \to a'+b'$. The unitarity relation [Eq. (79)] becomes

$$iT^*_{\nu_a\nu_b\nu_{a'}\nu_{b'}}(s_{b'}, u_{b'b}) - iT_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'}) = \sum_{N=1}^{\infty} A_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'}; N+1), \tag{80}$$

where the $(N+1)$-particle intermediate-state contribution is defined by

$$A_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'}; N+1) = \sum_{\lambda_i} \int d\Phi_{n+1} T_{\lambda_i\lambda_a\lambda_b}(\cdots; N+1, 2) T^*_{\lambda_i\lambda_{a'},\lambda_{b'}}(\cdots; N+1, 2). \tag{80$'$}$$

If there are bound states below threshold, one may include[35] their contribution in Eq. (80′). Diagonalization (or reduction) of integrations in Eq. (80′) depends on the coupling scheme one chooses; one special coupling scheme may be more suitable than others for handling a particular approximation. For convenience, we shall label the little groups and Lorentz transformation in the upper amplitude with a prime. We work in the frame $[b, l]$ in which the Wigner rotations in the two-to-two scattering amplitudes become multiples of the identity.

Suppose we choose the standard coupling scheme (see Fig. 6). The diagrammatical representation for the $(N+1)$-particle intermediate state is shown in Fig. 1(b). The 2-particle intermediate-state contribution to the unitarity relation is well known. We write it down with our notation and conventions as

$$A_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'}; 2) = [\rho^*(b)\rho(b')]^{-1/2} \sum_{j_b\lambda_i} [N(b)]^2 D^{j_b}_{\mu_{b'},\mu_b}(u_{bb'}{}^{-1}) T^{j_b}_{\lambda_0\lambda_1\nu_a\nu_b}(s_b) T^{j_b *}_{\lambda_0\lambda_1\lambda_{a'},\lambda_{b'}}(s_b)\Theta(b)\Theta(b')\Theta(2). \tag{81a}$$

The complex conjugate and the step functions are written explicitly in the hope that they may help further an understanding of the assumptions which are made in the helicity formalism in order to satisfy the axioms of an analytic S-matrix theory.

The calculation of the three-particle intermediate-state contribution is more involved. We first express both the lower and upper two-to-three helicity amplitudes in the frame $[b, l]$ and take the same reference frames $[i, l]$ $(i = 1, 2)$ for the intermediate state. The choice of $[2, l]$ as the reference frame of the particle p_2 for the upper helicity amplitude is especially suited to the treatment of unitarity. Previously, we took $[Q_2; p_2, -p_{b'}]$ $(=[2, l]')$ instead of $[Q_2; p_2, -p_b]$ $(=[2, l])$. However, we can make the above choice, since we have freedom in choosing a reference frame. With this choice, the group u_2' is no longer a y rotation. Substituting Eq. (51b) into Eq. (80′), and summing over the λ_i's, one can easily see that the Wigner rotations for the intermediate particles in the upper and lower amplitudes (after combined together) are reduced to unity. On the other hand, one can easily see that

$$u_2 = u_{bb'}u_2', \quad b_1' = b_1, \quad u_1' = u_1. \tag{82}$$

Substituting Eqs. (23) and (82) into Eq. (80′), and performing the integration over du_1 first and then over du_2, one obtains

$$\begin{aligned} A_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'}; 3) = {}& [\rho^*(b)\rho(b')]^{-1/2} \sum_{j_b} [N(b)]^2 D^{j_b}_{\mu_{b'}\mu_b}(u_{bb'}{}^{-1}) \\ & \times \sum_{j_1 m_1 \lambda_1} \int ds_1 T^{j_1 j_b}_{\lambda_i m_1 \nu_a\nu_b}(s_1, s_b; 3, 2) T^{j_1 j_b *}_{\lambda_i m_1 \nu_{a'}\nu_{b'}}(s_1, s_b; 3, 2)\Theta(b)\Theta(b')\Theta(3). \end{aligned} \tag{81b}$$

