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Configuration-Space View of High-Energy Scattering*

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We consider the amplitude $T(\nu=2q\cdot p, \kappa=q^2, \delta=2q\cdot\Delta;\, t=\Delta^2)$ for the reaction $A(q)+C(p)\to B(q')+D(p')$, where A and B are scalar currents and C and D are on-shell scalar particles. We assume scaling behavior, $T \underset{A}{\to} \nu^{-1}F(\omega,\tau;t)$ for $\nu\to\infty$, with $\omega\equiv\kappa/\nu$, $\tau=\delta/\nu$, and t fixed, and Regge behavior, $T\underset{R}{\to}\nu^{\alpha(t)}\beta(\kappa,\delta;t)$ for $\nu\to\infty$, with κ, δ, and t fixed and implement these behaviors on integral representations with suitable spectral functions. For $\delta=0$, we derive the commutativity relation $\lim_{\omega\to 0}\lim_A T=\lim_{\kappa\to\infty}\lim_R T$ and the asymptotic behavior $f(\lambda,0;t)\sim\lambda^{\alpha(t)}$ for the coefficient $f(x\cdot p, x\cdot\Delta;t)$ of the leading light-cone (LC) singularity of the Fourier transform of T. [f is related to F by $F(\omega,\tau;t)\propto\int_0^\infty d\lambda\, e^{i\lambda\omega}f(\lambda,\lambda\tau;t)$.] Thus, the leading LC singularity determines the trajectory function $\alpha(t)$ and thus provides a configuration-space view of high-energy behavior. This generalizes our previous results for $\Delta=0$. To see what can happen for $\delta\neq 0$, we use the ladder approximation for T. We again derive a commutativity relation $\lim_{\tau\to 0}\lim_{\omega\to 0}\lim_A T=\lim_{\kappa\to\infty}\lim_{\delta\to\infty}\lim_R T$, but now $f(\lambda,\lambda\tau;t)\sim \text{const}\neq 0$. Thus the large-$\lambda$ behavior of f is quite different for $\tau=0$ and for $\tau\neq 0$. This behavior can, in particular, be nice at $\tau=0$, and the assumption that $0<|\int_0^\infty d\lambda\, f(\lambda,0;t)|<\infty$ is seen to lead to interesting constraints on the Regge trajectory.

I. INTRODUCTION

We have previously shown that, in a large class of models,[1] the behavior of a forward off-shell scattering amplitude $T(q^2,\nu)$, in the Regge limit for large mass q^2, is the same as its behavior in the scaling limit for small q^2/ν.[2,3] This result is interesting because we further showed that the behavior in the scaling limit was determined by the leading light-cone (LC) singularity of the Fourier transform $\hat{T}(x^2, x\cdot p)$ of T.[3,4] This means that one can completely determine the $t=0$ intercepts $\alpha(0)$ of the Regge trajectories $\alpha(t)$ from a knowledge of the LC singularities, and thus obtain a configuration-space picture of high-energy behavior. The $\alpha(0)$ are specified by the functional behavior of $\hat{T}(x^2, x\cdot p)$ at the LC $x^2=0$ for large $x\cdot p$.[3]

The purpose of the present paper is to extend these considerations away from the forward direction $t\equiv\Delta^2=0$. This is important in order to establish consistency with the complete Regge picture. We find that the complete trajectory functions $\alpha(t)$ are determined by the leading LC singularities. Now, however, the *functional behavior* for large $x\cdot p$ does not in general specify $\alpha(t)$. This happens only at the point $\delta\equiv 2q\cdot\Delta=0$. This gives special meaning to this point and invites speculation about its significance for on-shell hadron-hadron scattering.

To sharpen the motivation for our study and establish some notation, we consider first the forward reaction $A(q)+C(p)\to A(q)+C(p)$. Here the A's represent scalar currents of dimension 2 [e.g., $A(x) = :\phi^\dagger(x)\phi(x):$] and variable mass squared $q^2\equiv\kappa$, and the C's represent on-shell scalar particles with mass squared $p^2=m^2$. The scattering amplitude $T(\kappa,\nu)$ is thus a function of the mass κ and the energy variable $\nu\equiv 2q\cdot p$. We are concerned with its behavior in the (scaling) A limit ($\nu\to\infty$, $\omega\equiv\kappa/\nu$ fixed) and in the (Regge) R limit ($\nu\to\infty$, κ fixed). We assume scaling behavior,[5]

$$T\underset{A}{\to}\nu^{-1}F(\omega), \tag{1.1}$$

and Regge-pole behavior,

$$T\underset{R}{\to}\nu^{\alpha}\beta(\kappa). \tag{1.2}$$

In the models of Ref. 3, one obtains

$$F(\omega)\xrightarrow[\omega\to 0]{}\beta\omega^{-\alpha-1} \tag{1.3}$$

and

$$\beta(\kappa)\xrightarrow[\kappa\to\infty]{}\beta\kappa^{-\alpha-1}. \tag{1.4}$$

Thus we have the symbolic commutativity relation[6]

$$\lim_{\omega\to 0}\lim_{A} T=\lim_{\kappa\to\infty}\lim_{R} T=\beta\nu^{-1}\omega^{-\alpha-1}=\beta\nu^{\alpha}\kappa^{-\alpha-1}. \tag{1.5}$$