Proceeding as above, one can extend this derivation to more general cases, and obtain the $(N+1)$-particle intermediate-state contribution:

$$A_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'}; N+1) = [\rho^*(b)\rho(b')]^{-1/2} \sum_{j_b} [N(b)]^2 D^{j_b}_{\mu_{b'},\mu_b}(u_{bb'}{}^{-1}) A^{j_b}_{\nu_{a'},\nu_{b'},\nu_a\nu_b}(S_b, N+1), \tag{81c}$$

where

$$A^{j_b}_{\nu_{a'}\nu_{b'},\nu_a\nu_b}(S_b, N+1) \equiv \sum_{j_i \lambda_i m_i} \left[\int \prod_{i=1}^{n-1} ds_i\right] T^{j_i j_b}_{\lambda_i m_i \nu_a \nu_b}(s_i, s_b; N+1, 2) T^{j_i j_b *}_{\lambda_i m_i \nu_{a'}, \nu_{b'}}(s_i, s_b; N+1, 2) \Theta(b)\Theta(b')\Theta(n+1). \tag{83}$$

We have written the step functions in Eqs. (81a)–(81c) because they determine the limits of the integrations in these expressions. Observe that Eqs. (81a)–(81c) have common factors. We define the total "absorptive" part of the h.p.w. amplitude ΔT^{j_b}:

$$[\rho^*(b)\rho(b')]^{-1/2}\sum_{j_b}[N(b)]^2 D^{j_b}_{\mu_{b'},\mu_b}(u_{bb'}{}^{-1})\Delta T^{j_b}_{\nu_{a'}\nu_{b'},\nu_a\nu_b}(s_b) \equiv iT^*_{\nu_a\nu_b\nu_{a'}\nu_{b'}}(s_{b'}; u_{b'b}) - iT_{\nu_{a'}\nu_{b'},\nu_a\nu_b}(s_b; u_{bb'}). \tag{84}$$

From Eqs. (80), (80′), (81a)–(81c), and (83), one can easily obtain

$$\Delta T^{j_b}_{\nu_{a'}\nu_{b'},\nu_a\nu_b}(s_b) = \sum_{N=1}^{\infty} A^{j_b}_{\nu_{a'}\nu_{b'},\nu_a\nu_b}(s_b; N+1). \tag{85}$$

Equation (85) provides us with the unitarity relation for the h.p.w. amplitude in the standard coupling scheme. The unitarity relation is completely diagonalized in this scheme in the whole physical region of the process $a+b \to a'+b'$.

The unitarity relations for the general process $a+b+\cdots+e \to a'+b'+\cdots+e'$, including decay processes, can also be obtained. We first define the generalized "absorptive" part of the h.p.w. amplitude $\Delta T^{j'_x j j_x}$ [see Eq. (55)]:

$$\sum(\cdots)\Delta T^{j'_x j j_x}_{\lambda'_x m'_x \lambda_x m_x}(s'_x, s, s_x; N_e, N_{e'}) \equiv iT^*_{\nu_x \nu'_x}(\cdots; N_e, N_{e'}) - iT_{\nu'_x \nu_x}(\cdots; N_{e'}, N_e), \tag{86}$$

where λ_x is a collective symbol for $(\lambda_a, \lambda_b, \ldots, \lambda_e)$, etc. Proceeding as above, one can obtain the diagonalized unitarity relation:

$$\Delta T^{j'_x j j_x}_{\lambda'_x m'_x \lambda_x m_x}(s'_x, s, s_x; N_{e'}, N_e) = \sum A^{j'_x j j_x}_{\lambda'_x m'_x \lambda_x m_x}(s'_x, s, s_x; N_{e'}, N_e; N+1), \tag{87a}$$

where the $(N+1)$-particle "absorptive" part is defined by

$$A^{j'_x j j_x}_{\lambda'_x m'_x \lambda_x m_x}(s'_x, s, s_x; N_{e'}, N_e; N+1)$$
$$\equiv \int \prod_{i=1}^{n-1} ds_i \sum_{j_i m_i \lambda_i} T^{j_i j j_x}_{\lambda_i m_i \lambda_x m_x}(s_i, s, s_x; N+1, N_e) T^{j_i j j'_x *}_{\lambda_i m_i \lambda'_x m'_x}(s_i, s, s'_x; N+1, N_{e'})\Theta(N_e)\Theta(N_{e'})\Theta(n+1). \tag{87b}$$

The diagrammatical representation for the $(N+1)$-particle intermediate state contribution is shown in Fig. 8.