Equation (1.3) tells us that if we know $F(\omega)$, we can determine α. Now, $F(\omega)$ has a simple configuration-space interpretation.[3] The scaling behavior (1.1) is equivalent to the LC behavior

$$\langle p|A(x)A(0)|p\rangle\xrightarrow[x^2\to 0]{}(x^2-i\epsilon x_0)^{-1}f(x\cdot p), \tag{1.6}$$

and $F(\omega)$ is essentially the Fourier transform of $f(\lambda)\theta(\lambda)$,[7]

$$F(\omega)=\int_0^\infty d\lambda\, e^{i\lambda\omega}\tilde{f}(\lambda), \tag{1.7}$$

where $\tilde{f}(\lambda)\propto f(\lambda)$. Equation (1.3) therefore gives

$$\tilde{f}(\lambda) \underset{\lambda\to\infty}{\longrightarrow} \lambda^{\alpha}. \tag{1.8}$$

This is how the LC behavior (1.6) determines α.[3] Thus, from (1.5), the leading LC singularity determines the large-κ piece of the Regge residue function $\beta(\kappa)$. Nonleading LC singularities give the rest of the residue.

In any local field theory, the behavior (1.6) can be understood in terms of LC operator-product expansions.[8,9] Assuming canonical dimensions,[10] the expansion of interest is ($A = B$ here)

$$A(x)B(0) \underset{x^2\to 0}{\longrightarrow} (x^2)^{-1} \sum_{n=0}^{\infty} x^{\alpha_1}\cdots x^{\alpha_n} O^{(n)}_{\alpha_1\cdots\alpha_n}(0), \tag{1.9}$$

where $\dim O^{(n)} = n+2$. Thus

$$\left\langle p \left| \sum_{n=0}^{\infty} x^{\alpha_1}\cdots x^{\alpha_n} O^{(n)}_{\alpha_1}\cdots_{\alpha_n}(0) \right| p \right\rangle = f(x\cdot p) + O(x^2). \tag{1.10}$$

Equations (1.8)–(1.10) provide configuration-space constraints on the field theory arising from the Regge-behavior requirement (1.2).

The above considerations do not, of course, exhaust the constraints of Regge behavior on the field theory. Regge theory also describes what happens away from $t = 0$, and we would now like to impose these constraints. We shall, for simplicity, assume pure Regge-pole behavior although the effects of cuts could be easily handled by the same methods.

We consider the general reaction $A(q) + C(p) \to B(q') + D(p')$, where A and B are scalar currents of dimension 2 and C and D are on-shell scalar particles with $p^2 = m_C^2$ and $p'^2 = m_D^2$. The scattering amplitude is

$$T(\nu, \kappa, \delta;\ t) = i\int d^4x\, e^{iq\cdot x}\langle p|TA(x)B(0)|p'\rangle, \tag{1.11}$$

where we have chosen as variables

$$\nu = 2q\cdot p, \quad \kappa = q^2, \quad t = \Delta^2 \equiv (p-p')^2, \qquad \delta = 2q\cdot\Delta = (q'^2 - q^2 - t). \tag{1.12}$$

We will also use the scaling variables $\omega = \kappa/\nu$ and $\tau = \delta/\nu$. We always keep fixed the masses $p^2 = m_C^2$ and $p'^2 = m_D^2$, and so the on-shell amplitude is $T(\nu, m_A^2, m_B^2 - m_A^2 - t; t)$. In the A limit we let $\nu\to\infty$ with ω, τ, and t fixed, and in the R limit we have $\nu\to\infty$ with κ, δ, and t fixed. The assumed behaviors are

$$T \underset{A}{\to} \nu^{-1}F(\omega, \tau; t) \tag{1.13}$$

and

$$T \underset{R}{\to} \nu^{\alpha(t)}\beta(\kappa, \delta; t). \tag{1.14}$$

In Sec. III we shall implement these behaviors with integral representations with suitable spectral functions. The point $\delta = 0$ will be seen to be the simplest, and the commutativity relation

$$\lim_{\omega\to 0}\lim_{A} T(\nu, \kappa, 0; t) = \lim_{\kappa\to\infty}\lim_{R} T(\nu, \kappa, 0; t) \tag{1.15}$$

will be seen to hold there in general. The configuration-space picture follows from the relation (Sec. II)

$$F(\omega, \tau; t) = \int_0^{\infty} d\lambda\, e^{i\lambda\omega}\tilde{f}(\lambda, \tau; t), \tag{1.16}$$

where $\tilde{f}$ is proportional to the function f defined by

$$\left\langle p \left| \sum_{n=0}^{\infty} x^{\alpha_1}\cdots x^{\alpha_n} O^{(n)}_{\alpha_1}\cdots_{\alpha_n}(0) \right| p' \right\rangle = f(x\cdot p,\ x\cdot\Delta;\ t) + O(x^2). \tag{1.17}$$

The especially simple form for $\tau = 0$,

$$F(\omega, 0; t) = \int_0^{\infty} d\lambda\, e^{i\lambda\omega}\tilde{f}(\lambda, 0; t), \tag{1.18}$$

leads to the behavior

$$\tilde{f}(\lambda, 0; t) \underset{\lambda\to\infty}{\longrightarrow} -iA_{\alpha}(t)\lambda^{\alpha(t)}. \tag{1.19}$$

Thus, at $\tau = 0$, (1.7)–(1.8) are simply generalized.