The unitarity relations for the helicity amplitude or the h.p.w. amplitude with other coupling schemes can also be diagonalized, at least in part of the physical region of the general production processes. A detailed discussion will be given elsewhere. We have to emphasize that the unitarity relations in Eqs. (85) and (87a) include the connected and the disconnected parts of the helicity amplitudes. As demonstrated by Mandelstam,[36] one can separate these if he wishes.

It has been shown by JW[15] that under time reversal the pseudoparticle state $|Q_b j_b \lambda_b \lambda_a\rangle$ in its rest frame (and thus in any frame) behaves as the helicity state of a real particle. By mathematical induction, one can easily show that the a.m. state[37] for the $(N+1)$-particle system has a similar property under time reversal U_t:

$$U_t|Q_n^{(n,l)} j_n m_n \cdots\rangle = \epsilon(-1)^{j_n - m_n}|Q_n^{(n,l)} j_n - m_n \cdots\rangle, \tag{88}$$

where $\epsilon = e^{-\pi/2}$ in our normalization of the helicity states. In the derivation of Eq. (88) we have assumed that under complex conjugation q_i changes to $e^{i\pi}q_i$. Assuming the scattering operator (and thus the T-matrix operator) to be invariant under time reversal, i.e., $U_t{}^{-1}SU_t = S^\dagger$ (and thus $U_t{}^{-1}TU_t = T^\dagger$), one can easily show that the h.p.w. amplitude is symmetric:

$$T^{j_x j j'_x}_{\lambda_x m_x \lambda'_x m'_x}(s_x, s, s'_x; N_e, N_{e'}) = T^{j'_x j j_x}_{\lambda'_x m'_x \lambda_x m_x}(s'_x, s, s_x; N_{e'}, N_e). \tag{89}$$

For the process $a+b\to a'+b'$, one can show from Eqs. (51a) and (89) that the helicity amplitude in the frame $[b,l]$ or $[b',l]'$ is skew-symmetric (note $u_{b'b}=u_{bb'}{}^{-1}$):

$$T_{\nu_a\nu_b\nu_{a'}\nu_{b'}}(s_{b'},u_{b'b})=-T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'}). \tag{90a}$$

In deriving Eq. (90a), we have assumed that under complex conjugation the factor $[\rho^*(b)\rho(b')]^{-1/2}$ changes sign, but the scattering angle $\beta_{bb'}$ is unchanged. The negative sign in the right-hand side of Eq. (90a) stems from the fact that $T_{\nu_a\nu_b\nu_{a'}\nu_{b'}}$ has a factor $[\rho(b)\rho^*(b')]^{-1/2}$ while $T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}$ a factor $[\rho^*(b)\rho(b')]^{-1/2}$. In usual treatment, these two factors are not distinguished in the derivation of the h.p.w. amplitude, and thus there are no negative signs in front of the right-hand side of Eq. (90a). On the other hand, one can show that for spinless particles the combination of time-reversal invariance and parity conservation implies that the helicity amplitude is either symmetric or skew-symmetric. Hence, Eq. (90a) may also be considered as a consistency condition for determining the relative phase factor of time reversal and parity when they act on the particle states. Assuming the helicity amplitude is Hermitian analytic,[30] one can show that it is "imaginary analytic" instead of "real analytic":

$$T^*_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'})=-T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'}) \tag{90b}$$

below threshold. From Eq. (90a), one can easily show that the discontinuity ΔT across physical cut is given by

$$\Delta T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'})\equiv T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'})+T^*_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'})$$

$$=2\,\mathrm{Re}\,T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'}). \tag{91a}$$

One can also show that the h.p.w. amplitude is "real analytic," and that its discontinuity across the physical cut is

$$\Delta T^{j_b}_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,s_{bb'})\equiv iT^{j_b*}_{\nu_a\nu_b\nu_{a'}\nu_{b'}}(s_{b'})-iT^{j_b}_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b)$$

$$=2\,\mathrm{Im}\,T^{j_b}_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b). \tag{91b}$$

From Eqs. (91a) and (80) [Eqs. (91b) and (85)], one can set up a dispersion relation for each helicity amplitude (each h.p.w. amplitude).