Away from $\tau = 0$ less general statements can be made and so, in Sec. III, we restrict our attention to the ϕ^3 ladder approximation to T. This model satisfies (1.13) and (1.14). We find that commutativity continues to hold in this model in the sense that

$$\lim_{\tau\to 0}\lim_{\omega\to 0}\lim_{A} T = \lim_{\kappa\to\infty}\lim_{\delta\to\infty}\lim_{R} T. \tag{1.20}$$

Now, however, $\tilde{f}(\lambda, \lambda\delta/\nu; t)$ is found to approach a constant for $\lambda\to\infty$ and so its functional form in λ gives no information about $\alpha(t)$. Thus the point $\delta = 0$ is seen to be rather privileged from this point of view.

The possible further significance of the point $0 = \delta = q'^2 - q^2 - t$ is discussed in Sec. IV. We exploit the possibility that at $\delta = 0$ the large-λ behavior of $\tilde{f}$ can be nice. A major advantage of our configuration-space formalism is that it lends itself well to the incorporation of such additional assumptions. We see, for example, that the assumption that $\tilde{f}(\lambda, 0; t)$ is integrable for some $t = t_0$ leads to significant restrictions on the Regge-trajectory functions at t_0. This is especially interesting if t_0 is the mass-shell value $m_B^2 - m_A^2$ of t.

II. FORMALISM

In this section we describe the class of models to be considered and present some formalism to be used in Sec. III. In order to clearly bring out the main points of interest, we shall limit our explicit consideration to some simple situations. Thus we shall take A and B to be scalar fields ϕ of dimension 1, rather than currents. Also, in order to make eventually an easy comparison with ϕ^3

ladder models, we will only consider graphs from $g\phi^3$ theory with no single-particle intermediate state in the t channel. These simplifications are introduced for convenience only. We could equally well work with general current operators and general theories satisfying scaling and Regge behavior.

Nakanishi[11,12,13] has shown that, to all orders of perturbation theory, the above amplitude can be written in the form[14]

$$T(\nu,\kappa,\delta;t)=\int_0^1 d\eta\int_{-1}^1 d\zeta\int d\gamma\, H(\eta,\zeta,\gamma;t)D^{-3}, \tag{2.1a}$$

where

$$D=\gamma+(1-\eta)[\tfrac{1}{2}(1+\zeta)\delta+\tfrac{1}{2}(1+\zeta)t - \tfrac{1}{2}(1+\zeta)m_C^2-\tfrac{1}{2}(1-\zeta)m_D^2]+\kappa+\nu\eta-i\epsilon. \tag{2.1b}$$

If we were considering all graphs in ϕ^3 theory, then D^{-3} in (2.1a) would be replaced by D^{-2}. Instead of (2.1), we could work with the Jost-Lehmann-Dyson representation, but this is less convenient for our purposes.

Let us now consider the ladder approximation to T in which the two scalar particles exchange a scalar meson of mass μ. Then the Bethe-Salpeter equation for T gives an integral equation for H.[12] The corresponding "asymptotic" ($\eta\to 0$) integral equation has a solution

$$\tilde H(\eta,\zeta,\gamma;t)=\eta^{-\alpha-1}H_\alpha(\zeta,\gamma;t),\quad \alpha=\alpha(t).$$

H_α satisfies the integral equation (suppressing t)[12]

$$H_\alpha(\zeta,\gamma)=g\int_{-1}^1 d\zeta'\int d\gamma' K_\alpha(\zeta,\gamma;\zeta',\gamma')H_\alpha(\zeta',\gamma'), \tag{2.2}$$

where

$$K_\alpha=\tfrac{1}{2}\int_0^1 dx\, x^{-2}G^{-\alpha-2}\frac{\partial}{\partial\gamma}\{\gamma^{\alpha+1}[\theta(\gamma)-\theta(\gamma-RG)]\}. \tag{2.3}$$

Here

$$G=x^{-1}[\gamma'+(1-x)\rho(\zeta')]+(1-x)^{-1}\mu^2, \tag{2.4}$$

with

$$\rho(\zeta)=\tfrac{1}{2}(1+\zeta)m_C^2+\tfrac{1}{2}(1-\zeta)m_D^2-\tfrac{1}{4}(1-\zeta^2)t \tag{2.5}$$

and

$$R=(1\mp\zeta)/(1\mp\zeta')\quad\text{for}\quad \zeta\gtrless\zeta'. \tag{2.6}$$

It can be shown that[12]

$$\lim_{\gamma\to\infty}H_\alpha(\zeta,\gamma)=0. \tag{2.7}$$

The above solution leads to Regge asymptotic behavior for T. The eigenvalue equation (2.2) determines α as a function of t. The asymptotic equation has, however, other solutions corresponding to nonpure Regge-pole behavior. The solution to the exact equation is nevertheless expected to have pure Regge-pole behavior. This is strongly suggested by the explicit Bethe-Salpeter l-plane analysis,[15] by the explicit asymptotic summations,[16] and by the fact that it is true for the exact zero-mass solution.[17]

The behavior of T in the A limit is, by the usual arguments,[3,4,8] determined by the LC behavior of the integrand in (1.11). The general leading behavior

$$\langle p|TA(x)B(0)|p'\rangle\xrightarrow[x^2\to 0]{}(x^2-i\epsilon)^{-r}f_r(x\cdot p,x\cdot\Delta;t) \tag{2.8}$$

gives

$$T\underset{A}{\to}K_r\nu^{r-2}\int_0^\infty d\lambda\, e^{i\lambda\omega}f_r(\lambda,\lambda\tau;t)\lambda^{1-r} \tag{2.9}$$

$$\equiv\nu^{r-2}\int_0^\infty d\lambda\, e^{i\lambda\omega}\tilde f_r(\lambda,\lambda\tau;t), \tag{2.10}$$

where K_r is a trivial numerical factor. Equation (2.9) is easily derived by substituting (2.8) in (1.11) and using

$$\int d^4x\, e^{iq\cdot x}(x^2-i\epsilon)^{-r}\propto(q^2+i\epsilon)^{r-2}$$

and

$$\int d\xi\, e^{-i\xi\sigma}(\xi+i\epsilon)^s\propto\theta(\sigma)\sigma^{-s-1}.$$

If r is a negative integer (say $-n$), then $(x^2)^{-r}$ must be interpreted as $(x^2)^n\ln x^2$ in order for the above relation to be valid.

The LC operator-product expansion for the models under consideration is

$$\phi(x)\phi(0)\xrightarrow[x^2\to 0]{}x^2\ln x^2 g^2\sum_{n=0}^\infty x^{\alpha_1}\cdots x^{\alpha_n}O^{(n)}_{\alpha_1\cdots\alpha_n}(0), \tag{2.11}$$

with $\dim O^{(n)}=n+2$ and $\dim g=1$. Thus $r=-1$ and so (2.10) becomes[18]

$$T\underset{A}{\to}\nu^{-3}\int_0^\infty d\lambda\, e^{i\lambda\omega}\tilde f(\lambda,\lambda\tau;t). \tag{2.12}$$

III. LIMITS AND RELATIONS

Our purpose now is to use the formalism of Sec. II to study the R and A limits defined in Sec. I. We assume the behaviors (1.13) and (1.14), which become here

$$T\underset{A}{\to}\nu^{-3}F(\omega,\tau;t) \tag{3.1}$$

and

$$T\underset{R}{\to}\nu^{\alpha(t)}\beta(\kappa,\delta;t). \tag{3.2}$$

A number of interesting results will emerge, including those stated in Sec. I.

Consider first the Nakanishi representation (2.1). Our basic extra assumption will be that the spectral function $H(\eta,\zeta,\gamma;t)$ is well behaved for large γ, uniformly in η, ζ, and t, at least so that the integral

$$H(\eta,\zeta;t)\equiv\int d\gamma\, H(\eta,\zeta,\gamma;t) \tag{3.3}$$

exists. This is true in the ladder models [cf. Eq. (2.7)], and is the generalization of the assumption on the Deser-Gilbert-Sudarshan spectral function used and motivated in Ref. 3 to obtain the $t=0$ results stated in Sec. I. Without this assumption, the scaling behavior (1.13) could only be obtained in a rather pathological way.

The A limit of (2.1) is thus

$$T \underset{A}{\to} \nu^{-3} \int_0^1 d\eta \int_{-1}^1 d\xi\, H(\eta, \xi; t) \times[(1-\eta)\tfrac{1}{2}(1+\xi)\tau+\omega+\eta]^{-3}, \tag{3.4}$$

in accordance with (3.1). Comparison with (2.12) gives

$$\tilde{f}(\lambda, \lambda\tau; t)=\tfrac{1}{2}i\lambda^2 \int_0^1 d\eta \int_{-1}^1 d\xi\, H(\eta, \xi; t) \times \exp\{i\lambda[(1-\eta)\tfrac{1}{2}(1+\xi)\tau+\eta]\}. \tag{3.5}$$

The simplicity of this relation, which conveniently exhibits H as essentially the Fourier transform of $\tilde{f}$, is another reason for our use of the Nakanishi representation. Note that, from (3.4) or, equally well, from (2.8)–(2.10), the $\tau \to 0$ (or, equivalently, $\delta \to 0$) limit always commutes with the A limit:

$$\lim_A T(\nu, \kappa, 0; t)=\lim_{\tau\to 0} \lim_A T(\nu, \kappa, \delta; t) =\nu^{-3} \int d\eta \int d\xi\, H(\eta, \xi; t)(\omega+\eta)^{-3}. \tag{3.6}$$

Note also that (3.5) shows that $\tilde{f}(z, z'; t)$ is an entire function of z and z'. This is essentially a consequence of the spectral properties of T built into the representation (2.1).