An alternative prescription is to assume that the momentum transfer t instead of the scattering angle $\beta_{bb'}$ is unchanged under complex conjugation. Hence, under the latter operation the factors $\cos\beta_{bb'}$ and $\sin\beta_{bb'}$ change to $e^{-2\pi i}\cos\beta_{bb'}$ and $e^{-2\pi i}\sin\beta_{bb'}$, respectively, because they both have a denominator $[G(b,(ab))G(b',(a'b'))]^{1/2}$. With this prescription, we have

$$T_{\nu_a\nu_b\nu_{a'}\nu_{b'}}(s_{b'},u_{b'b})=e^{-i\pi(2m-1)}T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,s_{bb'}) \tag{92a}$$

and

$$iT^*_{\nu_a\nu_b\nu_{a'}\nu_{b'}}(s_{b'},u_{b'b})-iT_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'})=-2i\,\mathrm{Re}\,T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'})\quad \text{if } m=\text{integer or zero} \tag{92b}$$

$$=2\,\mathrm{Im}\,T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b,u_{bb'})\qquad \text{if } m=\text{half-integer}, \tag{92c}$$

where m is the larger of $|\mu_b|$ and $|\mu_{b'}|$. From Eqs. (92b) and (92c), one can easily see that the helicity amplitude can be "real analytic" or "imaginary analytic," depending on whether m is a half-integer or an integer. The h.p.w. amplitude is real analytic as in the first prescription [see Eq. (91b)]. The p.w. unitarity relations in Eqs. (85) and (87a) have some additional phase factors in the second prescription. It seems that this prescription may have some defects. However, we shall not discuss it further in this paper. The purpose of this paragraph is to demonstrate the effect on a helicity amplitude (and an h.p.w. amplitude) of the different behavior of $\cos\beta_{bb'}$ under complex conjugation.

For the reaction processes involving more than four particles, their h.p.w. expansions are functions of the little groups (one Toller angle at least) and sometimes the Wigner rotations. Given Eq. (89), it is not clear, to the best of this author's knowledge, whether the full helicity amplitude are, in general, symmetric. In the spinless case the combination of time-reversal invariance and parity conservation implies that the full helicity amplitude is either symmetric or skew-symmetric. A careful study is needed for particles with spin.

Under the multi-Regge hypothesis, one may perform a Sommerfeld-Watson transform in j_i for Eqs. (83)

and (87b); these processes change the sums of products of the h.p.w. amplitudes into the sums of products of the Regge residues plus the background terms (if any). One can also carry out the Sommerfeld-Watson transform in j_x, j_x', and $j(j_b)$ so that the unitarity relations for the h.p.w. amplitudes become a relation for the Regge residues; the choice of one form or another depends upon the range of subenergies of interest.

Finally, we write down the explicit limits of integration in Eqs. (83) and (87b); these limits are determined by the step functions and sometimes also by the conservation of total momentum. For the standard coupling scheme, the limits are fixed only by the step functions

$$\int \prod_{i=1}^{n-1} ds_i \theta(n+1) = \int_{M_{n-1}^2}^{S_{n-1}} ds_{n-1} \int_{M_{n-2}^2}^{S_{n-2}} ds_{n-2} \cdots \int_{M_1^2}^{S_1} ds_1 , \tag{93}$$

where

$$M_i \equiv m_0 + m_1 + \cdots + m_i \tag{93'}$$

and

$$S_i \equiv (s_{i+1}{}^{1/2} - m_{i+1})^2 . \tag{93''}$$

VII. APPLICATIONS

As in the two-to-two scattering process, the helicity amplitude for a general process is useful for phenomenological analysis. Our normalization of single-particle state is different from JW's. For the process $a+b\to a'+b'$, we have the c.m. differential cross section

$$d\sigma = q_{b'} |T_{\nu_{a'}\nu_{b'}\nu_a\nu_b}(s_b, u_{bb'})|^2 du_{b'}/16(2\pi)^4 w_b{}^2 q_b , \tag{94}$$

where $du_{b'}$ is the solid angle of the final system. We can easily see that our amplitude differs[38] from L. L. Wang's by a factor of 8π. For the production process $a+b\to 0+1+\cdots+N$, we have the c.m. differential cross section