In order to obtain information about $\tilde{f}$, we consider next the R limit (1.14). To obtain (1.14), we take

$$H(\eta, \xi, \gamma; t) \underset{\eta\to 0}{\longrightarrow} \eta^{-\alpha-1} H_\alpha(\xi, \gamma; t)[\tfrac{1}{2}\Gamma(\alpha+3)\Gamma(-\alpha)]^{-1} \tag{3.7}$$

for some function H_α. Here α means $\alpha(t)$, and we have factored out the Γ functions for convenience. The trajectory α is determined as a function of t by an integral equation for H_α arising from the one for H.[13] With (3.7), as Nakanishi showed for $\alpha > -3$, one has

$$T \underset{R}{\to} \nu^\alpha \int_{-1}^1 d\xi \int d\gamma H_\alpha(\xi, \gamma; t)[D(\eta=0)]^{-\alpha-3}. \tag{3.8}$$

Equations (3.7) and (3.8) can be chosen to be correct for *all* α. The behavior (3.8) should, of course, be correct for all α by analytical continuation from $\alpha > -3$. In order for (2.1) to satisfy (3.8) for $\alpha \le -3$, however, the spectral function H must satisfy certain conditions in addition to (3.7); for example,

$$\int_0^1 d\eta \int_{-1}^1 d\xi \int d\gamma\, H(\eta, \xi, \gamma; t)\eta^{-3}=0 \quad \text{for } \alpha(t)<-3. \tag{3.9}$$

We will assume that such conditions are satisfied, since it will be of interest to us to consider $\alpha \le -3$.

Equation (3.6) only gives enough information to determine the large-λ behavior of $\tilde{f}$ for $\tau=0$. From (3.5), we have the especially simple relation

$$\tilde{f}(\lambda, 0; t)=\tfrac{1}{2}i\lambda^2 \int_0^1 d\eta \int_{-1}^1 d\xi\, H(\eta, \xi; t)e^{i\lambda\eta}. \tag{3.10}$$

Equation (3.7) thus gives

$$\tilde{f}(\lambda, 0; t) \underset{\lambda\to\infty}{\longrightarrow} -iA_\alpha(t)\lambda^{\alpha+2}, \tag{3.11}$$

apart from oscillatory terms,[19] where

$$A_\alpha(t)=-e^{-\pi i\alpha/2}[\Gamma(\alpha+3)]^{-1}I_\alpha(t), \tag{3.12}$$

with

$$I_\alpha(t)\equiv \int_{-1}^1 d\xi\, H_\alpha(\xi, t), \tag{3.13}$$

$$H_\alpha(\xi, t)\equiv \int d\gamma\, H_\alpha(\xi, \gamma; t). \tag{3.14}$$

According to Eqs. (1.18) and (3.11), the $\tau=0$ (or $\delta=0$)[20] point of the general amplitude gives the same type of results as in the forward $\Delta=0$ case. Note incidentally, that the behaviors of the f's are exactly the same for the $r=+1$ [cf. Eq. (2.8)] case discussed in Sec. I ($f\sim\lambda^{r-1}\tilde{f}\sim\tilde{f}\sim\lambda^\alpha$) and the $r=-1$ case discussed above ($f\sim\lambda^{r-1}\tilde{f}\sim\lambda^{-2}\tilde{f}\sim\lambda^\alpha$). In particular, we can again learn about the Regge trajectories from the leading LC singularity. Now in fact, we obtain from (3.11) the complete trajectory function $\alpha(t)$.

It follows from Eqs. (2.12) and (3.11) that

$$\lim_{\omega\to 0}\lim_{\tau\to 0}\lim_A T(\nu, \kappa, \delta; t)=\nu^{-3}\omega^{-\alpha-3}I_\alpha(t) =\nu^\alpha\kappa^{-\alpha-3}I_\alpha(t). \tag{3.15}$$

Exactly the same result follows from Eqs. (3.6) and (3.7). This is another way of seeing that, at $\delta=0$, the A limit determines the Regge trajectories. We can use the result (3.8) to take the limits in the "opposite" order and obtain (for all δ, including $\delta=0$)

$$\lim_{\kappa\to\infty}\lim_R T(\nu, \kappa, \delta; t)=\nu^\alpha\kappa^{-\alpha-3}I_\alpha(t). \tag{3.16}$$

We thus obtain the commutativity relation (1.15). In terms of the coefficient functions F and β, the leading LC singularity is seen to determine and equate their asymptotic behaviors,

$$F(\omega, 0; t) \underset{\omega\to 0}{\longrightarrow} I_\alpha(t)\omega^{-\alpha-3}, \tag{3.17}$$

$$\beta(\kappa, 0; t) \xrightarrow[\kappa\to\infty]{} I_\alpha(t)\kappa^{-\alpha-3}. \quad (3.18)$$

To get an idea of what happens away from $\tau=0$, we shall restrict ourselves to the ladder approximation for T. In this approximation, T is known to have the scaling[21] and Regge[15-17] behaviors (3.1) and (3.2). The moment (3.3) also exists so that all of the above results are valid. Since this ladder model is one of the few models known to satisfy these conditions, our use of it is hardly much of a restriction. Its compact structure, exemplified by Eqs. (2.2)–(2.7), will, however, enable us to obtain a detailed description of the limits of interest away from $\tau=0$.