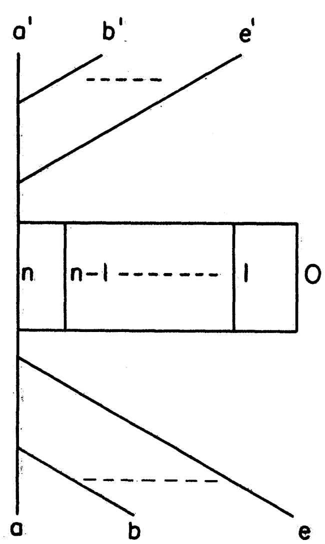

FIG. 8. Unitarity diagram for the process $a+b+\cdots+e\to a'+b'+\cdots+e'$.

$$d\sigma = |T_{\nu_i\nu_a\nu_b}(s_i, u_i; s_b, u_{bn}; N+1, 2)|^2 \, d\Phi_{n+1}/4(2\pi)^4 q_b w_b . \tag{95}$$

The parametrization of $d\Phi_{n+1}$ discussed earlier in Eq. (24) is particularly suitable for phenomenological analysis. In case one is interested in the total cross section without distinguishing the helicities of final particles, one obtains

$$d\sigma = \sum_{j_b j_i m_i \lambda_i} (2j_b+1) |T^{j_i j_b}_{\lambda_i m_i \nu_a \nu_b}(s_i, s_b; N+1, 2)|^2 \times \prod_{i=1}^{n-1} ds_i/(2\pi)^5 q_b{}^2 , \tag{96}$$

where the subenergies s_i and the total energy s_b (s_n) are defined in Eqs. (5b) and (5c) (see in Fig. 3). Each term in Eq. (96) gives the contribution from a "generalized" partial wave. Equation (96) is particularly useful if one can saturate the h.p.w. amplitudes with a few resonances. If one needs one internal line, or more generally r internal lines, for the amplitude to be saturated by Regge poles, one can achieve it by considering coupling schemes similar to those shown in Fig. 9(a) or 9(b). A detailed discussion of these will be made elsewhere. If one is interested in the decay rate or in the process $a+b\to 0+$(anything), one can use Eqs. (56) or (51b), and derive the proper formulas. In case there are more than two particles in each system, initial and final, we would have more coupling schemes; each has its own advantage for certain ranges of subenergies if one wants to saturate the amplitudes with a few resonances or Regge poles.

One of the most important properties for the unitarity relations corresponding to the standard coupling schemes shown in Fig. 1(b) and Fig. 8 is their diagonalization for any momenta p_x and p_x'. The expressions in Eqs. (83), (85), (87a), and

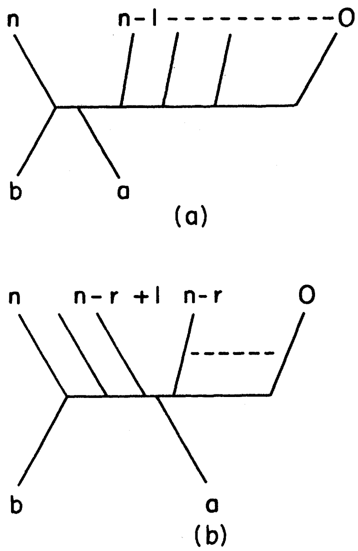

FIG. 9. Some coupling schemes.

(87b) hold for particles with arbitrary mass and spin, in the initial, final, and intermediate states. The simplicity of these expressions makes bootstrap calculations more manageable, particularly when $\Delta T^{j_x' jj_x}$ in Eq. (87a) is proportional to $\mathrm{Im}\, T^{j_x' jj_x}$ or $\mathrm{Re}\, T^{j_x' jj_x}$. If not, we can use dispersion relations for the full helicity amplitude instead of the h.p.w. amplitude. In general, the full helicity amplitude has a simpler analytic structure than that of the h.p.w. amplitude, and the unitarity relation for the latter is more compact in form. Hence, in bootstrap calculations, the full helicity amplitude may serve as a candidate for the dispersion relation, and the unitarity relations for the h.p.w. amplitude as a bridge relating the associated amplitudes. The N/D method of bootstrap calculation for spinless particles has been discussed by Mandelstam,[36] in particular for the three-to-three scattering amplitude. The unitarity relations in this paper are valid for the particles with spin. Thus one can generalize Mandelstam's method for the latter case. Finally, we remark that the h.p.w. expansions (51a), (51b), (55), and (56) and the helicity unitarity relations (83), (85), (87a) and (87b) may provide a simpler representation for studying the analytic structure of production amplitudes.

The unitarity relation for a reaction amplitude or an h.p.w. amplitude may be viewed as a counterpart of the perturbative expansion in local field theory. In the latter the expansion is a power series in a coupling constant, which cannot converge in strong interaction physics. A unitarity relation relates a reaction amplitude to a sum of terms, each coming from the contribution of one independent intermediate state. By definition, this sum exists in the physical region of the reaction amplitude considered, and thus provides, through dispersion relations, a representation of the reaction amplitude for strong interactions or others, as long as the latter can be governed by an analytic S-matrix theory. Unitarity can be utilized in two ways which are closely related. One way is to examine which models, with or without approximation, satisfy unitarity. Most of the recent work on the CGL model or multiperipheral models is in this category. This is somewhat a consistency calculation. As we pointed out in the introduction, the CGL model or multiperipheral models essentially take the coupling scheme shown in Fig. 1(a). Other coupling schemes, instead, play an important role if some of the subenergies in the intermediate states are not high; similar calculations in these schemes would probably prove themselves equally important.

One can also use the unitarity relations in another way, as a basis for proposing new models and possibly for improving some existing models by introducing inelastic contributions. One may introduce a model, as in local field theory, by choosing vertex functions in the tree diagrams (the coupling-scheme diagrams) for the helicity amplitude or the h.p.w. amplitude. Thus the helicity formalism provides a tool for extensive studies of unitarity. The investigation of the unitarity diagrams shown in Figs. 1(c) and 1(d) is in progress.

Finally, we demonstrate the usefulness of unitarity in a simple (and probably unrealistic) model. We also illustrate (a) the effect of using different coupling schemes with similar approximation and (b) the kind of approximation that must be made in order to transform the unitarity relation into a linear integral equation. Consider a scattering process $a+b\to a'+b'$. All particles in the initial, final, and intermediate states are scalar mesons with mass μ. Suppose that the amplitudes involved in the unitarity relations are dominated by this scalar meson pole. In this model, the h.p.w. amplitude for the process $a+b\to 0+1+\cdots+(2N+1)$ is given by

$$T(s_i, s; 2N, 2)=g\prod_{i=1}^{n}[g/(s_{2i}-\mu^2)]. \qquad (97)$$

This amplitude can be used as the full amplitude after multiplying it by a factor $[\rho(b)\rho(2N+2)]^{-1/2}$, since all partial waves other than the s wave vanish. The diagrammatical representation of unitarity in the standard coupling scheme is shown in Fig. 10(a). If we adopt the coupling scheme shown in Fig. 10(b), we have the simplest version of the ABFST model.[3] In field-theory language, the production amplitudes in these two coupling schemes are the complementary part in the lowest approximation (Born term),

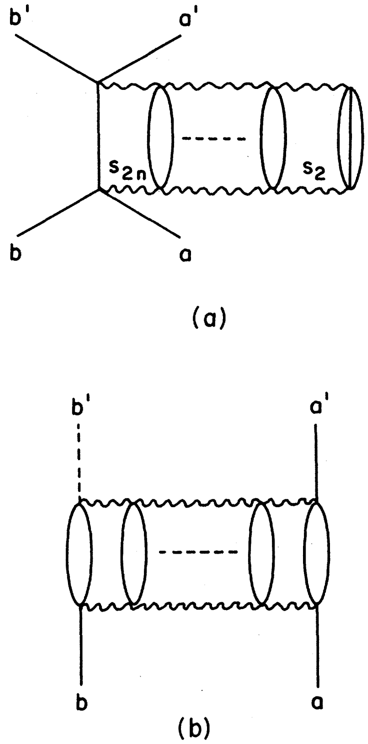

FIG. 10. Unitarity diagrams for the pion system.

while in the language of s-matrix theory, they are related by crossing and continuation. From Eqs. (83), (85), and (97), one has

$$T(s_b)=\sum_{N=1}^{\infty}A(s_b;\,2N+2) \tag{98a}$$

and

$$A(s_b;\,2N+2)=g^2\int\prod_{i=1}^{2n}ds_i\prod_{i=1}^{n}\left(\frac{g}{s_{2i}-\mu^2}\right)^2. \tag{98b}$$