We begin by deriving the behavior of $H_\alpha(\zeta, \gamma; t)$ for $\zeta\to\pm1$. If we take $\alpha<-1$, we easily find from (2.2)–(2.6) the result

$$H_\alpha(\zeta, \gamma; t) \xrightarrow[\zeta\to\pm1]{} C_\alpha(\gamma; t)(1-\zeta^2)^{\alpha+1}. \quad (3.19)$$

Equation (3.19) exhibits the leading singularity at $\zeta=\pm1$. We have taken $\alpha<-1$ in order that a singularity exists. The limit (3.19) is uniform in γ, so that we can write

$$H_\alpha(\zeta; t) \xrightarrow[\zeta\to\pm1]{} C_\alpha(t)(1-\zeta^2)^{\alpha+1}, \quad (3.20)$$

with

$$C_\alpha(t)=\int d\gamma\, C_\alpha(\gamma; t). \quad (3.21)$$

It is only to derive the behavior (3.19) that we need the explicit ladder model. It is necessary to know this threshold behavior, however, in order to proceed.

Using (3.7) and (3.20) in (3.5), we obtain the result[22]

$$\tilde f(\lambda, \lambda\tau; t) \xrightarrow[\lambda\to\infty]{} -iB_\alpha(t)\tau^{-\alpha-2}, \quad (3.22)$$

apart from oscillatory terms, where

$$B_\alpha(t)=-C_\alpha(t)e^{-\pi i\alpha/2}(\alpha+2)^{-1}. \quad (3.23)$$

Thus, quite contrary to the $\tau=0$ case (3.11), we now obtain a *constant* asymptotic behavior for $\tilde f$. It is no longer true that the *functional form* in λ of $\tilde f$, for large λ, determines α. It remains true, of course, that knowledge of the complete large-λ value of $\tilde f$ is sufficient to determine $\alpha(t)$.

Let us comment further on the sharp contrast between the $\tau=0$ behavior (3.11) and the $\tau\neq0$ behavior (3.22). We have seen that for $\tau\neq0$, $\tilde f\to$ const $\neq0$ for $\lambda\to\infty$, for all α. For $\tau=0$, on the other hand, $\tilde f\to$ const λ^α for $\lambda\to\infty$. The $\tau=0$ behavior for large λ can thus be better or worse than the $\tau\neq0$ behavior, depending on $\alpha(t)$. Namely, if $\alpha(t)>-2$, the $\tau=0$ behavior is worse, and if $\alpha(t)<-2$, the $\tau=0$ behavior is better. We will discuss these possibilities in more detail in Sec. IV.

Even though the large-λ behavior of $\tilde f$ for $\tau\neq0$ is so different from that for $\tau=0$, commutativity-type relations are still valid. To see this, we first use (3.22) to take the $\omega\to0$ limit of (2.12) to obtain

$$\lim_{\omega\to0}\lim_{A} T=B_\alpha(t)\nu^\alpha\delta^{-\alpha-2}\kappa^{-1}. \quad (3.24)$$

The same result of course follows from using (3.7) and (3.20) in (3.4). The behavior of T in the opposite order is, by using (3.20) in (3.8), easily found to be

$$\lim_{\kappa\to\infty}\lim_{\delta\to\infty}\lim_{R} T=B_\alpha(t)\nu^\alpha\delta^{-\alpha-2}\kappa^{-1}. \quad (3.25)$$

Thus we obtain the equality (1.20).[23,24] In terms of the coefficient functions, we have

$$F(\omega, \tau; t) \xrightarrow[\omega\to0]{} B_\alpha(t)\tau^{-\alpha-2}\omega^{-1} \quad (3.26)$$

and

$$\beta(\kappa, \delta; t) \xrightarrow[\delta\to\infty,\,\kappa\to\infty]{} B_\alpha(t)\delta^{-\alpha-2}\kappa^{-1}. \quad (3.27)$$

All of the above results can be explicitly verified for Nakanishi's[17] exact solution to the ladder approximation at $t=0$, for the case of equal-mass $m=1$ legs and zero-mass rungs, except for the first of mass $\mu=2m=2$:

$$H(\eta, \zeta, \gamma; 0)=\delta(\gamma)(1-\eta)P_a\left(1+\frac{(1-\eta)^2(1-\zeta)^2}{\eta(1+\eta)}\right), \quad (3.28)$$

where

$$a=(g+\tfrac14)^{1/2}-\tfrac12\equiv\alpha_0+1 \quad (3.29)$$

and P_a is a Legendre function. Thus

$$H_\alpha(\zeta; 0)=\text{const}(1-\zeta^2)^{\alpha_0+1}. \quad (3.30)$$

We mention, finally, that our arguments for the significance of the point $\tau=0$ do not depend on our specific choice of variables (1.12) or functional form (1.17). If we worked with any other variables or other function $g(x\cdot P, x\cdot Q; t)=f(x\cdot p, x\cdot\Delta; t)$, with P and Q linear combinations of p and p', we would still conclude that $\tau=0$ is the point where the functional form of f is determined by the Regge trajectory. If we maintain explicit crossing symmetry, then the natural variables are Q^2, $Q\cdot P$, and $Q\cdot\Delta$, with $Q=q+q'$ and $P=p+p'$. We then find again that $\tau=Q\cdot\Delta/Q\cdot P\to0$ is the special point.[25]

IV. INTEGRABILITY ASSUMPTION AND CONSEQUENCES

Motivated by the results of Sec. III, we will study here further the significance of the point $\tau=0$. We are particularly interested in the fact that at $\tau=0$, and only there, the large-λ behavior of $\tilde f$ can be nice. This invites us to explore consequences of the assumption that this behavior is, in fact, nice.

In the previous sections, we have derived a number of results both from the Nakanishi representation (3.4) and from the "configuration-space" representation (2.12). It is useful to have these (and other) different representations because they emphasize different aspects of the amplitude. This is important because, in order to go beyond the essentially purely kinematical considerations of Sec. III, further (dynamical) assumptions must be introduced, and such assumptions might be easier to discover or formulate in a particular representation. Configuration space is, we feel, especially suited to this purpose, since that is where we all live and where our intuition from classical physics, quantum mechanics, and quantum electrodynamics[26] can be brought to bear. Configuration-space ideas, such as current algebra, short-distance expansions, and light-cone expansions have already increased our understanding of some aspects of particle physics. It therefore seems worthwhile to search for further configuration-space principles.

Now, current algebra makes statements about $\lambda=0$ behavior. Equation (1.6), for example, implies the typical current-algebraic relation

$$\langle p|[\dot{A}(0,\vec{x}),A(0)]|p\rangle=2\pi i\delta(\vec{x})f(0). \qquad (4.1)$$

Light-cone expansions specify further the *singularity* structure at the LC, as in (1.6), but do not give much information about $f(\lambda)$. We have seen above that the large-λ behavior of $f(\lambda)$ is determined by Regge theory. Conversely, assumptions about this behavior will give constraints on the Regge limit.

Let us, for example, suppose that $\tilde{f}(\lambda)$ in (1.7) is integrable. Then

$$F(\omega)\xrightarrow[\omega\to 0]{}\int_0^\infty d\lambda\,\tilde{f}(\lambda)\equiv b. \qquad (4.2)$$

Comparison with (1.3) thus gives the constraint $\alpha\leq -1$. If we further assume that $b\neq 0$, then we learn that $\alpha=-1$. Thus simple, *a priori* not unreasonable assumptions about the function (1.10) lead to interesting predictions about Regge trajectories. A look at Eq. (1.8), however, tends to weaken the analysis since it says that $\tilde{f}(\lambda)$ is integrable if and only if $\alpha\leq -1$.[27] If α is, in fact, less than -1, then $b=0$. Nevertheless, if we consider the configuration-space function $\tilde{f}(\lambda)$ to be the fundamental quantity, and do not assume Regge behavior to begin with, then the assumption that $0<|b|<\infty$ implies Regge behavior with $\alpha=-1$, so that $T\underset{R}{\to}\nu^{-1}\beta(\kappa)$.

As an example, let us consider the electroproduction structure functions, following the notation of Ref. 3. We have

$$\nu W_2(\kappa,\nu)\underset{A}{\to}F_2(\omega) \qquad (4.3)$$

with

$$F_2(\omega)=i\int d\lambda\,e^{i\lambda\omega}\hat{f}_2(\lambda). \qquad (4.4)$$

The integrability assumption for $\hat{f}_2(\lambda)$ leads to $F_2(\omega)\to$ const as $\omega\to 0$, which is precisely the experimental behavior and the behavior suggested by Pomeranchukon dominance of the R limit.[2,3]

Our analysis in Sec. III enables us to extend the above considerations away from $\Delta=0$. We consider the general A limit (2.12). We have seen in Sec. III that only for $\tau=0$ can we assume that the large-λ behavior of $f(\lambda,\lambda\tau;t)$ is nice. Indeed, since $\tilde{f}(\lambda,\lambda\tau;t)\to$ const $\neq 0$ as $\lambda\to\infty$, for $\tau\neq 0$, $\tilde{f}(\lambda,\lambda\tau;t)$ can never be integrable for $\tau\neq 0$. We can, however, assume that $\tilde{f}(\lambda,0;t)$ is integrable for some $t=t_0$. Then the representation

$$F(\omega,0;t)=\int_0^\infty d\lambda\,e^{i\lambda\omega}\tilde{f}(\lambda,0;t) \qquad (4.5)$$

leads to the behavior

$$F(\omega,0;t_0)\xrightarrow[\omega\to 0]{}\int_0^\infty d\lambda\,\tilde{f}(\lambda,0;t_0)\equiv b(t_0). \qquad (4.6)$$

Comparison with (3.17) thus gives $\alpha(t_0)\leq -3$.[28] Again, if $b(t_0)$ is assumed not to vanish, we have the equality $\alpha(t_0)=-3$. Thus, for $\tau=0$ and for this value t_0 of t, the ν behavior of the A limit (3.4) is the same as the ν behavior of the R limit (3.8) and

$$\lim_{\omega\to 0}F(\omega,0;t_0)=\lim_{\kappa\to\infty}\beta(\kappa,0;t_0)=I_{-3}(t_0)=b(t_0).$$

If $b(t_0)=0$, then $\alpha(t_0)<-3$. The vanishing of the moment (4.6) in this case is, in fact, an immediate consequence of (3.5) and (3.9).