We make the further assumption, as in the ABFST model,[3] that the mass μ is negligible. From Eq. (93), one can see that the integration limit becomes very simple:

$$\int\prod_{i=1}^{2n}ds_i\sim\int_0^{s_b}ds_{2n}\int_0^{s_{2n}}ds_{2n-1}\cdots\int_0^{s_2}ds_1, \tag{99a}$$

$$=\int_0^{s_b}ds_1\int_{s_1}^{s_b}ds_{2n}\int_{s_1}^{s_{2n}}ds_{2n-1}\cdots\int_{s_1}^{s_3}ds_2. \tag{99b}$$

As in the CGL model[1] and in the ABFST model,[3] we define

$$A(s_b,\,2N+2)\equiv\int_{M^2}^{s_b}ds_1\int_{s_1}^{s_b}ds_2B(s_b,\,s_2;\,2N)(g/s_2)^2; \tag{100a}$$

then

$$B(s_b,\,s_2;\,2N)=\int_{s_2}^{s_b}ds'(s'-s_2)B(s_b,\,s';\,2N-2)(g/s')^2. \tag{100b}$$

Here we have interchanged the order of the integration variables and then performed one integration in Eq. (100b). In Eq. (100a), we have replaced the lower limit of the last integration by M^2 (>0) in order to avoid divergences. Defining

$$B(s_b,\,s)\equiv\sum_{N=1}^{\infty}B(s_b,\,s;\,2N), \tag{101a}$$

one obtains, from Eq. (100b), the integral equation

$$B(s_b,\,s)=B(s_b,\,s;\,2)+g^2\int_s^{s_b}ds'\frac{(s'-s)B(s_b,\,s')}{s'^2}, \tag{101b}$$

where

$$B(s_b,\,s;\,2)=g^2. \tag{101c}$$

One can easily see that Eq. (101b) can be transformed into an integral equation in which both the kernel and the inhomogeneous term are functions of s/s_b, and thus so is $B(s_b,\,s)$. Changing the variable s/s_b to r, one obtains

$$B(r)=g^2+g^2\int_r^1dr'\frac{(r'-r)B(r')}{r'^2}, \tag{101d}$$

where

$$B(r)\equiv B(s_b,\,s)\text{ and }r=s/s_b. \tag{101e}$$

The integral equation (101d) is readily transformed into a differential equation, the solution of which is

$$B(s_b,\,s)=C[a_-(s/s_b)^{a_+}-a_+(s/s_b)^{a_-}], \tag{102}$$

with

$$2a_\pm=1\pm(1+4g^2)^{1/2},\quad C=-4g^2/(1+4g^2)^{1/2}. \tag{102$'$}$$

From Eqs. (98a), (100a), and (102), one can obtain the total absorptive part of $\Delta T(s_b)$. We write the leading term for large s_b,

$$\Delta T(s_b)\sim Ca_+M^{2a_-}s_b^{-a_-}. \tag{103}$$

The total absorptive part $\Delta T(s_b)$ behaves as a power in s_b. It is interesting to note that if $g\leq(\frac{21}{16})^{1/2}$, then $\Delta T(s_b)$ does not violate the Froissart bound[39] for nonforward scattering.

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[21]We do not follow the historical development of local field theory. We take the perturbative expansion as a general framework, and consider the choice of a particular interaction Hamiltonian as proposing a model. In this sense, quantum electrodynamics is an excellent model (and thus a realistic theory).

[22]In this convention, the inverses of u_y and b are $u_y^{-1} = u_z(\pi)u_y u_z(-\pi)$, and $b^{-1} = u_y(\pi)bu_y(-\pi)$. It is obvious that $[u_y(\pi)]^2 = 1$.

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[26]In the scattering amplitude, if a pole in s_i appears, the pseudoparticle becomes a real particle or a dynamical system.

[27]Different signs for $\sin\phi$ contribute constant phase factors which are independent of the 4-momenta p_i. Hence, in some cases one need not fix the signs for $\sin\phi_i$ in the helicity partial-wave expansion if the corresponding phase factors can be put into the helicity partial-wave amplitude.

[28]For example, see Ref. 14.

[29]Since the normalization of a particle state, defined in Eq. (28c), is invariant under Poincaré group, so is the helicity amplitude if all particles in the total system are spinless. Note that the normalization here differs from JW's.

[30]In the analytic S-matrix theory, one assumes that these steps functions can be ignored.

[31]It does not mean that the amplitudes are invariant under the Lorentz group, since it is true only for some Lorentz frames, not all.

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