It is tempting to speculate further and make an assumption about the values t_0 for which $0<|b(t_0)|<\infty$. A natural choice is the mass-shell value $t_0=m_B{}^2-m_A{}^2$ (assuming δ is privileged over other variables giving the same τ) so that $\alpha(m_B{}^2-m_A^2)=-3$. These assumptions, and others concerning the dimension of relevant current operators, were recently made in realistic cases where they lead to the quantization conditions $\alpha_R(m_B{}^2-m_A{}^2)=N_{AB}^R$ = half integer, for any particles A and B which couple to the Reggeon R.[29] These conditions were found to be in good agreement with experiment. Although additional strong assumptions are required here, the results further suggest the special significance of the point $\delta=0$.

Let us conclude by summarizing our main results:

(i) The leading LC singularity determines the A limit (3.2) and the Regge-trajectory function $\alpha(t)$ by (3.26) or, more directly, by (3.22). It also determines the behavior (3.27) of the Regge residue function for $\delta\to\infty$, $\kappa\to\infty$.

(ii) The situation for $0=\delta=2q\cdot\Delta$ is a simple generalization of the forward $\Delta=0$ case and we have (3.17), (3.18), and (3.11).

(iii) The function $\tilde{f}(\lambda,\lambda\tau;t)$ is never integrable in λ. If $\tilde{f}(\lambda,0;t_0)$ is assumed to be integrable, we have $\alpha(t_0)\leq -3$; if its integral is further assumed not to vanish, we have $\alpha(t_0) = -3$; and if it is still further assumed that all this happens on shell, we have $\alpha(m_B^2 - m_A^2) = -3$.

These results suggest that it may be useful to study further the significance of the interesting point $\tau = 0$.

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[1]These models are defined by Deser-Gilbert-Sudarshan representations with suitable spectral functions.

[2]R. A. Brandt, Phys. Rev. Letters 22, 1149 (1969).

[3]R. A. Brandt, Phys. Rev. D 1, 2808 (1970). The treatment for the general scalar amplitudes, of interest in the present paper, is given in the Appendix of this reference.

[4]R. A. Brandt, Phys. Rev. Letters 23, 1260 (1969).

[5]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

[6]Such relations were originally suggested by H. Abarbanel, M. Goldberger, and S. Treiman, Phys. Rev. Letters 22, 500 (1969). See G. Preparata (unpublished) for a derivation using mass dispersion relations.

[7]The factor $\theta(\lambda)$ is present because we are working with the time-ordered product. If we use its absorptive part, as in Ref. 3, the factor would not be present.

[8]R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).

[9]W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory, 1970 Brandeis Summer Institute*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1971).

[10]We assume as usual, consistently with exact scaling, that logarithmic factors are not present. See Ref. 8.

[11]N. Nakanishi, Progr. Theoret. Phys. (Kyoto) 26, 377 (1961).

[12]N. Nakanishi, Phys. Rev. 133, B214 (1964).

[13]N. Nakanishi, Phys. Rev. 133, B1224 (1964).

[14]Equation (2.1) is obtained from the representation of Refs. 11–13 by simply changing to our variables.

[15]B. W. Lee and R. F. Sawyer, Phys. Rev. 127, 1266 (1962).

[16]P. G. Federbush and M. T. Grisaru, Ann. Phys. (N.Y.) 22, 263 (1963); J. C. Polkinghorne, J. Math. Phys. 4, 503 (1963).

[17]N. Nakanishi, Phys. Rev. 135, B1430 (1964).

[18]Strictly speaking, since (2.11) is not singular at $x^2=0$, an assumption about the effect of the omitted terms in (2.11) (which vanish for $x^2 \to 0$ faster than the exhibited term) must be made in order to obtain (2.12).

[19]These oscillatory terms can come from the end point $\eta=1$, but they do not contribute to the leading behaviors of interest and so they will always be neglected.

[20]As far as the leading term in the A limit is concerned, the point $\delta = 0$ is, of course, equivalent to any point $\delta = \text{const}$.

[21]See, for example, H. Abarbanel *et al.*, Ref. 6.

[22]The large-λ behavior of interest comes from the integration region near $\eta = 0$ and $\xi = 1$. We have assumed this is given by first letting $\eta \sim 0$ and then $\xi \sim -1$. This can be justified by using more explicit properties of the model, but we omit the details.

[23]We can actually derive this equality under much more general circumstances than for the ladder model. Any assumed behavior of the type (3.19) leads to it, but not necessarily to the explicit form (3.25). It would be interesting to determine the threshold behavior of $H_\alpha(\xi;t)$ in a model-independent way.

[24]Other types of limits are also of interest. For example, G. Altarelli and G. Preparata (unpublished) have, in our language, used the limit $\omega = \tau \to 0$.

[25]I thank Roger Dashen for raising this question.

[26]We have in mind the formulation based on finite local-field equations: R. A. Brandt, Ann. Phys. (N.Y.) 52, 122 (1969); Fortschr. Physik 18, 249 (1970).

[27]For our purposes, λ^{-1} is considered as an integrable function. This can be understood in several ways, the most simple being in terms of an analytic continuation of an integral of $\lambda^{-1-\epsilon}$.

[28]The shift from -1 to -3 here is, of course, due to our choice of fields and models. Using more realistic currents and models, we would encounter larger (physical) values of α.

[29]R. A. Brandt and C. A. Orzalesi, Phys. Letters 34B, 641 (1971).